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General integral formulas involving Humbert hypergeometric functions of two variables

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Abstract. In this paper, we have established two general integral formulas involving Humbert hypergeometric functions of two variables Φ_2 and Ψ_2 . The results are obtained with the help of a generalization of classical Kummer's summation theorem on the sum of the series ${}_2F_1(-1)$ due to Lavoie *et al.* [5]. Some interesting applications are also presented.

1. Introduction

The confluent hypergeometric functions Φ_2 and Ψ_2 are defined and represented as follows [2,8]:

$$\Phi_2[a,a';b;x;y]=\sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n\ x^m\ y^n}{(b)_{m+n}m!\ n!} \tag{1.1}$$

and

$$\Psi_2[a;b,b';x;y]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\ x^m\ y^n}{(b)_m(b')_n m!\ n!}, \tag{1.2}$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by [8]

$$(\lambda)_n=\begin{cases}1, & \text{if } n=0\\ \lambda(\lambda+1)...(\lambda+n-1), & \text{if } n=1,2,3,...\end{cases} \tag{1.3}$$

Exton [3,4] gave the definitions and the Laplace integral representations of the quadruple hypergeometric functions K_5 and K_{12} as follows:

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$$\begin{aligned}
&K_5(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;x,y,z,t)\\
&=\sum_{p,q,r,s=0}^{\infty}\frac{(a)_{p+q+r+s}(b_1)_{p+q}(b_2)_{r+s}\ x^p\ y^q\ z^r\ t^s}{(c_1)_p(c_2)_q(c_3)_r(c_4)_s\ p!q!r!s!}\\
&=\frac{1}{\Gamma(a)}\int_0^{\infty}e^{-s}s^{a-1}\Psi_2(b_1;c_1,c_2;xs,ys)\Psi_2(b_2;c_3,c_4;zs,ts)\mathrm{d}s
\end{aligned}\tag{1.4}$$

and

$$\begin{aligned}
&K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;x,y,z,t)\\
&=\sum_{p,q,r,s=0}^{\infty}\frac{(a)_{p+q+r+s}(b_1)_p(b_2)_q(b_3)_r(b_4)_s\ x^p\ y^q\ z^r\ t^s}{(c_1)_{p+q}(c_2)_{r+s}\ p!q!r!s!}\\
&=\frac{1}{\Gamma(a)}\int_0^{\infty}e^{-s}s^{a-1}\Phi_2(b_1,b_2;c_1;xs,ys)\Phi_2(b_3,b_4;c_2;zs,ts)\mathrm{d}s.
\end{aligned}\tag{1.5}$$

The Kampé de Fériet function of two variables $F^{p:q;k}_{l:m;n}[x,y]$ is defined and represented as follows [8]:

$$\begin{aligned}
&F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k);\\ & x,y\\ (\alpha_l):(\beta_m);(\gamma_n);\end{matrix}\right]\\
&=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\ \frac{x^r}{r!}\ \frac{y^s}{s!}.
\end{aligned}\tag{1.6}$$

In the present investigation, we shall require the following generalization of the classical Kummer's theorem for the series ${}_2F_1(-1)$ [5]:

$$\begin{aligned}
{}_2F_1\left[\begin{matrix}a\ ,\ b\ ;\\ 1+a-b+i;\end{matrix}-1\right]&=\frac{\Gamma(\frac{1}{2})\Gamma(1+a-b+i)\Gamma(1-b)}{2^a\Gamma(1-b+\frac{1}{2}(i+|i|))}\\
&\times\{\frac{A_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1+\frac{1}{2}a-b+\frac{1}{2}i)}\\
&\quad+\frac{B_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i)}\}
\end{aligned}$$

for (i=0,±1,±2,±3,±4,±5), where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x . The coefficients A_i *and* B_i are given respectively in [5]. When i=0, (1.7) reduces immediately to the classical Kummer's theorem [1],(see also [6])

$${}_2F_1\left[\begin{matrix}a\ ,\ b\ ;\\ 1+a-b;\end{matrix}-1\right]=\frac{\Gamma(1+a-b)\Gamma(\frac{1}{2})}{2^a\Gamma(1+\frac{1}{2}a-b)\Gamma(\frac{1}{2}a+\frac{1}{2})}.\tag{1.7}$$

The following results will be required also [8]:

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},\ \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}=\frac{(1-)^n}{(1-\alpha)_n}, \tag{1.8}$$

$$\Gamma\left(\frac{1}{2}\right)\Gamma(1+\alpha) = 2^{\alpha}\ \Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(1+\frac{1}{2}\alpha\right), \tag{1.9}$$

$$(\alpha)_{2n}=2^{2n}\left(\frac{1}{2}\alpha\right)_n\ \left(\frac{1}{2}\alpha+\frac{1}{2}\right)_n \tag{1.10}$$

and

$$(2n)! = 2^{2n}\left(\frac{1}{2}\right)_n n!,\ (2n+1)! = 2^{2n}\left(\frac{3}{2}\right)_n n!. \tag{1.11}$$

2. Main Integrals Formulas

First Integral

$$\frac{1}{\Gamma(a)}\int_0^{\infty} e^{-s}s^{a-1}\Psi_2(b_1;c_1,c_1+i;xs,-xs)\Psi_2(b_2;c_2,c_2+i;ys,-ys)\mathrm{d}s$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2}(b_1)_{2m_1}(b_2)_{2m_2}\ x^{2m_1}\ y^{2m_2}}{(c_1)_{2m_1}(c_2)_{2m_2}(2m_1)!(2m_2)!}$$

$$\times(A_i^{(1)}C_1+B_i^{(1)}D_1)(A_i^{(2)}C_2+B_i^{(2)}D_2)$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+1}(b_1)_{2m_1+1}(b_2)_{2m_2}\ x^{2m_1+1}\ y^{2m_2}}{(c_1)_{2m_1+1}(c_2)_{2m_2}(2m_1+1)!(2m_2)!}$$

$$\times(A_i^{(3)}\ E_1+B_i^{(3)}\ F_1)(A_i^{(2)}\ C_2+B_i^{(2)}\ D_2)$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+1}(b_1)_{2m_1}(b_2)_{2m_2+1}\ x^{2m_1}\ y^{2m_2+1}}{(c_1)_{2m_1}(c_2)_{2m_2+1}(2m_1)!(2m_2+1)!}$$

$$\times(A_i^{(1)}\ C_1+B_i^{(1)}\ D_1)(A_i^{(4)}\ E_2+B_i^{(4)}\ F_2)$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+2}(b_1)_{2m_1+1}(b_2)_{2m_2+1}\ x^{2m_1+1}\ y^{2m_2+1}}{(c_1)_{2m_1+1}(c_2)_{2m_2+1}(2m_1+1)!(2m_2+1)!}$$

$$\times(A_i^{(3)}\ E_1+B_i^{(3)}\ F_1)(A_i^{(4)}\ E_2+B_i^{(4)}\ F_2), \tag{2.1}$$

where

$$C_r = \frac{2^{2m_r}\Gamma(\frac{1}{2})\Gamma(c_r+i)\Gamma(c_r+2m_r)}{\Gamma(c_r+2m_r+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(m_r+c_r+\frac{1}{2}i)}$$

$$D_r = \frac{2^{2m_r}\Gamma(\frac{1}{2})\Gamma(c_r+i)\Gamma(c_r+2m_r)}{\Gamma(c_r+2m_r+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i-[\frac{i}{2}])\Gamma(m_r+c_r-\frac{1}{2}+\frac{1}{2}i)}$$

$$E_r=\frac{2^{2m_r+1}\Gamma(\frac{1}{2})\Gamma(c_r+i)\Gamma(c_r+2m_r+1)}{\Gamma(c_r+2m_r+1+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i-[\frac{1+i}{2}])\Gamma(m_r+\frac{1}{2}+c_r+\frac{1}{2}i)}$$

$$F_r=\frac{2^{2m_r+1}\Gamma(\frac{1}{2})\Gamma(c_r+i)\Gamma(c_r+2m_r+1)}{\Gamma(c_r+2m_r+1+\frac{1}{2}(i+|i|))\Gamma(-m_r-\frac{1}{2}+\frac{1}{2}i-[\frac{i}{2}])\Gamma(m_r+c_r+\frac{1}{2}i)}$$

(for i=0,±1,±2,±3,±4,±5 ; r=1,2).

The coefficients $A_i^{(1)}$ and $B_i^{(1)}$ can be obtained from the tables of A_i and B_i given in [5] by replacing a and b by $-2m_1$ and $1-c_1-2m_1$, the coefficients $A_i^{(2)}$ and $B_i^{(2)}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_2$ and $1-c_2-2m_2$, the coefficients $A_i^{(3)}$ and $B_i^{(3)}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_1-1$ and $-c_1-2m_1$ and the coefficients $A_i^{(4)}$ and $B_i^{(4)}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_2-1$ and $-c_2-2m_2$.

Second Integral

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Phi_2(b_1-i,b_1;c_1;xs,-xs)\Phi_2(b_2-i,b2;c_2;ys,-ys)\mathrm{d}s$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2}(b_1-i)_{2m_1}(b_2-i)_{2m_2}\ x^{2m_1}\ y^{2m_2}}{(c_1)_{2m_1}(c_2)_{2m_2}(2m_1)!(2m_2)!}$$

$$\times(A_i'^{(1)}C_1'+B_i'^{(1)}D_1')(A_i'^{(2)}C_2'+B_i'^{(2)}D_2')$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+1}(b_1-i)_{2m_1+1}(b_2-i)_{2m_2}\ x^{2m_1+1}\ y^{2m_2}}{(c_1)_{2m_1+1}(c_2)_{2m_2}(2m_1+1)!(2m_2)!}$$

$$\times(A_i'^{(3)}E_1'+B_i'^{(3)}F_1')(A_i'^{(2)}C_2'+B_i'^{(2)}D_2')$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+1}(b_1-i)_{2m_1}(b_2-i)_{2m_2+1}\ x^{2m_1}\ y^{2m_2+1}}{(c_1)_{2m_1}(c_2)_{2m_2+1}(2m_1)!(2m_2+1)!}$$

$$\times(A_i'^{(1)}C_1'+B_i'^{(1)}D_1')(A_i'^{(4)}E_2'+B_i'^{(4)}F_2')$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+2}(b_1-i)_{2m_1+1}(b_2-i)_{2m_2+1}\ x^{2m_1+1}\ y^{2m_2+1}}{(c_1)_{2m_1+1}(c_2)_{2m_2+1}(2m_1+1)!(2m_2+1)!}$$

$$\times\ (A_i'^{(3)}E_1'+B_i'^{(3)}F_1')(A_i'^{(4)}E_2'+B_i'^{(4)}F_2'), \tag{2.2}$$

where

$$C'_r = \frac{2^{2m_r}\Gamma(\frac{1}{2})\Gamma(1-2m_r-b_r+i)\Gamma(1-b_r)}{\Gamma(1-b_r+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1-m_r-b_r+\frac{1}{2}i)}$$

$$D'_r = \frac{2^{2m_r}\Gamma(\frac{1}{2})\Gamma(1-2m_r-b_r+i)\Gamma(1-b_r)}{\Gamma(1-b_r+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i-[\frac{i}{2}])\Gamma(-m_r+\frac{1}{2}-b_r+\frac{1}{2}i)}$$

$$E'_r = \frac{2^{2m_r+1}\Gamma(\frac{1}{2})\Gamma(-2m_r-b_r+i)\Gamma(1-b_r)}{\Gamma(1-b_r+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i-[\frac{1+i}{2}])\Gamma(-m_r+\frac{1}{2}-b_r+\frac{1}{2}i)}$$

$$F'_r = \frac{2^{2m_r+1}\Gamma(\frac{1}{2})\Gamma(-2m_r-b_r+i)\Gamma(1-b_r)}{\Gamma(1-b_r+\frac{1}{2}(i+|i|))\Gamma(-m_r-\frac{1}{2}+\frac{1}{2}i-[\frac{i}{2}])\Gamma(-m_r-b_r+\frac{1}{2}i)}$$

for i=0,±1,±2,±3,±4,±5 ; r=1,2.

The coefficients $A_i'^{(1)}$ and $B_i'^{(1)}$ can be obtained from the tables of A_i and B_i given in [5] by replacing a by $-2m_1$, the coefficients $A_i'^{(2)}$ and $B_i'^{(2)}$ can be obtained from the tables of A_i and B_i by replacing a by $-2m_2$, the coefficients $A_i'^{(3)}$ and $B_i'^{(3)}$ can be obtained from the tables of A_i and B_i by replacing a by $-2m_1-1$ and the coefficients $A_i'^{(4)}$ and $B_i'^{(4)}$ can be obtained from the tables of A_i and B_i by replacing a by $-2m_2-1$.

Proof of the first integral:

Denoting the left hand side of (2.1) by I, then from the definition (1.4), we have

(2.3)

$$I=\sum_{m_1,p_1,m_2,p_2=0}^{\infty} \frac{(a)_{m_1+p_1+m_2+p_2}(b_1)_{m_1+p_1}(b_2)_{m_2+p_2}\ x^{m_1}(-x)^{p_1}y^{m_2}(-y)^{p_2}}{(c_1)_{m_1}(c_1+i)_{p_1}(c_2)_{m_2}(c_2+i)_{p_2}\ m_1!p_1!m_2!p_2!}.$$

Now, using the well-known results [8]

$$(\alpha)_{m+n} = (\alpha)_m(\alpha+m)_n$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A(n,m) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}A(n,m-n),$$

$$(\alpha)_{m-n} = \frac{(-1)^n(\alpha)_m}{(1-\alpha-m)_n},\ 0\leq n\leq m$$

$$(m-n)!=\frac{(-1)^n m!}{(-m)_n},\ 0\leq n\leq m,$$

then after a little simplification, we have

$$I=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{(a)_{m_1+m_2}(b_1)_{m_1}(b_2)_{m_2}\ x^{m_1}\ y^{m_2}}{(c_1)_{m_1}(c_2)_{m_2}\ m_1!m_2!}$$

$$ {}_2F_1\left[\begin{matrix}-m_1\ ,\ 1-c_1-m_1\ ; & \\ & -1 \\ c_1+i\quad ; & \end{matrix}\right] {}_2F_1\left[\begin{matrix}-m_2\ ,\ 1-c_2-m_2\ ; & \\ & -1 \\ c_2+i\quad ; & \end{matrix}\right]. $$

Separating into its even and odd terms, we have

$$ I=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2}(b_1)_{2m_1}(b_2)_{2m_2}\ x^{2m_1}\ y^{2m_2}}{(c_1)_{2m_1}(c_2)_{2m_2}\ (2m_1)!(2m_2)!} $$

$$ \times {}_2F_1\left[\begin{matrix}-2m_1,1-c_1-2m_1; & \\ & -1 \\ c_1+i\quad ; & \end{matrix}\right] {}_2F_1\left[\begin{matrix}-2m_2,1-c_2-2m_2; & \\ & -1 \\ c_2+i\quad ; & \end{matrix}\right] $$

$$ +\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+1}(b_1)_{2m_1+1}(b_2)_{2m_2}\ x^{2m_1+1}\ y^{2m_2}}{(c_1)_{2m_1+1}(c_2)_{2m_2}\ (2m_1+1)!(2m_2)!} $$

$$ \times {}_2F_1\left[\begin{matrix}-2m_1-1,-c_1-2m_1; & \\ & -1 \\ c_1+i\quad ; & \end{matrix}\right] {}_2F_1\left[\begin{matrix}-2m_2,1-c_2-2m_2; & \\ & -1 \\ c_2+i\quad ; & \end{matrix}\right] $$

$$ +\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+1}(b_1)_{2m_1}(b_2)_{2m_2+1}\ x^{2m_1}\ y^{2m_2+1}}{(c_1)_{2m_1}(c_2)_{2m_2+1}\ (2m_1)!(2m_2+1)!} $$

$$ \times {}_2F_1\left[\begin{matrix}-2m_1,1-c_1-2m_1; & \\ & -1 \\ c_1+i\quad ; & \end{matrix}\right] {}_2F_1\left[\begin{matrix}-2m_2-1,-c_2-2m_2; & \\ & -1 \\ c_2+i\quad ; & \end{matrix}\right] $$

$$ +\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+2}(b_1)_{2m_1+1}(b_2)_{2m_2+1}\ x^{2m_1+1}\ y^{2m_2+1}}{(c_1)_{2m_1+1}(c_2)_{2m_2+1}\ (2m_1+1)!(2m_2+1)!} $$

$$ \times {}_2F_1\left[\begin{matrix}-2m_1-1,-c_1-2m_1; & \\ & -1 \\ c_1+i\quad ; & \end{matrix}\right] {}_2F_1\left[\begin{matrix}-2m_2-1,-c_2-2m_2; & \\ & -1 \\ c_2+i\quad ; & \end{matrix}\right]. $$

Finally , if we use the generalized Kummer's theorem (1.7), then, after a little simplification, we readily arrive at the right hand side of (2.1).This completes the proof of the first integral . The proof of the second integral is similar to that of the first integral with the only difference that we use here the result (1.5).

3. Applications

(i) In (2.1), if we take i=0 and use the results (1.9)–(1.12), we get after a little simplification the following integral formula:

$$\begin{aligned}&\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Psi_2(b_1;c_1,c_1;xs,-xs)\Psi_2(b_2;c_2,c_2;ys,-ys)\mathrm{d}s\\ &=F^{2:2;2}_{0:3;3}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}: & \frac{1}{2}b_1,\frac{1}{2}b_1+\frac{1}{2}; & \frac{1}{2}b_2,\frac{1}{2}b_2+\frac{1}{2}; \\ - & :c_1,\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2}; & c_2,\frac{1}{2}c_2,\frac{1}{2}c_2+\frac{1}{2};\end{matrix} -4x^2,-4y^2\right].\end{aligned} \tag{3.1}$$

Now, in (3.1) using the result [7]

$$\Psi_2(a;b,b;x,-x)={}_2F_3\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}; \\ b,\frac{1}{2}b,\frac{1}{2}b+\frac{1}{2};\end{matrix} -x^2\right], \tag{3.2}$$

we get

$$\begin{aligned}&\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}{}_2F_3\left[\begin{matrix}\frac{1}{2}b_1,\frac{1}{2}b_1+\frac{1}{2}; \\ c_1,\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2};\end{matrix} -(xs)^2\right]\\ &\times{}_2F_3\left[\begin{matrix}\frac{1}{2}b_2,\frac{1}{2}b_2+\frac{1}{2}; \\ c_2,\frac{1}{2}c_2,\frac{1}{2}c_2+\frac{1}{2};\end{matrix} -(ys)^2\right]\mathrm{d}s\\ &=F^{2:2;2}_{0:3;3}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}: & \frac{1}{2}b_1,\frac{1}{2}b_1+\frac{1}{2}; & \frac{1}{2}b_2,\frac{1}{2}b_2+\frac{1}{2}; \\ - & :c_1,\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2}; & c_2,\frac{1}{2}c_2,\frac{1}{2}c_2+\frac{1}{2};\end{matrix} -4x^2,-4y^2\right],\end{aligned} \tag{3.3}$$

which for $c_1=b_1$, $c_2=b_2$ reduces to the following integral in terms of Appell function F_4 [8]

$$\begin{aligned}&\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}{}_0F_1(-;b_1;-x^2s^2){}_0F_1(-;b_2;-y^2s^2)\mathrm{d}s\\ &=F_4[\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2};b_1,b_2;-4x^2,-4y^2].\end{aligned} \tag{3.4}$$

(ii) In (2.1), if we take i=1 and use the results (1.9)–(1.12), we get after a little simplification the following integral formula :

$$\begin{aligned}&\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Psi_2(b_1;c_1,c_1+1;xs,-xs)\Psi_2(b_2;c_2,c_2+1;ys,-ys)\mathrm{d}s\\ &=F^{2:2;2}_{0:3;3}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}: & \frac{1}{2}b_1,\frac{1}{2}b_1+\frac{1}{2}; & \frac{1}{2}b_2,\frac{1}{2}b_2+\frac{1}{2}; \\ - & :c_1,\frac{1}{2}c_1+\frac{1}{2},\frac{1}{2}c_1+1; & c_2,\frac{1}{2}c_2+\frac{1}{2},\frac{1}{2}c_2+1;\end{matrix} -4x^2,-4y^2\right]\\ &+\frac{ab_1x}{c_1(c_1+1)}\end{aligned}$$

$$\times F^{2:2;2}_{0:3;3}\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1: & \frac{1}{2}b_1+\frac{1}{2},\frac{1}{2}b_1+1 & ; & \frac{1}{2}b_2,\frac{1}{2}b_2+\frac{1}{2} & ; \\ - & :c_1+1,\frac{1}{2}c_1+1,\frac{1}{2}c_1+\frac{3}{2} & ; & c_2,\frac{1}{2}c_2+\frac{1}{2},\frac{1}{2}c_2+1 & ;\end{matrix} -4x^2,-4y^2\right]$$

$$+\frac{ab_2y}{c_2(c_2+1)}$$

$$\times F^{2:2;2}_{0:3;3}\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1: & \frac{1}{2}b_1,\frac{1}{2}b_1+\frac{1}{2} & ; & \frac{1}{2}b_2+\frac{1}{2},\frac{1}{2}b_2+1 & ; \\ - & :c_1,\frac{1}{2}c_1+\frac{1}{2},\frac{1}{2}c_1+1 & ; & c_2+1,\frac{1}{2}c_2+1,\frac{1}{2}c_2+\frac{3}{2} & ;\end{matrix} -4x^2,-4y^2\right]$$

$$+\frac{a(a+1)b_1b_2xy}{c_1c_2(c_1+1)(c_2+1)}$$

(3.5)

$$\times F^{2:2;2}_{0:3;3}\left[\begin{matrix}\frac{1}{2}a+1,\frac{1}{2}a+\frac{3}{2}: & \frac{1}{2}b_1+\frac{1}{2},\frac{1}{2}b_1+1 & ; & \frac{1}{2}b_2+\frac{1}{2},\frac{1}{2}b_2+1 & ; \\ - & :c_1+1,\frac{1}{2}c_1+1,\frac{1}{2}c_1+\frac{3}{2} & ; & c_2+1,\frac{1}{2}c_2+1,\frac{1}{2}c_2+\frac{3}{2} & ;\end{matrix} -4x^2,-4y^2\right].$$

Further, taking $c_1=b_1-1$, $c_2=b_2-1$ in (3.5) , we get

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Psi_2(b_1;b_1-1,b_1;xs,-xs)\Psi_2(b_2;b_2-1,b_2;ys,-ys)\mathrm{d}s$$

$$=F_4\left[\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2};b_1-1,b_2-1;-4x^2,-4y^2\right]$$

$$+\frac{ax}{b_1-1}F_4\left[\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1;b_1,b_2-1;-4x^2,-4y^2\right]$$

$$+\frac{ay}{b_2-1}F_4\left[\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1;b_1-1,b_2;-4x^2,-4y^2\right]$$

(3.6)
$$+\frac{a(a+1)xy}{(b_1-1)(b_2-1)}F_4\left[\frac{1}{2}a+1,\frac{1}{2}a+\frac{3}{2};b_1,b_2;-4x^2,-4y^2\right].$$

(iii) In (2.2), if we take $i=0$ and use the results (1.9)–(1.12), we get after a little simplification the following integral formula :

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Phi_2(b_1,b_1;c_1;xs,-xs)\Phi_2(b_2,b_2;c_2;ys,-ys)\mathrm{d}s$$

(3.7)
$$=F^{2:1;1}_{0:2;2}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}: & b_1 & ; & b_2 & ; \\ - & :\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2} & ; & \frac{1}{2}c_2,\frac{1}{2}c_2+\frac{1}{2} & ;\end{matrix} x^2,y^2\right].$$

Further, taking $c_1{=}2b_1$, $c_2{=}2b_2$ in (3.7) and using the result [7]

$$\Phi_2(a,a;2a;x,y)=e^y{}_1F_1(a;2a;x,-y), \tag{3.8}$$

we get

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-s-xs-ys}s^{a-1}{}_1F_1(b_1;2b_1;2xs){}_1F_1(b_2;2b_2;2ys)\mathrm{d}s$$

$$=F_4[\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2};b_1+\frac{1}{2},b_2+\frac{1}{2};x^2,y^2]. \tag{3.9}$$

(iv) In (2.2), if we take $i{=}1$ and use the results (1.9)–(1.12), we get after a little simplification the following integral formula :

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Phi_2(b_1-1,b_1;c_1;xs,-xs)\Phi_2(b_2-1,b_2;c_2;ys,-ys)\mathrm{d}s$$

$$=F^{2:1;1}_{0:2;2}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}: & b_1 & ; & b_2 & ; \\ & & & & x^2,y^2 \\ - & :\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2}; & & \frac{1}{2}c_2,\frac{1}{2}c_2+\frac{1}{2}; & \end{matrix}\right]$$

$$-\frac{ax}{c_1}F^{2:1;1}_{0:2;2}\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1: & b_1 & ; & b_2 & ; \\ & & & & x^2,y^2 \\ - & :\frac{1}{2}c_1+\frac{1}{2},\frac{1}{2}c_1+1; & & \frac{1}{2}c_2,\frac{1}{2}c_2+\frac{1}{2}; & \end{matrix}\right]$$

$$-\frac{ay}{c_2}F^{2:1;1}_{0:2;2}\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1: & b_1 & ; & b_2 & ; \\ & & & & x^2,y^2 \\ - & :\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2}; & & \frac{1}{2}c_2+\frac{1}{2},\frac{1}{2}c_2+1; & \end{matrix}\right]$$

$$+\frac{a(a+1)xy}{c_1c_2}$$

$$F^{2:1;1}_{0:2;2}\left[\begin{matrix}\frac{1}{2}a+1,\frac{1}{2}a+\frac{3}{2}: & b_1 & ; & b_2 & ; \\ & & & & x^2,y^2 \\ - & :\frac{1}{2}c_1+\frac{1}{2},\frac{1}{2}c_1+1; & & \frac{1}{2}c_2+\frac{1}{2},\frac{1}{2}c_2+1; & \end{matrix}\right]. \tag{3.10}$$

Further, taking $c_1{=}2b_1{-}1$, $c_2{=}2b_2{-}1$ in (3.10) , we get

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Phi_2(b_1-1,b_1;2b_1-1;xs,-xs)$$

$$\times\Phi_2(b_2-1,b_2;2b_2-1;ys,-ys)\mathrm{d}s$$

$$=F_4\left[\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2};b_1-\frac{1}{2},b_2-\frac{1}{2};x^2,y^2\right]$$

$$-\frac{ax}{2b_1-1}F_4\left[\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1;b_1+\frac{1}{2},b_2-\frac{1}{2};x^2,y^2\right]$$

$$-\frac{ay}{2b_2-1}F_4\left[\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1;b_1-\frac{1}{2},b_2+\frac{1}{2};x^2,y^2\right]$$

$$+\frac{a(a+1)xy}{(2b_1-1)(2b_2-1)}F_4\left[\frac{1}{2}a+1,\frac{1}{2}a+\frac{3}{2};b_1+\frac{1}{2},b_2+\frac{1}{2};x^2,y^2\right]. \tag{3.11}$$

The other special cases of the integrals (2.1) and (2.2) can also be obtained in the similar manner.

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