

Well-posedness and asymptotic stability for the Lamé system with internal distributed delay

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ABSTRACT. In this work, we consider the Lamé system in 3-dimension bounded domain with distributed delay term. We prove, under some appropriate assumptions, that this system is well-posed and stable. Furthermore, the asymptotic stability is given by using an appropriate Lyapunov functional.

1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^3 with smooth boundary $\partial\Omega$. Let us consider the following Lamé system with a distributed delay term:

$$(1.1) \quad \begin{cases} u''(x, t) - \Delta_e u(x, t) \\ \quad + \int_{\tau_1}^{\tau_2} \mu_2(s) u'(x, t - s) ds + \mu_1 u'(x, t) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases}$$

with initial conditions

$$(1.2) \quad \begin{cases} u(x) = u_0(x), & u'(x, 0) = u_1(x), & \text{in } \Omega, \\ u'(x, -t) = f_0(x, -t), & & \text{in } \Omega \times (0, \tau_2), \end{cases}$$

where (u_0, u_1, f_0) are given history and initial data. Here Δ denotes the Laplacian operator and Δ_e denotes the elasticity operator, which is the 3×3 matrix-valued differential operator defined by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu) \nabla(\operatorname{div} u), \quad u = (u_1, u_2, u_3)^T$$

and μ and λ are the Lamé constants which satisfy the conditions

$$(1.3) \quad \mu > 0, \quad \lambda + \mu \geq 0.$$

Moreover, $\mu_2 : [\tau_1, \tau_2] \rightarrow \mathbb{R}$ is a bounded function and $\tau_1 < \tau_2$ are two positive constants.

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In the particular case $\lambda + \mu = 0$, $\Delta_e = \mu\Delta$ gives a vector Laplacian, that is, (1.1) describes the vector wave equation.

In recent years, the control of partial differential equations with time delay effects has become an active and attractive area of research see ([1, 7, 9, 14, 15, 16] and [21]), and the references therein. Recently, S. A. Messaoudi et al. [21] considered the following problem with a strong damping and a strong distributed delay:

$$(1.4) \quad \begin{cases} u_{tt} - \Delta_x u(x, t) - \mu_1 \Delta u_t(x, t) \\ \quad - \int_{\tau_1}^{\tau_2} \mu_2(s) \Delta u_t(x, t-s) ds = 0, & \text{in } \Omega \times (0, +\infty), \\ u = 0, & \text{on } \Gamma \times [0, +\infty), \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x), & \text{on } \Omega, \\ u_t(x, -t) = f_0(x, -t), & 0 < t \leq \tau_2, \end{cases}$$

and under the assumption

$$(1.5) \quad \mu_1 > \int_{\tau_1}^{\tau_2} \mu_2(s) ds.$$

The authors proved that the solution is exponentially stable.

In [3], the authors considered the Bresse system in bounded domain with internal distributed delay

$$(1.6) \quad \begin{cases} \rho_1 \varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x - l\varphi) \\ \quad + \mu_1 \varphi_t + \mu_2 \varphi_t(x, t - \tau_1) = 0, \\ \rho_2 \psi_{tt} - El\psi_{xx} - Gh(\varphi_x - lw + \psi) \\ \quad + \int_{\tau_1}^{\tau_2} \mu(s) \psi_t(x, t-s) ds = 0, \\ \rho_1 w_{tt} - Eh(w_x - l\varphi)_x + lGh(\varphi_x + lw + \psi) \\ \quad + \widetilde{\mu}_1 w_t + \widetilde{\mu}_2 w_t(x, t - \tau_2) = 0, \end{cases}$$

where $(x, t) \in]0, L[\times \mathbb{R}_+$, the authors proved, under suitable conditions, that the system is well-posed and its energy converges to zero when time goes to infinity. For Timoshenko-type system with thermoelasticity of second sound, in the presence of a distributed delay Apalara [1] considered the following system:

$$(1.7) \quad \begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \mu \varphi_t \\ \quad + \int_{\tau_1}^{\tau_2} \mu_2(s) \varphi_t(x, t-s) ds = 0, & \text{in } (0, 1) \times (0, +\infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma \theta_x = 0, & \text{in } (0, 1) \times (0, \infty), \\ \rho_3 \theta_t + q_x + \delta \psi_{tx} = 0, & \text{in } (0, 1) \times (0, \infty), \\ \tau q_t + Bq + \theta_x = 0, & \text{in } (0, 1) \times (0, \infty), \end{cases}$$

and proved an exponential decay result under the assumption

$$(1.8) \quad \mu > \int_{\tau_1}^{\tau_2} \mu_2(s) ds.$$

In [4], Beniani and al. considered the following Lamé system with time varying delay term:

$$(1.9) \quad \begin{cases} u''(x, t) - \Delta_e u(x, t) + \mu_1 g_1(u'(x, t)) \\ \quad + \mu_2 g_2(u'(x, t - \tau(s))) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases}$$

the authors proved, under suitable conditions, that energy converges to zero when time goes to infinity.

The paper is organized as follows. In Section 2, we prove the global existence and uniqueness of solutions of (1.1)-(1.2). In Section 3, we prove the stability results.

2. WELL-POSEDNESS

In this section, we prove the existence and uniqueness of solutions of (1.1)-(1.2) using semigroup theory.

As in [20], we introduce the variable

$$z(x, \rho, t, s) = u'(x, t - \rho s), \quad (x, \rho, t, s) \in \Omega \times (0, 1) \times (0, \infty) \times (\tau_1, \tau_2).$$

Then, it is easy to check that

$$(2.1) \quad sz_t(x, \rho, t, s) + z_\rho(x, \rho, t, s) = 0, \quad (x, \rho, t, s) \in \Omega \times (0, 1) \times (0, \infty) \times (\tau_1, \tau_2).$$

Thus, system (1.1) becomes

$$(2.2) \quad \begin{cases} u''(x, t) - \Delta_e u(x, t) + \mu_1 u'(x, t) \\ \quad + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, t, s) ds = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ sz_t(x, \rho, t, s) + z_\rho(x, \rho, t, s) = 0, & \text{in } \Omega \times (0, 1) \times (0, \infty) \times (\tau_1, \tau_2) \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x), & \text{in } \Omega \times (0, 1) \times (\tau_1, \tau_2), \\ z(x, \rho, 0, s) = f_0(x, -\rho s), & \text{in } \Omega \times (0, 1) \times (\tau_1, \tau_2), \end{cases}$$

Next, we will formulate the system (1.1)-(1.2) in the following abstract linear first-order system:

$$(2.3) \quad \begin{cases} \mathcal{U}_t(t) = \mathcal{A}\mathcal{U}(t), \text{ for } t > 0, \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases}$$

where $\mathcal{U} = (u, u_t, z)^T$, $\mathcal{U}_0 = (u_0, u_1, f_0)^T \in \mathcal{H}$

$$\mathcal{H} = H_0^1(\Omega)^3 \times (L^2(\Omega))^3 \times L^2((0, 1), H)$$

We define the inner product in \mathcal{H} ,

$$\begin{aligned} \langle V, \bar{V} \rangle_{\mathcal{H}} &= \int_{\Omega} v \bar{v} dx + \mu \int_{\Omega} \nabla u \nabla \bar{u} dx + (\lambda + \mu) \int_{\Omega} \operatorname{div} u \cdot \operatorname{div} \bar{u} dx \\ &\quad + \int_{\Omega} \int_{\tau_1}^{\tau_2} s \mu_2(s) \int_0^1 z(x, \rho, t, s) \bar{z}(x, \rho, t, s) d\rho ds dx. \end{aligned}$$

The operators \mathcal{A} is linear and given by

$$(2.4) \quad \mathcal{A} \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} v \\ \Delta_e u(x, t) - \mu_1 v(x, t) - \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, t, s) ds \\ -\frac{1}{s} z_{\rho}(x, \rho, t, s) \end{pmatrix}$$

The domain $D(\mathcal{A})$ of \mathcal{A} is given by

$$D(\mathcal{A}) = \{V = (u, v, z)^T \in \mathcal{H}, \mathcal{A}V \in \mathcal{H}, z(\cdot, 0) = v\}.$$

The well-posedness of problem (2.3) is ensured by the following theorem.

Theorem 2.1. *Assume that*

$$(2.5) \quad \mu_1 > \int_{\tau_1}^{\tau_2} \mu_2(s) ds.$$

Then, for any $\mathcal{U}_0 \in \mathcal{H}$, the system (2.3) has a unique weak solution

$$\mathcal{U} \in C(\mathbb{R}^+, \mathcal{H}).$$

Moreover, if $\mathcal{U} \in D(\mathcal{A})$, then the solution of (2.3) satisfies (classical solution)

$$\mathcal{U} \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C(\mathbb{R}^+, D(\mathcal{A})).$$

Proof. We prove that $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}$ is a maximal monotone operator, that is, \mathcal{A} is dissipative and $Id - \mathcal{A}$ is surjective. Indeed, a simple calculation implies that, for any $V = (u, v, z)^T \in D(\mathcal{A})$,

$$\begin{aligned}
 \langle \mathcal{A}V, V \rangle_{\mathcal{H}} &= \mu \int_{\Omega} \nabla v(x, t) \nabla u(x, t) dx \\
 &\quad + (\lambda + \mu) \int_{\Omega} dv(x, t) \cdot du(x, t) dx \\
 &\quad + \int_{\Omega} \left\{ \Delta_e u(x, t) - \mu_1 v(x, t) \right. \\
 &\quad \left. - \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, t, s) ds \right\} v(x, t) dx \\
 (2.6) \quad &\quad - \int_{\Omega} \int_{\tau_1}^{\tau_2} \mu_2(s) \int_0^1 z(x, \rho, t, s) z_{\rho}(x, \rho, t, s) d\rho ds dx \\
 &= -\mu_1 \int_{\Omega} v^2(x, t) dx \\
 &\quad - \int_{\Omega} v(x, t) \left(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, t, s) ds \right) dx \\
 &\quad - \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} \mu_2(s) \int_0^1 \frac{\partial}{\partial \rho} |z(x, \rho, t, s)|^2 d\rho ds dx
 \end{aligned}$$

Using Young's inequality and taking into account that $z(\cdot, 0, \cdot, \cdot) = v$, we get

$$(2.7) \quad \langle \mathcal{A}V, V \rangle_{\mathcal{H}} = -\left(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s) ds \right) \int_{\Omega} v^2(x, t) dx$$

by virtue of (2.5). Therefore, \mathcal{A} is dissipative. On the other hand, we prove that $Id - \mathcal{A}$ is surjective. Indeed, let $F = (f, g, h)^T \in \mathcal{H}$ we show that there exists $V = (u, v, z)^T \in D(\mathcal{A})$ satisfying

$$(2.8) \quad (Id - \mathcal{A})V = F$$

which is equivalent to

$$(2.9) \quad \begin{cases} u - v = f, \\ v - \Delta_e u + \mu_1 v + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, t, s) ds = g, \\ sz(x, \rho, t, s) + z_{\rho}(x, \rho, t, s) = hs, \end{cases}$$

Using the equation in (2.9), we obtain

$$z(x, t, \rho, s) = (u - f)e^{-\rho s} + e^{-\rho s} \int_0^{\rho} sh(x, \sigma) e^{\sigma s} d\sigma.$$

Replacing v by $u - f$ in the second equation of (2.9), we get

$$(2.10) \quad Ku - \Delta_e u = G.$$

where

$$(2.11) \quad K = 1 + \mu_1 + \int_{\tau_1}^{\tau_2} e^{-s} \mu_2(s) ds > 0$$

and

$$G = g + \left(1 - \mu_1 - \int_{\tau_1}^{\tau_2} e^{-s} \mu_2(s) ds\right) f \\ + \int_{\tau_1}^{\tau_2} s e^{-s} \mu_2(s) \int_0^1 h(x, \sigma) \cdot e^{\sigma s} d\sigma ds.$$

So, we multiply (2.10) by a test function $\varphi \in (H_0^1(\Omega))^3$ and we integrate by using Green's equality, obtaining the following variational formulation of (2.10):

$$(2.12) \quad a(u, \varphi) = L(\varphi) \quad \text{for } \varphi \in (H_0^1(\Omega))^3,$$

where

$$(2.13) \quad a(u, \varphi) = \int_{\Omega} (Ku \cdot \varphi + \mu \nabla u \cdot \nabla \varphi + (\lambda + \mu) \operatorname{div} u \cdot \operatorname{div} \varphi) dx$$

and

$$(2.14) \quad L(\varphi) = \int_{\Omega} G \varphi dx.$$

It is clear that a is a bilinear and continuous form on $(H_0^1(\Omega))^3 \times (H_0^1(\Omega))^3$, and L is a linear and continuous form on $(H_0^1(\Omega))^3$. On the other hand, (1.3) and (2.11) imply that there exists a positive constant a_0 such that

$$a(u, u) \geq a_0 \|u\|_{(H_0^1(\Omega))^3}, \quad \text{for each } u \in (H_0^1(\Omega))^3,$$

which implies that a is coercive. Therefore, using the Lax-Milgram Theorem, we conclude that (2.12) has a unique solution $u \in (H_0^1(\Omega))^3$. By classical regularity arguments, we conclude that the solution u of (2.12) belongs into $(H^2(\Omega) \cap H_0^1(\Omega))^3$. Consequently, we deduce that (2.8) has a unique solution $V \in D(\mathcal{A})$. This proves that $Id - \mathcal{A}$ is surjective. Finally, (2.6) and (2.8) mean that $-\mathcal{A}$ is maximal monotone operator. Then, using Lummer-Phillips theorem (see [23]), we deduce that \mathcal{A} is an infinitesimal generator of a linear C_0 -semigroup on \mathcal{H} . \square

3. STABILITY

In this section, we investigate the asymptotic behaviour of the solution of problem (2.3). In fact, using the energy method to produce a suitable Lyapunov functional, we define the energy associated with the solution of (1.1)-(1.2) by

$$(3.1) \quad E_u(t) = \frac{1}{2} \int_{\Omega} (\mu |\nabla u|^2 + (\lambda + \mu) |\operatorname{div} u|^2 + |u'|^2) dx \\ + \frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, t, \rho, s) ds d\rho dx.$$

Theorem 3.1. *Assume that (1.3) and (2.5) hold. Then, for any $\mathcal{U}_0 \in \mathcal{H}$, there exist positive constants δ_1 and δ_2 , such that the solution of (2.3) satisfies*

$$(3.2) \quad E(t) \leq \delta_2 e^{-\delta_1 t}, \quad \text{for } t \in \mathbb{R}^+.$$

We carry out the proof of Theorem 3.1. Firstly, we will estimate several Lemmas.

Lemma 3.2. *Suppose that μ_1, μ_2 satisfy (2.5). Then energy functional satisfies, along the solution u of (1.1)-(1.2),*

$$(3.3) \quad E'(t) \leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s) ds\right) \int_{\Omega} u'^2(x, t) dx \leq 0$$

Proof. A differentiation of $E(t)$ gives

$$(3.4) \quad \begin{aligned} E'(t) = & \int_{\Omega} (\mu \nabla u \nabla u' + (\lambda + \mu) \operatorname{div} u \operatorname{div} u' + u' u'') dx \\ & + \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z'(x, t, \rho, s) z(x, t, \rho, s) ds d\rho dx. \end{aligned}$$

Using (2.2) and integrating by parts, we get

$$(3.5) \quad \begin{aligned} E'(t) = & -\mu_1 \int_{\Omega} u'^2(x, t) dx - \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x, t, 1, s) u'(x, t) ds dx \\ & - \frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \frac{\partial}{\partial \rho} (z^2(x, t, \rho, s)) ds d\rho dx \\ = & -\mu_1 \int_{\Omega} u'^2(x, t) dx - \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x, t, 1, s) u'(x, t) ds dx \\ & - \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, t, 1, s) ds dx \\ & + \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \int_{\Omega} u'^2(x, t) dx. \end{aligned}$$

Young's inequality leads to the desired estimate. □

Lemma 3.3. *The functional*

$$(3.6) \quad \phi(t) = \int_{\Omega} u \cdot u' dx, \quad \text{for } t \in \mathbb{R}^+$$

satisfies, along the solution u of (1.1)-(1.2)

$$(3.7) \quad \begin{aligned} \phi'(t) \leq & c \int_{\Omega} |u'|^2 dx - (\mu - c) \int_{\Omega} |\nabla u|^2 dx - (\lambda + \mu) \int_{\Omega} |\operatorname{div} u|^2 dx \\ & + c \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, t, 1, s) ds dx, \end{aligned}$$

for a positive constant c .

Proof. By differentiating (3.6) and using (2.2), yields

$$(3.8) \quad \begin{aligned} \phi'(t) = & \int_{\Omega} |u'|^2 dx - \mu \int_{\Omega} |\nabla u|^2 dx - (\lambda + \mu) \int_{\Omega} |\operatorname{div} u|^2 dx \\ & - \mu_1 \int_{\Omega} uu' dx - \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)| uz(x, t, 1, s) ds dx. \end{aligned}$$

By using Young's inequality, we obtain

$$(3.9) \quad \begin{aligned} \phi'(t) \leq & \left(\frac{\mu_1^2}{2} + 1 \right) \int_{\Omega} |u'|^2 dx - \mu \int_{\Omega} |\nabla u|^2 dx \\ & - (\lambda + \mu) \int_{\Omega} |\operatorname{div} u|^2 dx + \frac{1}{2} \int_{\Omega} u^2(x, t) dx \\ & + \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \int_{\Omega} u^2(x, t) dx \\ & + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, t, 1, s) ds dx. \end{aligned}$$

Then, Poincaré's inequality leads to the desired estimate. \square

Lemma 3.4. *The functional*

$$(3.10) \quad I(t) = \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)| z^2(x, t, \rho, s) ds d\rho dx, \quad \text{for } t \in \mathbb{R}^+$$

satisfy

$$(3.11) \quad \begin{aligned} I'(t) \leq & -e^{-\tau_2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, t, 1, s) ds dx \\ & + \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \int_{\Omega} u'^2(x, t) dx \\ & - e^{-\tau_2} \int_{\Omega} \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, t, \rho, s) ds d\rho dx. \end{aligned}$$

Proof. Using (2.1), the derivative of I entails

$$(3.12) \quad \begin{aligned} I'(t) = & 2 \int_{\Omega} \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)| z'(x, t, \rho, s) z(x, t, \rho, s) ds d\rho dx \\ = & - \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s\rho} \frac{\partial}{\partial \rho} \left(z^2(x, t, \rho, s) \right) ds d\rho dx \\ = & - \int_{\Omega} \int_{\tau_1}^{\tau_2} se^{-s} |\mu_2(s)| z^2(x, t, 1, s) ds dx \\ & + \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \int_{\Omega} u'^2(x, t) dx \\ & - \int_{\Omega} \int_{\tau_1}^{\tau_2} s |\mu_2(s)| \int_0^1 e^{-s\rho} z^2(x, t, \rho, s) d\rho ds dx, \end{aligned}$$

and the desired estimate follows immediately. \square

Now, we prove our main stability results (3.2).

Proof of Theorem 3.1. Let

$$(3.13) \quad L(t) = NE(t) + \epsilon\phi(t) + I(t),$$

where N and ϵ are positive constants that will be fixed later. Taking the derivative of $L(t)$ with respect to t and making use of (3.3), (3.6) and (3.11), we obtain

$$(3.14) \quad \begin{aligned} L'(t) \leq & -\left\{ \left(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s) ds \right) N - c\epsilon - \int_{\tau_1}^{\tau_2} \mu_2(s) ds \right\} \int_{\Omega} |u'|^2 dx \\ & - (\lambda + \mu) \int_{\Omega} |\operatorname{div} u|^2 dx - (\mu - c)\epsilon \int_{\Omega} |\nabla u|^2 dx \\ & - (e^{-\tau_2} - c\epsilon) \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, t, 1, s) ds dx \\ & - e^{-\tau_2} \int_{\Omega} \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, t, \rho, s) ds d\rho dx. \end{aligned}$$

At this point, we choose our constants in (3.14), carefully, such that all the coefficients in (3.14) will be negative. It suffices to choose ϵ so small such that

$$e^{-\tau_2} - c\epsilon > 0,$$

then pick N large enough such that

$$\left(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s) ds \right) N - c\epsilon - \int_{\tau_1}^{\tau_2} \mu_2(s) ds > 0.$$

Consequently, recalling (3.1), we deduce that there exist also $\eta_2 > 0$, such that

$$(3.15) \quad \frac{dL(t)}{dt} \leq -\eta_2 E(t), \quad \text{for } t \geq 0.$$

On the other hand, it is not hard to see that from (3.13) and for N large enough, there exist two positive constants β_1 and β_2 such that

$$(3.16) \quad \beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \quad \text{for } t \geq 0.$$

Combining (3.15) and (3.16), we deduce that there exists $\Lambda > 0$ for which the estimate

$$(3.17) \quad \frac{dL(t)}{dt} \leq -\Lambda L(t), \quad \forall t \geq 0,$$

holds. Integrating (3.17) over $(0, t)$ and using (3.16) once again, then (3.2) holds. Then, the proof is complete. \square

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