Common Fixed Points for Faintly Compatible Mappings

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ABSTRACT. In this paper, we obtain a generalized common fixed point theorem for four mappings using the conditions of non-compatibility and faint compatibility.

1. INTRODUCTION AND PRELIMINARIES

Generalizing Banach contraction principle, Jungck [9] initiated the study of common fixed points for a pair of commuting mappings satisfying contractive type conditions. In 1982, Sessa [14] introduced the weaker notion of commutativity which is generally known as *Weak Commutativity* and established some interesting results on the existence of common fixed points for the pair of mappings. Further, Jungck [10] generalized the concept of weak commutativity by introducing the notion of compatible mappings. Throughout this section (f, g) denotes a pair of mapping on a metric space X .

Definition 1.1 ([10]). The pair of mappings (f, g) is said to be compatible iff $\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$ for some $t \in X$.

Definition 1.2 ([10]). The pair (f, g) is said to be non-compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$ for some $t \in X$ but $\lim_{n \rightarrow \infty} d(fgx_n, gfx_n)$ is either non-zero or non-existent.

Again in 1996, Jungck [8] generalized the the concept of compatibility by introducing weakly compatible mappings.

Definition 1.3 ([8]). The pair (f, g) is said to be weakly compatible if the pair commutes on the set of coincidence points, i.e., $fgx = gfx$ whenever $fx = gx$ for some $x \in X$.

Al-Thagafi and Shahzad [2] introduced the concept of occasionally weakly compatible mappings by weakened the notion of weakly compatible mappings.

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Definition 1.4 ([2]). The pair (f, g) is said to be occasionally weakly compatible if there exists a coincidence point $x \in X$ such that $fx = gx$ implies $fgx = gfx$.

In 2010, Pant et al. [12] redefined the concept of occasionally weakly compatible mappings by introducing conditional commutativity.

Definition 1.5 ([12]). The pair (f, g) is said to be conditionally commuting if the pair commutes on a nonempty subset of the set of coincidence points whenever the set of coincidences is nonempty.

Again, Pant et al. [13] gave the concept of conditional compatibility which is independent of compatibility condition and proved that in case of existence of unique common fixed/coincidence point, conditional compatibility can not be reduced to the compatibility condition. Further, they also proved that conditional compatibility need not imply commutativity at the coincidence points.

Definition 1.6 ([13]). The pair (f, g) is said to be conditionally compatible iff whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n$ is nonempty, there exists a sequence $\{y_n\}$ such that

$$\lim_{n \rightarrow \infty} fy_n = \lim_{n \rightarrow \infty} gy_n = t \text{ and } \lim_{n \rightarrow \infty} d(fgy_n, gfy_n) = 0.$$

Over the last two decades, there are a number of common fixed/coincidence point theorems for the pair of mappings under different contractive conditions with compatibility and its weaker versions imposed on the mappings (for more details, see [4, 5, 6, 7, 1, 3] and references therein).

In a recent work, Bisht and Shahzad [3] gave a new notion of conditionally compatible mappings in a slightly different settings and named it as faintly compatible mappings.

Definition 1.7 ([3]). The pair (f, g) is said to be faintly compatible iff (f, g) is conditionally compatible and (f, g) commutes on a nonempty subset of coincidence points whenever the set of coincidences is nonempty.

Bisht et al. [3] proved some interesting common fixed point theorems using the concept of faintly compatible mappings on non-complete metric spaces under different contractive conditions. Complementing the work of Bisht et al. [3], we give following examples for the comparative discussions on the above concepts.

(i) *Compatibility implies faint compatibility but converse may not be true.*

Example 1.1. Let $X = [2, 4]$ and d be the usual metric on X . Define self mappings f and g on X as follows:

$$f(x) = \begin{cases} 2 & \text{if } x = 2 \text{ or } x > 3 \\ x + 1 & \text{if } 2 < x \leq 3 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2 & \text{if } x = 2 \\ \frac{x+4}{2} & \text{if } 2 < x \leq 3 \\ \frac{x+1}{2} & \text{if } x > 3. \end{cases}$$

In this example, f and g are faintly compatible but not compatible. For if, we consider the constant sequence $\{x_n = 2\}$, then f and g are faintly compatible. On the other hand, if we choose a sequence $\{y_n = 3 + \frac{1}{n}\}$, then $\lim_{n \rightarrow \infty} f y_n = \lim_{n \rightarrow \infty} g y_n = 2$ and $\lim_{n \rightarrow \infty} d(f g y_n, g f y_n) = 1 (\neq 0)$. Hence f and g are not compatible.

- (ii) *Faint compatibility and non-compatibility are independent concepts.*

Example 1.2. Let $X = [2, 8]$ and d be the usual metric on X . Define self mappings f and g on X as follows:

$$f(x) = \begin{cases} 6 & \text{if } 2 \leq x \leq 4 \\ 2 & \text{if } x > 4 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2 & \text{if } 2 \leq x < 4 \\ x - 2 & \text{if } x \geq 4. \end{cases}$$

In this example, f and g are non-compatible but not faintly compatible. To see this, we consider a sequence $\{x_n = 4 + \frac{1}{n}\}$, then $\lim_{n \rightarrow \infty} f x_n = 2 = \lim_{n \rightarrow \infty} g x_n$ but $\lim_{n \rightarrow \infty} d(f g x_n, g f x_n) = 4$. So, f and g are non-compatible.

Example 1.3. Let $X = [1, \infty)$ and let d be the usual metric on X . Define self mappings f and g on X as follows:

$$f(x) = x \quad \forall x \in X \quad \text{and} \quad g(x) = 3x - 2 \quad \forall x \in X.$$

In this one, f and g are faintly compatible but not non-compatible.

- (iii) *Weakly compatible implies faint compatibility, but converse is not true in general.*

Example 1.4. Let $X = [0, \frac{2}{3}]$ with the usual metric d . Define self mappings f and g on X as follows:

$$f(x) = \frac{1}{3} - \left| \frac{1}{3} - x \right| \quad \text{and} \quad g(x) = \frac{2 - 3x}{9}.$$

In this example the mappings f and g are faintly compatible but not weakly compatible. To see this, we take a constant sequence $\{x_n = \frac{1}{12}\}$ and they are commuting at the coincidence point $x = \frac{1}{12}$. On the other hand, f and g do not commute at the coincidence point $x = \frac{2}{3}$, hence they are not weakly compatible.

- (iv) *Occasionally weakly compatible implies faintly compatible but the converse may not be true.*

Example 1.5. Let $X = [0, \infty)$ with usual metric d on X . Define self mappings f and g on X as follows:

$$f(x) = \frac{x}{2} \quad \forall x \in X \quad \text{and} \quad g(x) = \frac{x + 3}{2} \quad \forall x \in X.$$

In this example, mappings f and g are trivially faintly compatible but not occasionally compatible.

In one of the interesting paper, Jungck [11] established a common fixed point theorem for four mappings in a complete metric space. Now, we prove our main result for the existence of common fixed point for four mappings in a non-complete metric space using the concept of faintly compatible mappings which is analogous to the result of Jungck [11].

2. MAIN RESULTS

Theorem 2.1. *Let A, B, S and T be continuous self mappings of a metric space (X, d) . Suppose*

- (i) *pairs (A, S) and (B, T) are non-compatible and faintly compatible,*
- (ii) *$AX \subset TX$ and $BX \subset SX$.*

If there exists $k \in (0, 1)$ such that

$$(1) \quad d(Ax, By) \leq k \max \{d(Ax, Sx), d(By, Ty), d(Sx, Ty), \\ \frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\}$$

for $x, y \in X$. Then there is a unique point $z \in X$ such that $Az = Bz = Sz = Tz = z$.

Proof. As the pair (A, S) is non-compatible, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$ for some $t \in X$ but $\lim_{n \rightarrow \infty} d(ASx_n, SAsx_n)$ is either non-zero or non-existent. Since A and S are faintly compatible and $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$, there exists a sequence $\{z_n\}$ in X satisfying $\lim_{n \rightarrow \infty} Az_n = \lim_{n \rightarrow \infty} Sz_n = u$ (say) such that

$$(2) \quad \lim_{n \rightarrow \infty} d(ASz_n, SAz_n) = 0.$$

Further, since A is continuous, $\lim_{n \rightarrow \infty} AAz_n = Au$ and $\lim_{n \rightarrow \infty} ASz_n = Au$. These last three limits together imply $\lim_{n \rightarrow \infty} SAz_n = Au$. The inclusion $AX \subset TX$ implies that $Au = Tv$ for some $v \in X$ and $\lim_{n \rightarrow \infty} AAz_n = Tv$, $\lim_{n \rightarrow \infty} SAz_n = Tv$.

Similarly, non-compatibility of the pair B, T implies that there exists a sequence $\{y_n\}$ in X such that $\lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = t'$ for some $t' \in X$ but $\lim_{n \rightarrow \infty} d(BTy_n, TBy_n)$ is either non-zero or non-existent. Now faint compatibility of B and T will imply that there exists a sequence $\{w_n\}$ in X satisfying $\lim_{n \rightarrow \infty} Bw_n = \lim_{n \rightarrow \infty} Tw_n = u'$ (say) such that

$$(3) \quad \lim_{n \rightarrow \infty} d(BTw_n, TBw_n) = 0.$$

Again, B is continuous so $\lim_{n \rightarrow \infty} BBw_n = Bu'$ and $\lim_{n \rightarrow \infty} BTw_n = Bu'$. These last three limits together imply $\lim_{n \rightarrow \infty} TBw_n = Bu'$. The inclusion $BX \subset SX$ implies that $Bu' = Sv'$ for some $v' \in X$ and $\lim_{n \rightarrow \infty} BBw_n = Sv'$, $\lim_{n \rightarrow \infty} TBw_n = Sv'$.

Using the condition (1), we get

$$\begin{aligned}
 d(u, u') &= \lim_{n \rightarrow \infty} d(Az_n, Bw_n) \\
 &\leq k \lim_{n \rightarrow \infty} \max \{d(Az_n, Sz_n), d(Bw_n, Tw_n), d(Sz_n, Tw_n), \\
 &\quad \frac{1}{2}[d(Az_n, Tw_n) + d(Bw_n, Sz_n)]\} \\
 &= k \max \{d(u, u'), d(u', u'), d(u, u'), \frac{1}{2}[d(u, u') + d(u', u)]\}.
 \end{aligned}$$

Thus

$$d(u, u') \leq k d(u, u') \Rightarrow d(u, u') = 0 \Rightarrow u = u'$$

So, $Au = Tv$ and $Bu = Sv'$.

Now, $\lim_{n \rightarrow \infty} Az_n = \lim_{n \rightarrow \infty} Sz_n = \lim_{n \rightarrow \infty} Bw_n = \lim_{n \rightarrow \infty} Tw_n = u$.

Continuity of S and T together with conditions (2) and (3) imply

$$\begin{aligned}
 \lim_{n \rightarrow \infty} SSz_n &= \lim_{n \rightarrow \infty} SAz_n = Su \Rightarrow \lim_{n \rightarrow \infty} SSz_n = \lim_{n \rightarrow \infty} ASz_n = Su, \\
 \text{and } \lim_{n \rightarrow \infty} TBw_n &= \lim_{n \rightarrow \infty} TTW_n = Tu \Rightarrow \lim_{n \rightarrow \infty} TTW_n = \lim_{n \rightarrow \infty} BTW_n = Tu.
 \end{aligned}$$

Now,

$$\begin{aligned}
 d(ASz_n, BTW_n) &\leq k \max \{d(ASz_n, SSz_n), d(BTW_n, TTW_n), d(SSz_n, TTW_n), \\
 &\quad \frac{1}{2}[d(ASz_n, TTW_n) + d(BTW_n, SSz_n)]\}.
 \end{aligned}$$

Taking $n \rightarrow \infty$, we get

$$\begin{aligned}
 d(Su, Tu) &\leq k \max \{d(Su, Su), d(Tu, Tu), d(Su, Tu), \frac{1}{2}[d(Su, Tu) + d(Tu, Su)]\} \\
 &\Rightarrow d(Su, Tu) \leq k d(Su, Tu) \Rightarrow d(Su, Tu) = 0
 \end{aligned}$$

$$(4) \quad \Rightarrow Su = Tu.$$

Also,

$$\begin{aligned}
 d(Au, Tu) &= \lim_{n \rightarrow \infty} d(Au, BTW_n) \\
 &\leq \max \lim_{n \rightarrow \infty} \{d(Au, Su), d(BTW_n, TTW_n), d(Su, TTW_n), \\
 &\quad \frac{1}{2}[d(Au, TTW_n) + d(BTW_n, Su)]\} \\
 \Rightarrow d(Au, Tu) &\leq k \max \{d(Au, Su), d(Tu, Tu), d(Su, Tu), \\
 &\quad \frac{1}{2}[d(Au, Tu) + d(Tu, Su)]\} \\
 \Rightarrow d(Au, Tu) &\leq k \max \{d(Au, Tu), \frac{1}{2}d(Au, Tu)\} \\
 \Rightarrow d(Au, Tu) &= 0
 \end{aligned}$$

$$(5) \quad \Rightarrow \quad Au = Tu.$$

Using (2.1) with $x = y = u$, we get

$$\begin{aligned} d(Au, Bu) &\leq k \max \{d(Au, Su), d(Bu, Tu), d(Su, Tu), \\ &\quad \frac{1}{2}[d(Au, Tu) + d(Bu, Su)]\} \\ \Rightarrow d(Au, Bu) &\leq k \max \{d(Bu, Au), \\ &\quad \frac{1}{2}d(Bu, Au)\} \\ \Rightarrow d(Au, Bu) &= 0 \end{aligned}$$

$$(6) \quad \Rightarrow \quad Au = Bu.$$

From (4), (5) and (6), we have $Au = Bu = Su = Tu$. In fact, u is a common fixed point A, B, S and T . To see this,

$$\begin{aligned} d(u, Bu) &= \lim_{n \rightarrow \infty} d(Az_n, Bu) \\ &\leq k \max \lim_{n \rightarrow \infty} \{d(Az_n, Sz_n), d(Bu, Tu), d(Sz_n, Tu), \\ &\quad \frac{1}{2}[d(Az_n, Tu) + d(Bu, Sz_n)]\} \\ \Rightarrow d(u, Bu) &\leq k \max \{d(u, Tu), \frac{1}{2}[d(u, Tu) + d(Bu, u)]\} \\ \Rightarrow d(u, Bu) &\leq k \max \{d(u, Bu), \frac{1}{2}d(u, Bu)\} \\ \Rightarrow d(u, Bu) &= 0 \Rightarrow Bu = u. \end{aligned}$$

Hence, $Au = Bu = Su = Tu = u$.

For the uniqueness of the common fixed point, let w be another common fixed point of A, B, S and T , i.e., $Aw = Bw = Sw = Tw = w$. We have

$$\begin{aligned} d(u, w) &= A(Au, Bw) \leq k \max \{d(Au, Bu), d(Bw, Tw), d(Su, Tw), \\ &\quad \frac{1}{2}[d(Au, Tw) + d(Bw, Su)]\} \\ \Rightarrow d(u, w) &\leq k d(u, w) \Rightarrow u = w. \end{aligned}$$

□

Remark 2.1. In Theorem 2.1, taking (A, S) and (B, T) as compatible pairs of mappings and metric space as complete, we get the result of Jungck [11].

Now, we give an example in the support of our main result.

Example 2.1. Let $X = [0, 10]$ with usual metric d on X . Define self mappings A, B, S and T on X as follows:

$$Ax = \begin{cases} 5 & \text{if } x \leq 5 \\ 10 - x & \text{if } x > 5 \end{cases} \quad Bx = \begin{cases} \frac{3x+10}{5} & \text{if } x \leq 5 \\ 10 - x & \text{if } x > 5 \end{cases}$$

$$Sx = \begin{cases} \frac{-3x+40}{5} & \text{if } x \leq 5 \\ 10 - x & \text{if } x > 5 \end{cases} \quad Tx = 10 - x \quad \forall x \in X.$$

In this example, pairs (A, S) and (B, T) on X are faintly compatible mappings. For this, we consider the constant sequence $\{x_n = 5\}$ and $AS(5) = SA(5)$. Also, pairs (A, S) and (B, T) are non-compatible mappings. To see this, consider the sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} x_n = 10$, then

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = 0$$

and

$$\lim_{n \rightarrow \infty} d(ASx_n, SAx_n) \neq 0, \quad \lim_{n \rightarrow \infty} d(BTx_n, TBx_n) \neq 0.$$

It can be verified that the mappings A, B, S and T on X are satisfying the condition (1) with $k = \frac{3}{8}$ and 5 is the only common fixed point of A, B, S and T .

Taking $A = B$ and $S = T$ in Theorem 2.1, we obtain following corollary.

Corollary 2.1. *Let A, S be self mappings of a metric space (X, d) . Suppose*

- (i) *A and S are continuous,*
- (ii) *pairs (A, S) is non-compatible faintly compatible,*
- (iii) *$AX \subset SX$.*

If there exists $k \in (0, 1)$ such that

$$d(Ax, Ay) \leq k \max \{d(Ax, Sx), d(Ay, Sy), d(Sx, Sy), \frac{1}{2}[d(Ax, Sy) + d(Ay, Sx)]\}$$

for $x, y \in X$. Then A and S have unique common fixed point in X .

Remark 2.2. Observe that Corollary 2.1 is a generalization of the result due to Bisht et al. ([3], Theorem 2.1) .

Corollary 2.2. *Let A, B, S and T be continuous self mappings of a metric space (X, d) . Suppose the pairs (A, S) and (B, T) are non-compatible faintly compatible and $AX \subset TX$ and $BX \subset SX$. If there exists $k \in (0, 1)$ such that any of the following inequalities holds*

- (i) $d(Ax, By) \leq k \max \{d(Ax, Sx), d(By, Ty)\}$
- (ii) $d(Ax, By) \leq k \max \{d(Ax, Sx), d(By, Ty), d(Sx, Ty)\}$
- (iii) $d(Ax, By) \leq k \max \{d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(Ax, Ty) + d(By, Sx)]\}$

for all $x, y \in X$. Then there is a unique point $z \in X$ such that $Az = Bz = Sz = Tz = z$.

If we consider $A = T$ and $B = S$ in Theorem 2.1, we have following corollary.

Corollary 2.3. *Let A and B be continuous self mappings of a metric space (X, d) , the pair (A, B) be non-compatible faintly compatible and $AX \subset BX$. If there exists $k \in (0, 1)$ such that*

$$d(Ax, By) \leq k \max \{d(Ax, Bx), d(Ay, By), d(Bx, Ay), \\ \frac{1}{2}[d(Ax, Ay) + d(Bx, By)]\}$$

for all $x, y \in X$. Then A and B have a unique common fixed point in X .

Taking $T = S = I$ (identity map) in Theorem 2.1, we obtain following result as corollary.

Corollary 2.4. *Let A and B be continuous self mappings of a metric space (X, d) , the pair (A, B) be non-compatible faintly compatible and $AX \subset BX$. If there exists $k \in (0, 1)$ such that*

$$d(Ax, By) \leq k \max \{d(x, y), d(x, Ax), d(y, By), \frac{1}{2}[d(y, Ax) + d(x, By)]\}$$

for all $x, y \in X$. Then A and B have a unique common fixed point in X .

REFERENCES

- [1] M. Abbas, Lj. Ćirić, B. Damjanović, M. A. Khan, *Coupled coincidence and common fixed point theorems for hybrid pair of mappings*, Fixed Point Theory Appl., (2012). doi: 10.1186/1687-1812-2012-4.
- [2] M. A. Al-Thagafi, N. Shahzad, *Generalized I -nonexpansive self maps and invariant approximations*, Act. Math. Sin., 24(2008), 867-876.
- [3] R. K. Bisht, N. Shahzad, *Faintly compatible mappings and common fixed points*, Fixed Point Theory Appl., (2013). doi: 10.1186/1687-1812-2013-156.
- [4] Lj. Ćirić, B. Samet, C. Vetro, *Common fixed point theorems for families of occasionally weakly compatible mappings*, Math. Comput. Model., 53(5-6)(2011), 631-636.
- [5] Lj. Ćirić, B. Samet, N. Cakić, B. Damjanović, *Generalized (ψ, ω) -weak nonlinear contractions in ordered K -metric spaces*, Comput. Math. Appl., 62(2011), 3305-3316.
- [6] Lj. Ćirić, A. Razani, S. Radenović, J. S. Ume, *Common fixed point theorems for families of weakly compatible maps*, Comput. Math. Appl., 55(11)(2008), 2533-2543.
- [7] Lj. Ćirić, N. T. Nikolić, J. S. Ume, *Common fixed point theorems for weakly compatible quasi contraction mappings*, Acta Math. Hung., 113(4)(2006), 257-267.
- [8] G. Jungck, *Common fixed points for non-continuous non-self maps on non-metric spaces*, Far East J. Math. Sci., 4(1996), 199-215.
- [9] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly, 83(1976), 261-263.
- [10] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci., 9(4)(1986), 771-779.
- [11] G. Jungck, *Compatible mappings and common fixed points (2)*, Int. J. Math. Math. Sci., 11(2)(1988), 285-288.

- [12] V. Pant, R. P. Pant, *Common fixed points of conditionally commuting maps*, Fixed Point Theory, 1(2010), 113-118.
- [13] R. P. Pant, R. K. Bisht, *Occasionally weakly compatible mappings and fixed points*, Bull. Belg. Math. Soc. Simon Stevin, 19(2012), 655-661.
- [14] S. Sessa, *On a weak commutativity condition of mappings in fixed point consideration*, Publ. Inst. Math., 32(1982), 149-153.

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