Mathematica Moravica
Vol. **21**, No. **1** (2017), 87–94

A generalization of modules with the property (P^*)

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Abstract. I.A- Khazzi and P.F. Smith called a module M have the property (P^*) if every submodule N of M there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} \subseteq Rad(\frac{M}{K})$. Motivated by this, it is natural to introduce another notion that we called modules that have the properties (GP^*) and $(N-GP^*)$ as proper generalizations of modules that have the property (P^*). In this paper we obtain various properties of modules that have properties (GP^*) and $(N-GP^*)$. We show that the class of modules for which every direct summand is a fully invariant submodule that have the property (GP^*) is closed under finite direct sums. We completely determine the structure of these modules over generalized f-semiperfect rings.

1. Introduction

Throughout this paper, all rings are associative with identity element and all modules are unital right R-modules. Let R be a ring and let M be an R-module. The notation $N \leq M$ means that N is a submodule of M. A module M is called *extending* if every submodule of M is essential in a direct summand of M [4]. Here a submodule $L \leq M$ is said to be *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap N \neq 0$ for every nonzero submodule $N \leq M$. Dually, a submodule S of M is called *small (in M)*, denoted as $S << M$, if $M \neq S + L$ for every proper submodule L of M. By $Rad(M)$, we denote the intersection of all maximal submodules of M. An R-module M is called *supplemented* if every submodule N of M has a *supplement*, that is a submodule K minimal with respect to $M = N + K$. Equivalently, $M = N + K$ and $N \cap K << K$ [11]. M is called *(f-) supplemented* if every (finitely generated) submodule of M has a supplement in M (see [11]). On the other hand, M is called *amply supplemented* if, for any submodules N and K of M with $M = N + K$, K contains a supplement of N in M. Accordingly a module M is called *amply f-supplemented* if every finitely generated submodule of M satisfies same condition. It is clear that (amply) f-supplemented modules are a proper generalization of (amply) supplemented modules.

2010 *Mathematics Subject Classification.* 16D10, 16N80.

Key words and phrases. Generalized f-semiperfect ring, the properties (P^*), (GP^*) and $(N\text{-}GP^*)$.

A module M is called *lifting* if for every submodule N of M there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} << \frac{M}{K}$ (i.e. K is a *coessential submodule* of N in M) as a dual notion of extending modules. Mohamed and Müller has generalized the concept of lifting modules to $\oplus$-supplemented modules. M is called *$\oplus$-supplemented* if every submodule of M has a supplement that is a direct summand of M [6].

Let M be an R-module and let N and K be any submodules of M. If $M = N + K$ and $N \cap K \subseteq Rad(K)$, then K is called a *Rad-supplement* of N in M [12](according to [10], generalized supplement). It is clear that every supplement is Rad-supplement. M is called *Rad-supplemented* (according to [10], generalized supplemented) if every submodule of M has a Rad-supplement in M, and M is called *amply Rad-supplemented* if, for any submodules N and K of M with $M = N+K$, K contains a Rad-supplement of N in M. An R-module M is called *f-Rad-supplemented* if every finitely generated submodule of M has a Rad-supplement in M, and a module M is called *amply f-Rad-supplemented* if every finitely generated submodule of M has ample Rad-supplements in M (see [7]). A module M is called *Rad-$\oplus$-supplemented* if every submodule has a Rad-supplement that is a direct summand of M [3] and [5].

Recall from Al-Khazzi and Smith [1] that a module M is said to have *the property (P^*)* if for every submodule N of M there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} \subseteq Rad(\frac{M}{K})$. The authors have obtained in the same paper the various properties of modules with the property (P^*). Radical modules have the property (P^*). It is clear that every lifting module has the property (P^*) and every module with the property (P^*) is Rad-$\oplus$-supplemented.

Let $f : P \longrightarrow M$ be an epimorphism. If $Ker(f) << P$, then f is called *cover*, and if P is a projective module, then a cover f is called a *projective cover* [11]. Xue [12] calls f a *generalized cover* if $Ker(f) \leq Rad(P)$, and calls a generalized cover f a *generalized projective cover* if P is a projective module. In the spirit of [12] , a module M is said to be *(generalized) semiperfect* if every factor module of M has a (generalized) projective cover. A module M is said to be *f-semiperfect* if, for every finitely generated submodule $U \leq M$, the factor module $\frac{M}{U}$ has a projective cover in M [11]. Let M be an R-module. M is called *generalized f-semiperfect module* if, for every finitely generated submodule $U \leq M$, the factor module $\frac{M}{U}$ has a generalized projective cover in M [8].

In this study, we obtain some elementary facts about the properties (GP^*) and $(N - GP^*)$ which are a proper generalizations of the property (P^*). Especially, we give a relation for G^*-supplemented modules. We prove that every direct summand of a module that have the property (GP^*) has the property (GP^*). We show that a module M has the property $(N - GP^*)$ if and only if, for all direct summands M' and a coclosed submodule N' of N,

M' has the property $(N' - GP^*)$ for right R-modules M and N. We obtain that Let $M = \oplus_{i=1}^{n} M_i$ be a module and M_i is a fully invariant submodule of M for all $i \in \{1, 2, \ldots, n\}$. Then M has the property (GP^*) if and only if M_i has the property (GP^*) for all $i \in \{1, 2, \ldots, n\}$. We illustrate a module with the property (GP^*) which doesn't have the property (P^*). We give a characterization of generalized f-semiperfect rings via the property (GP^*).

2. Modules with the Properties of (GP^*) and $(N - GP^*)$

Definition 2.1. A module M *has the property* (GP^*) if, for every $\gamma \in End_R(M)$ there exists a direct summand N of M such that $N \subseteq Im(\gamma)$ and $\frac{Im\gamma}{N} \subseteq Rad(\frac{M}{N})$.

Proposition 2.1. *The following conditions are equivalent for a module M.*

1. *M has the property (GP^*).*
2. *For every $\gamma \in End_R(M)$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq Im(\gamma)$ and $M_2 \cap Im(\gamma) \subseteq Rad(M_2)$.*
3. *For every $\gamma \in End_R(M)$, $Im(\gamma)$ can be represented as $Im\gamma = N \oplus N'$, where N is a direct summand of M and $N' \subseteq Rad(M)$.*

Proof. (1) ⇒ (2) By the hypothesis, there exist direct summands M_1, M_2 of M such that $M_1 \subseteq Im(\gamma)$, $M = M_1 \oplus M_2$ and $\frac{Im(\gamma)}{M_1} \subseteq Rad(\frac{M}{M_1})$. Since M_2 is a Rad-supplement of M_1 in M, $Rad(\frac{M}{M_1}) = \frac{Rad(M)+M_1}{M_1}$ (See [13, Lemma 1.1]). Then $\frac{Im(\gamma)}{M_1} \subseteq \frac{Rad(M)+M_1}{M_1}$. So we have $Im(\gamma) \subseteq Rad(M_2) + M_1$. By the modular law, $M_2 \cap Im(\gamma) \subseteq Rad(M_2)$.

(2) ⇒ (3) For every $\gamma \in End_R(M)$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq Im(\gamma)$ and $M_2 \cap Im(\gamma) \subseteq Rad(M_2)$. So $Im(\gamma) = M_1 \oplus (Im(\gamma) \cap M_2)$ by the modular law. Say $N = M_1$ and $N' = Im(\gamma) \cap M_2$. Therefore $Im(\gamma) = N \oplus N'$, where N is a direct summand of M and $N' \subseteq Rad(M)$.

(3) ⇒ (1) By the hypothesis, for every $\gamma \in End_R(M)$, $Im(\gamma) = N \oplus N'$ where N is a direct summand of M and $N' \subseteq Rad(M)$. Thus there exists a direct summand N of M such that $N \subseteq Im(\gamma)$. We have $\frac{Im(\gamma)}{N} = \frac{N \oplus N'}{N} \subseteq \frac{N + Rad(M)}{N} \subseteq Rad(\frac{M}{N})$. □

Definition 2.2. A module M *has the property* $(N - GP^*)$ if, for every homomorphism $\gamma : M \longrightarrow N$, there exists a direct summand L of N such that $L \subseteq Im(\gamma)$ and $\frac{Im\gamma}{L} \subseteq Rad(\frac{N}{L})$.

It is clear that a right module M has the property (GP^*) if and only if M has the property $(M - GP^*)$.

Recall from [4, 3.6] that a submodule N of M is called *coclosed* in M if, N has no proper submodule K for which $K \subset N$ is cosmall in M, that is, $\frac{N}{K} \ll \frac{M}{K}$. Obviously any direct summand N of M is coclosed in M.

Theorem 2.1. *Let M and N be right R-modules. Then M has the property $(N - GP^*)$ if and only if, for all direct summands $M^{'}$ and a coclosed submodule $N^{'}$ of N, $M^{'}$ has the property $(N^{'} - GP^*)$.*

Proof. ($\Longrightarrow$) Let $M^{'} = eM$ for some $e^2 = e \in End_R(M)$ and let $N^{'}$ be a coclosed submodule of N. Assume that $\alpha \in Hom(M^{'}, N^{'})$. Since $\alpha(eM) = \alpha(M^{'}) \subseteq N^{'} \subseteq N$ and M has the property $(N - GP^*)$, there exists a decomposition $N = N_1 \oplus N_2$ such that $N_1 \subseteq Im(\alpha(e))$ and $N_2 \cap Im(\alpha(e)) \subseteq Rad(M_2) \subseteq Rad(N)$. Then we have $N^{'} = N_1 \oplus (N_2 \cap N^{'})$ by the modular law. Since $N^{'}$ is a coclosed submodule of N, then $Rad(N^{'}) = Rad(N) \cap N^{'}$ by [4, 3.7(3)]. So $N_2 \cap N^{'} \cap Im(\alpha) \subseteq Rad(N^{'})$. By using [4, 3.7(3)] once again, we get $N_2 \cap N^{'} \cap Im(\alpha) \subseteq Rad(N_2 \cap N^{'})$. Therefore $M^{'}$ has the property $(N^{'} - GP^*)$.

($\Longleftarrow$) Clear. □

Corollary 2.1. *The following conditions are equivalent for a module M.*

(1) *M has the property (GP^*).*
(2) *For any coclosed submodule N of M, every direct summand L of M has the property $(N - GP^*)$.*

Corollary 2.2. *Every direct summand of a module that have the property (GP^*) has the property (GP^*).*

Proposition 2.2. *Let M be an indecomposable module. Assume that, for $\delta \in End_R(M)$, $Im(\delta) \subseteq Rad(M)$ implies $\delta = 0$. Then, M has the property (GP^*) if and only if every nonzero endomorphism $\delta \in End_R(M)$ is an epimorphism.*

Proof. Assume that $0 \neq \delta \in End_R(M)$. Since M has the property (GP^*), there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq Im(\delta)$ and $M_2 \cap Im(\delta) \subseteq Rad(M_2)$. Since M is indecomposable, $M_1 = 0$ or $M_1 = M$. If $M_1 = 0$, then $Im(\delta) \subseteq Rad(M)$. By the hypothesis $\delta = 0$; a contradiction. Thus, $M_1 = M$ and hence, δ is epimorphism. The converse is clear. □

Recall from [4, 4.27] that a module M is said to be *Hopfian* if every surjective endomorphism of M is an isomorphism.

Proposition 2.3. *Let M be a noetherian module that has the property (GP^*). If every endomorphism γ of M, $Im(\gamma) \subseteq Rad(M)$ implies that $\gamma = 0$. Then there exists a decomposition $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$, where M_i is an indecomposable noetherian modules that has the property (GP^*) for which $End_R(M_i)$ is a division ring.*

Proof. Since M is noetherian, it has a finite decomposition noetherian direct summands. By Corollary 2.2, every direct summand has the property (GP^*). By Proposition 2.2, in view of the fact that every noetherian module is Hopfian, each indecomposable direct summand has a division ring. □

Definition 2.3. A module M is called *G^*-supplemented* if, for every $\gamma \in End_R(M)$, $Im(\gamma)$ has a *Rad*-supplement in M, and a module M is called *amply G^*-supplemented* if, for every $\gamma \in End_R(M)$, $Im(\gamma)$ has ample *Rad*-supplements in M.

It is clear that every module that has the property (GP^*) is G^*-supplemented by the Definition 2.3.

Proposition 2.4. *Let M be an amply G^*-supplemented R-module. Then every direct summand of M is amply G^*-supplemented.*

Proof. Let N be a direct summand of M. Then $M = N \oplus N'$ for some $N' \subseteq M$. Suppose that $f \in End_R(N)$ and $N = Im(f) + K$. Thus, $M = Im(f) + K + N'$. Note that $Im(f) = Im(\iota f \pi)$, where ι is the injection map from N to M and π is the projection map from M onto N. Since M is amply G^*-supplemented, there exists a *Rad*-supplement L of $N' + K$ with $L \subseteq Im(f)$. We get $K \cap L \subseteq (N' + K) \cap L \subseteq Rad(L)$ and $M = L + N' + K$. Thus $N = K + L$ by the modular law. So $K + L = N$ and $K \cap L \subseteq Rad(L)$. Therefore N is amply G^*-supplemented. □

Proposition 2.5. *Let M be an amply G^*-supplemented distributive module and let N be a direct summand of M for every Rad-supplement submodule N of M. Then M is a G^*-supplemented module.*

Proof. Let $f \in End_R(M)$, let K be a *Rad*-supplement of $Im(f)$ in M, and let N a *Rad*-supplement of K in M with $N \subseteq Im(f)$. By the hypothesis, $M = N \oplus N'$ for some $N' \leq M$. $Im(f) = Im(f) \cap (N + K) = N + (Im(f) \cap K)$. Since $Im(f) \cap K \subseteq Rad(K)$, then we have $Im(f) \cap K \cap N' \subseteq Rad(K)$. As M is distributive, $Im(f) + K \cap N' = N + K = M$ and $K = K \cap (N \oplus N') = (K \cap N) \oplus (K \cap N')$. So $K \cap N'$ is a direct summand of K. Since $Im(f) \cap K \cap N' \subseteq K \cap N'$, $Im(f) \cap K \cap N' \subseteq Rad(K \cap N')$. Therefore M is G^*-supplemented. □

Definition 2.4. A module M is called *$N - G^*-$supplemented* if, for every homomorphism $\phi : M \longrightarrow N$, there exists $L \leq N$ such that $Im(\phi) + L = N$ and $Im(\phi) \cap L \subseteq Rad(L)$. It is clear that the right module M is G^*-supplemented if and only if M is $M - G^*-$supplemented.

Recall from [11] that a submodule U of an R-module M is called *fully invariant* if $f(U)$ is contained in U for every R-endomorphism f of M. A module M is called *duo*, if for every submodule of M is fully invariant [9].

Theorem 2.2. *Let M_1,M_2 and N be modules. If N is $M_i - G^*-$supplemented for $i = 1, 2$, then N is $M_1 \oplus M_2 - G^*-$supplemented. The converse is true if $M_1 \oplus M_2$ is a duo module.*

Proof. Suppose that N is $M_i - G^*-$supplemented for $i = 1, 2$. We prove that N is $M_1 \oplus M_2 - G^*-$supplemented. Let $\phi = (\pi_1\phi, \pi_2\phi)$ be any homomorphism from N to $M_1 \oplus M_2$, where π_i is the projection map from $M_1 \oplus M_2$

into M_i for $i = 1, 2$. Since N is $M_i - G^*-$supplemented, there exists a submodule K_i of M_i such that $\pi_i\phi N + K_i = M_i$ and $\pi_i\phi N \cap K_i \subseteq Rad(K_i)$ for $i = 1, 2$. Let $K = K_1 \oplus K_2$. Then $M_1 \oplus M_2 = \pi_1\phi N + \pi_2\phi N + K_1 + K_2 = \phi N + K$. Since $\phi N \cap (K_1 + K_2) \subseteq (\phi N + K_1) \cap K_2 + (\phi N + K_2) \cap K_1$, we get $\phi N \cap (K_1 + K_2) \subseteq (\phi N + M_1) \cap K_2 + (\phi N + M_2) \cap K_1$. Since $\phi N + M_1 = \pi_2\phi N \oplus M_1$ and $\phi N + M_2 = \pi_1\phi N \oplus M_2$, we conclude that $\phi N \cap K \subseteq (\pi_2\phi N \cap K_2) + (\pi_1\phi N + K_1)$. Since $\pi_i\phi N \cap K_i \subseteq Rad(K_i)$ for $i = 1, 2$, we get $\phi N \cap K \subseteq Rad(K)$. Hence, N is $M_1 \oplus M_2 - G^*-$supplemented.

Conversely, let N be $M_1 \oplus M_2 - G^*-$supplemented. Let ϕ be a homomorphism from N to M_1. Then $Im(\iota\phi) = Im(\phi)$, where ι is the canonical inclusion from M_1 to $M_1 \oplus M_2$. Since N is $M_1 \oplus M_2 - G^*-$supplemented, there exists $K \subseteq M_1 \oplus M_2$ such that $M_1 \oplus M_2 = Im(\phi) + K$ and $Im(\phi) \cap K \subseteq Rad(K)$. Thus, $M_1 = Im(\phi) + (K \cap M_1)$ and $Im(\phi) \cap K \cap M_1 = Im(\phi) \cap K \subseteq Rad(K)$. As $M_1 \oplus M_2$ is a duo module and $K = K_1 \oplus K_2 \leq M_1 \oplus M_2$, $K \cap M_1$ is a direct summand of K. Hence $Im(\phi) \cap K \cap M \subseteq Rad(K \cap M_1)$. Therefore N is an $M_1 - G^*-$supplemented. □

Corollary 2.3. *Suppose that $M = M_1 \oplus M_2$ and M is a G^*-supplemented module for $i = 1, 2$. Then M is G^*-supplemented and, for every $f \in End_R(M)$, $Im(f)$ has a Rad-supplement of the form $K_1 + K_2$ with $K_1 \subseteq M_1$ and $K_2 \subseteq M_2$.*

Proof. Follows from the proof of Theorem 2.2. □

Theorem 2.3. *Let $M = \oplus_{i=1}^n M_i$ be a module and M_i be a fully invariant submodule of M for all $i \in \{1, 2, \ldots, n\}$.Then M has the property (GP^*) if and only if M_i has the property (GP^*) for all $i \in \{1, 2, \ldots, n\}$.*

Proof. The necessity follows from Theorem 2.1. Conversely, let N_i be a module that have the property (GP^*) for all $i \in \{1, 2, \ldots, n\}$. Also let $\phi = (\phi_{ij})_{i,j\in\{1,2,\ldots,n\}} \in End(M)$ be arbitrary, where $(\phi_{ij}) \in Hom(M_j, M_i)$. Since M_i is a fully invariant submodule of M for all $i \in \{1, 2, \ldots, n\}$, we get $Im(\phi) = \oplus_{i=1}^n Im(\phi_{ii})$. As M_i has the property (GP^*), there exists a direct summand N_i of M_i and a submodule K_i of M_i with $N_i \subseteq Im(\phi_{ii})$, $Im(\phi_{ii}) = N_i + K_i$ and $K_i \subseteq Rad(M_i)$. We say $N = \oplus_{i=1}^n N_i$. Then N is a direct summand of M. Moreover, $Im(\phi) = \oplus_{i=1}^n Im(\phi_{ii}) = \sum_{i=1}^n N_i + \sum_{i=1}^n K_i$ and $\oplus_{i=1}^n K_i \subseteq Rad(\oplus_{i=1}^n M_i) = Rad(M)$. Therefore M has the property (GP^*). □

Theorem 2.4. *The following assertions are equivalent for a ring R.*

(1) *R is generalized f-semiperfect.*
(2) *R_R is f-Rad-supplemented.*
(3) *Every cyclic right ideal has a Rad-supplement in R_R.*
(4) *R_R is a G^*-supplemented module.*
(5) *R_R has the property (GP^*).*

Proof. (1) ⇔ (2) ⇔ (3) By [8, Theorem 2.22].

(3) $\Rightarrow$ (4) is clear because $Im(\gamma)$ is cyclic for every $\gamma \in End_R(R_R)$.

(4) $\Rightarrow$ (3) Assume that $I = aR$ is any cyclic right ideal of R. Consider the R-homomorphism $\phi : R_R \longrightarrow R_R$ defined by $\phi(r) = ar$; where $r \in R$. Then $Im(\phi) = I$. By (4), $Im(\phi) = I$ has a Rad-supplement in R_R.

(5) $\Rightarrow$ (4) is clear.

(5) $\Rightarrow$ (3) Suppose that R_R has the property (GP^*). Let $J = bR$ is any cyclic right ideal of R. Consider the R-homomorphism $\phi : R_R \longrightarrow R_R$ defined by $\phi(r) = br$; where $r \in R$. Then $Im(\phi) = J$. By (5), there exists submodules R_1, R_2 of R_R such that $R_R = R_1 \oplus R_2$, $R_1 \subseteq Im(\phi) = J$ and $R_2 \cap Im(\phi) \subseteq Rad(R_2)$. So $R_R = J + R_2$ and $J \cap R_2 \subseteq Rad(R_2)$. Thus R_2 is a Rad-supplement of J in R_R. □

The equivalent condition for the property (P^*) if every submodule N of M there exist submodules K, K of M such that $K \leq N$, $M = K \oplus K'$ and $N \cap K' \subseteq Rad(K')$ (See [1]).

Proposition 2.6. *Let M be a module which has the property (P^*). Then M has the property (GP^*).*

Proof. Let $\phi : M \longrightarrow M$ be any homomorphism. Since M has the property (P^*), there exist submodules K, K of M such that $K \leq Im(\phi)$, $M = K \oplus K'$ and $Im(\phi) \cap K' \subseteq Rad(K')$. So M has the property (GP^*). □

Example 2.1. (See [2]) Let F be any field. Consider the commutative ring R which is the direct product $\prod_{i=0}^{\infty} F_i$, where $F_i = F$. So R_R is a regular ring which is not semisimple. The right R-module R is f-Rad-supplemented but not Rad-supplemented. Since R_R is f-Rad-supplemented, R_R has the property (GP^*) by Theorem 2.4. As R_R is not Rad-supplemented, R_R has not the property (P^*).

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