Ostrowski's inequalities for functions whose first derivatives are s -logarithmically preinvex in the second sense

BADREDDINE MEFTAH

ABSTRACT. In this paper, some Ostrowski's inequalities for functions whose first derivatives are s -logarithmically preinvex in the second sense are established.

1. INTRODUCTION

In 1938, A. M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows

Theorem 1.1 ([9]). *Let $I \subseteq \mathbb{R}$ be an interval. Let $f : I \rightarrow \mathbb{R}$, be a differentiable mapping in the interior I° of I , and $a, b \in I^\circ$ with $a < b$. If $|f'| \leq M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right], \quad x \in [a, b].$$

This is well-known Ostrowski's inequality. In recent years, a number of authors have written about generalizations, extensions and variants of such inequalities one can see [4, 5, 6, 7, 8, 15] and the reference cited therein.

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. Hanson in [3], introduced a new class of generalized convex functions, called invex functions. In [2], the authors gave the concept of preinvex functions which is special case of invexity. Pini [13], Noor [10, 11], Yang and Li [18] and Weir [17], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems.

2010 *Mathematics Subject Classification.* Primary: 26D10; Secondary: 26D15, 26A51.

Key words and phrases. Ostrowski inequality, midpoint inequality, Hölder inequality, power mean integral inequality.

Full paper. Received 8 December 2017, revised 12 July 2018, accepted 17 August 2018, available online 15 December 2018.

Meftah [8] established the following Ostrowski's inequality for functions whose derivative are log-preinvex.

Theorem 1.2. *Let $K \subseteq [0, \infty)$ be an invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$, $a, b \in K^\circ$ (K° interior of K) with $\eta(b, a) > 0$ and $[a, a + \eta(b, a)] \subset K$. Let $f : [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be a differentiable function such that $f' \in L([a, a + \eta(b, a)])$ and $f'(a) \neq 0$. If $|f'|$ is logarithmically preinvex function, then the following inequality holds:*

$$\left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) \, du \right| \leq \frac{\eta(b, a) |f'(a)|}{2} \\ \times \begin{cases} \left(\left(\frac{x-a}{\eta(b, a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b, a)} \right)^2 \right), & \text{if } A = 1, \\ 2 \left[\left(2 \frac{x-a}{\eta(b, a)} - 1 \right) \frac{A \frac{x-a}{\eta(b, a)}}{\ln A} + \frac{1-2A \frac{x-a}{\eta(b, a)} + A}{\ln^2 A} \right], & \text{if } A \neq 1, \end{cases}$$

for all $x \in [a, a + \eta(b, a)]$, where $A = \frac{|f'(b)|}{|f'(a)|}$.

Theorem 1.3. *Let $K \subseteq [0, \infty)$ be an invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$, $a, b \in K^\circ$ (K° interior of K) with $\eta(b, a) > 0$ and $[a, a + \eta(b, a)] \subset K$. Let $f : [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be a differentiable function such that $f' \in L([a, a + \eta(b, a)])$ and $f'(a) \neq 0$, let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f'|^q$ is a logarithmically preinvex function, then the following inequality holds:*

$$\left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) \, du \right| \leq \frac{\eta(b, a) |f'(a)|}{(p+1)^{\frac{1}{p}}} \\ \times \begin{cases} \left(\frac{x-a}{\eta(b, a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b, a)} \right)^2, & \text{if } A = 1, \\ \left(\frac{x-a}{\eta(b, a)} \right)^{1+\frac{1}{p}} \left(\frac{A^q \frac{x-a}{\eta(b, a)} - 1}{q \ln A} \right)^{\frac{1}{q}} + \left(1 - \frac{x-a}{\eta(b, a)} \right)^{1+\frac{1}{p}} \left(\frac{A^q - A^q \frac{x-a}{\eta(b, a)}}{q \ln A} \right)^{\frac{1}{q}}, & \text{if } A \neq 1, \end{cases}$$

for all $x \in [a, a + \eta(b, a)]$, where $A = \frac{|f'(b)|}{|f'(a)|}$.

Theorem 1.4. *Let $K \subseteq [0, \infty)$ be an invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$, $a, b \in K^\circ$ (K° interior of K) with $\eta(b, a) > 0$ and $[a, a + \eta(b, a)] \subset K$. Let $f : [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be a differentiable function such that $f' \in L([a, a + \eta(b, a)])$ and $f'(a) \neq 0$, let $q > 1$. If $|f'|^q$ is a logarithmically preinvex function, then the following inequality holds:*

$$\left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) \, du \right| \leq \frac{\eta(b, a)}{2^{1-\frac{1}{q}}} |f'(a)|$$

$$\times \begin{cases} \frac{1}{2^{\frac{1}{q}}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right), & \text{if } A = 1, \\ \left(\left(\frac{x-a}{\eta(b,a)} \right)^{2-\frac{2}{q}} \left(\frac{x-a}{\eta(b,a)} \frac{A^q \frac{x-a}{\eta(b,a)}}{\ln A} + \frac{1-A^q \frac{x-a}{\eta(b,a)}}{\ln^2 A} \right)^{\frac{1}{q}} + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{2-\frac{2}{q}} \right. \\ \left. \times \left(\frac{A^q - A^q \frac{x-a}{\eta(b,a)}}{\ln^2 A} - \left(1 - \frac{x-a}{\eta(b,a)} \right) \frac{A^q \frac{x-a}{\eta(b,a)}}{\ln A} \right)^{\frac{1}{q}} \right), & \text{if } A \neq 1, \end{cases}$$

for all $x \in [a, a + \eta(b, a)]$, where $A = \frac{|f'(b)|}{|f'(a)|}$.

Sarikaya et al. [14] established the following midpoint inequalities for differentiable log-preinvex functions.

Theorem 1.5. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$. Suppose that $f : K \rightarrow \mathbb{R}$ is a differentiable function. If $|f'|$ is log-preinvex on K then, for every $a, b \in K$ the following inequality holds:*

$$\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du - f\left(\frac{2a+\eta(b,a)}{2}\right) \right| \leq \eta(b,a) \left(\frac{|f'(b)|^{\frac{1}{2}} - |f'(a)|^{\frac{1}{2}}}{\log|f'(a)| - \log|f'(b)|} \right)^2.$$

Theorem 1.6. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$. Suppose that $f : K \rightarrow \mathbb{R}$ is a differentiable function. Assume $q \in \mathbb{R}$ with $q \geq 1$. If $|f'|^q$ is log-preinvex on K then, for every $a, b \in K$ the following inequality holds:*

$$\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du - f\left(\frac{2a+\eta(b,a)}{2}\right) \right| \leq \eta(b,a) \frac{|f'(a)|^{\frac{1}{2}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}q^{\frac{1}{q}}} \left(\frac{|f'(b)|^{\frac{q}{2}} - |f'(a)|^{\frac{q}{2}}}{\log|f'(a)| - \log|f'(b)|} \right)^{\frac{1}{q}}.$$

Motivated by the above results, in this paper we establish some new Ostrowski type inequalities for functions whose first derivatives are logarithmically s -preinvex in the second sense.

2. PRELIMINARIES

In this section we recall some concepts of convexity that are well known in the literature. Throughout this section I is an interval of $\mathbb{R}$.

Definition 2.1 ([12]). A positive function $f : I \rightarrow \mathbb{R}$ is said to be logarithmically convex, if

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2.2. A positive function $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ is said to be s -logarithmically convex function in the second sense on I , if the following inequality

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{(1-t)^s}$$

holds for some $s \in (0, 1]$, all $x, y \in I$ and $t \in [0, 1]$.

Definition 2.3 ([17]). A set K is said to be invex at x with respect to η , if

$$x + t\eta(y, x) \in K$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

K is said to be an invex set with respect to η if K is invex at each $x \in K$.

Definition 2.4 ([10]). A positive function f on the invex set K is said to be logarithmically preinvex function with respect to η , if

$$f(x + t\eta(y, x)) \leq [f(x)]^{1-t} [f(y)]^t$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

Definition 2.5 ([16]). A positive function f on the invex set $K \subseteq [0, \infty)$ is said to be s -logarithmically preinvex function in the second sense with respect to η , if

$$f(x + t\eta(y, x)) \leq [f(x)]^{(1-t)^s} [f(y)]^{ts}$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

The following lemmas are essential to establishing our main results.

Lemma 2.1 ([1]). Let $0 < \phi \leq 1 \leq \psi$ and $t, s \in (0, 1]$, then

$$\begin{aligned} \phi^{ts} &\leq \phi^{st}, \\ \psi^{ts} &\leq \psi^{st+1-s}. \end{aligned}$$

Lemma 2.2 ([4]). Let $A \subset \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function. If f' is integrable on $[a, a + \eta(b, a)]$, then the following equality

$$\begin{aligned} &f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) \, du \\ &= \eta(b, a) \left(\int_0^{\frac{x-a}{\eta(b, a)}} t f'(a + t\eta(b, a)) \, dt + \int_{\frac{x-a}{\eta(b, a)}}^1 (t-1) f'(a + t\eta(b, a)) \, dt \right) \end{aligned}$$

holds for all $x \in [a, a + \eta(b, a)]$.

3. MAIN RESULTS

In what follows we assume that $K \subseteq [0, \infty)$ be an invex subset with respect to the bifunction $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K^\circ$ interior of K with $a < a + \eta(b, a)$ such that $[a, a + \eta(b, a)] \subset K$.

Theorem 3.1. *Let $f : [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be a differentiable function such that $f' \in L([a, a + \eta(b, a)])$. If $|f'|$ is s -logarithmically preinvex function in the second sense for some fixed $s \in (0, 1]$ with $|f'(a)| \neq 0$, then for all $x \in [a, a + \eta(b, a)]$ we have the following inequality*

$$\left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) \, du \right| \leq \eta(b, a) \times \begin{cases} N_{(s, \lambda)} \left(\frac{\left(2 \frac{x-a}{\eta(b, a)} - 1\right) \lambda^s \frac{x-a}{\eta(b, a)}}{s \ln \lambda} + \frac{1+\lambda^s-2\lambda^s \frac{x-a}{\eta(b, a)}}{(s \ln \lambda)^2} \right), & \text{if } \lambda \neq 1; \\ \frac{|f'(a)|^s}{2} \left(\left(\frac{x-a}{\eta(b, a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b, a)} \right)^2 \right), & \text{if } |f'(a)| < 1 = \lambda; \\ \frac{|f'(a)| |f'(b)|^{1-s}}{2} \left(\left(\frac{x-a}{\eta(b, a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b, a)} \right)^2 \right), & \text{if } |f'(a)| > 1 = \lambda; \end{cases}$$

where

$$(1) \quad \lambda = \frac{|f'(b)|}{|f'(a)|},$$

and

$$(2) \quad N_{(s, \lambda)} = \begin{cases} |f'(a)|^s, & \text{if } |f'(a)|, |f'(b)| < 1; \\ |f'(a)|^s |f'(b)|^{1-s}, & \text{if } |f'(a)| \leq 1 \leq |f'(b)|; \\ |f'(a)|, & \text{if } |f'(b)| \leq 1 \leq |f'(a)|; \\ |f'(a)| |f'(b)|^{1-s}, & \text{if } |f'(a)|, |f'(b)| > 1. \end{cases}$$

Proof. From Lemma 2.2, and property of modulus, we have

$$\left| f(x) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u) \, du \right| \leq \eta(b, a) \left(\int_0^{\frac{x-a}{\eta(b, a)}} t |f'(a + t\eta(b, a))| \, dt + \int_{\frac{x-a}{\eta(b, a)}}^1 (1-t) |f'(a + t\eta(b, a))| \, dt \right).$$

Since $|f'|$ is s -logarithmically preinvex function, we deduce

$$\begin{aligned}
 & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du \right| \\
 & \leq \eta(b,a) \left(\int_0^{\frac{x-a}{\eta(b,a)}} t |f'(a)|^{(1-t)^s} |f'(b)|^{ts} \, dt \right. \\
 & \quad \left. + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) |f'(a)|^{(1-t)^s} |f'(b)|^{ts} \, dt \right).
 \end{aligned}
 \tag{3}$$

From Lemma 2.1, we have

$$|f'(a)|^{(1-t)^s} |f'(b)|^{ts} \leq N_{(s,\lambda)} \times \lambda^{st},$$

where λ and $N_{(s,\lambda)}$ are defined by (1) and (2) respectively.

Substituting (4) into (3), we obtain

$$\begin{aligned}
 & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du \right| \leq \eta(b,a) \times N_{(s,\lambda)} \\
 & \times \left(\int_0^{\frac{x-a}{\eta(b,a)}} t \lambda^{st} \, dt + \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) \lambda^{st} \, dt \right).
 \end{aligned}
 \tag{5}$$

Clearly, in the case where $\lambda \neq 1$, we have

$$\int_0^{\frac{x-a}{\eta(b,a)}} t \lambda^{st} \, dt = \frac{\frac{x-a}{\eta(b,a)} \lambda^{s \frac{x-a}{\eta(b,a)}}}{s \ln \lambda} + \frac{1 - \lambda^{s \frac{x-a}{\eta(b,a)}}}{(s \ln \lambda)^2},
 \tag{6}$$

and

$$\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) \lambda^{st} \, dt = \frac{\lambda^s - \lambda^{s \frac{x-a}{\eta(b,a)}}}{(s \ln \lambda)^2} - \frac{(1 - \frac{x-a}{\eta(b,a)}) \lambda^{s \frac{x-a}{\eta(b,a)}}}{s \ln \lambda}.
 \tag{7}$$

Substituting (6) and (7) into (5), we obtain

$$(8) \quad \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du \right| \leq \eta(b,a) \times N_{(s,\lambda)} \left(\frac{\left(2 \frac{x-a}{\eta(b,a)} - 1\right) \lambda^s \frac{x-a}{\eta(b,a)}}{s \ln \lambda} + \frac{1+\lambda^s - 2\lambda^s \frac{x-a}{\eta(b,a)}}{(s \ln \lambda)^2} \right).$$

Now, we assume that $\lambda = 1$, then (5) gives

$$(9) \quad \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du \right| \leq \eta(b,a) \times \begin{cases} \frac{|f'(a)|^s}{2} \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right), & \text{if } |f'(a)| < 1; \\ \frac{|f'(a)||f'(b)|^{1-s}}{2} \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right), & \text{if } |f'(a)| > 1. \end{cases}$$

The desired result follows from (8) and (9). $\square$

Corollary 3.1. *In Theorem 3.1, if we choose $x = \frac{2a+\eta(b,a)}{2}$, then we obtain the following midpoint inequality*

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du \right| \leq \eta(b,a) \begin{cases} N_{(s,\lambda)} \left(\frac{1-\lambda^{\frac{s}{2}}}{s \ln \lambda} \right)^2, & \text{if } \lambda \neq 1; \\ \frac{|f'(a)|^s}{4}, & \text{if } |f'(a)| < 1 = \lambda; \\ \frac{|f'(a)||f'(b)|^{1-s}}{4}, & \text{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Remark 3.1. Theorem 3.1 will be reduces to Theorem 7 from [8] and Corollary 1 will be reduces to Corollary 8 from [8] and Theorem 4.1 from [14] if we put $s = 1$.

Corollary 3.2. *In Theorem 3.1, if we choose $\eta(b,a) = b - a$, we have the following inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq (b-a)$$

$$\times \begin{cases} N_{(s,\lambda)} \left(\frac{\left(\frac{2x-(b+a)}{b-a}\right)\lambda^s \frac{x-a}{b-a}}{s \ln \lambda} + \frac{1+\lambda^s-2\lambda^s \frac{x-a}{b-a}}{(s \ln \lambda)^2} \right), & \text{if } \lambda \neq 1; \\ \frac{|f'(a)|^s}{2} \left(\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2 \right), & \text{if } |f'(a)| < 1 = \lambda; \\ \frac{|f'(a)||f'(b)|^{1-s}}{2} \left(\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2 \right), & \text{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Moreover if we choose $s = 1$ we get the following inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq (b-a) \times \begin{cases} |f'(a)| \left(\frac{\left(\frac{2x-(b+a)}{b-a}\right)\lambda^{\frac{x-a}{b-a}}}{\ln \lambda} + \frac{1+\lambda-2\lambda^{\frac{x-a}{b-a}}}{(\ln \lambda)^2} \right), & \text{if } \lambda \neq 1; \\ \frac{|f'(a)|}{2} \left(\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2 \right), & \text{if } 1 = \lambda. \end{cases}$$

Corollary 3.3. In Theorem 3.1, if we choose $\eta(b, a) = b - a$, and $x = \frac{a+b}{2}$ we have the following inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq (b-a) \times \begin{cases} N_{(s,\lambda)} \left(\frac{1+\lambda^s-2\lambda^{\frac{s}{2}}}{(s \ln \lambda)^2} \right), & \text{if } \lambda \neq 1; \\ \frac{|f'(a)|^s}{4}, & \text{if } |f'(a)| < 1 = \lambda; \\ \frac{|f'(a)||f'(b)|^{1-s}}{4}, & \text{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Moreover if we choose $s = 1$ we get the following inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq (b-a) |f'(a)| \times \begin{cases} \left(\frac{1-\lambda^{\frac{1}{2}}}{\ln \lambda} \right)^2, & \text{if } \lambda \neq 1, \\ \frac{|f'(a)|}{4}, & \text{if } 1 = \lambda. \end{cases}$$

Remark 3.2. The case $\lambda \neq 1$ in the last inequality of Corollary 3.3 represent Corollary 4.3 from [14].

Theorem 3.2. Let $f : K \rightarrow (0, \infty)$ be a differentiable function such that $f' \in L([a, a + \eta(b, a)])$, and let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f'|^q$ is s -logarithmically preinvex function in the second sense for some fixed $s \in (0, 1]$ with $|f'(a)| \neq 0$, we have the following inequality

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du \right| \leq \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}}$$

$$\times \begin{cases} N_{(s,q,\lambda)}^{\frac{1}{q}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{1+\frac{1}{p}} \left(\frac{\lambda^{qs} \frac{x-a}{\eta(b,a)} - 1}{qs \ln \lambda} \right)^{\frac{1}{q}} \right. \\ \quad \left. + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1+\frac{1}{p}} \left(\frac{\lambda^{qs} - \lambda^{qs} \frac{x-a}{\eta(b,a)}}{qs \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1; \\ |f'(a)|^s \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right), & \text{if } |f'(a)| \leq 1 = \lambda; \\ |f'(a)|^{2-s} \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right), & \text{if } |f'(a)| > 1 = \lambda, \end{cases}$$

where λ is defined as in (1), and

$$(10) \quad N_{(s,q,\lambda)} = \begin{cases} |f'(a)|^{qs}, & \text{if } |f'(a)|, |f'(b)| < 1; \\ |f'(a)|^{qs} |f'(b)|^{q-qs}, & \text{if } |f'(a)| \leq 1 \leq |f'(b)|; \\ |f'(a)|^q, & \text{if } |f'(b)| \leq 1 \leq |f'(a)|; \\ |f'(a)|^q |f'(b)|^{q-qs}, & \text{if } |f'(a)|, |f'(b)| > 1, \end{cases}$$

Proof. From Lemma 2.2, property of modulus, and Hölder's inequality, we have

$$\begin{aligned} & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du \right| \\ & \leq \eta(b,a) \left(\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^p \, dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} |f'(a + t\eta(b,a))|^q \, dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^p \, dt \right)^{\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 |f'(a + t\eta(b,a))|^q \, dt \right)^{\frac{1}{q}} \right) \\ & = \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{1+\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} |f'(a + t\eta(b,a))|^q \, dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1+\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 |f'(a + t\eta(b,a))|^q \, dt \right)^{\frac{1}{q}} \right). \end{aligned}$$

Using the fact that $|f'|^q$ is s -logarithmically preinvex and Lemma 2.1, we obtain

$$\begin{aligned}
 & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du \right| \\
 & \leq \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{1+\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} |f'(a)|^{q(1-t)^s} |f'(b)|^{qt^s} \, dt \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1+\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 |f'(a)|^{q(1-t)^s} |f'(b)|^{qt^s} \, dt \right)^{\frac{1}{q}} \right) \\
 & \leq \frac{\eta(b,a) N_{(s,q,\lambda)}^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{1+\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} \lambda^{qst} \, dt \right)^{\frac{1}{q}} \right. \\
 (11) \quad & \left. + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1+\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 \lambda^{qst} \, dt \right)^{\frac{1}{q}} \right).
 \end{aligned}$$

For $\lambda \neq 1$, (11) becomes

$$\begin{aligned}
 & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du \right| \\
 & \leq \frac{\eta(b,a) N_{(s,q,\lambda)}^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{1+\frac{1}{p}} \left(\frac{\lambda^{qs \frac{x-a}{\eta(b,a)}} - 1}{qs \ln \lambda} \right)^{\frac{1}{q}} \right. \\
 (12) \quad & \left. + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{1+\frac{1}{p}} \left(\frac{\lambda^{qs} - \lambda^{qs \frac{x-a}{\eta(b,a)}}}{qs \ln \lambda} \right)^{\frac{1}{q}} \right),
 \end{aligned}$$

where λ and $N_{(s,q,\lambda)}$ are defined as in (1) and (10) respectively, and the fact that

$$\int_0^{\frac{x-a}{\eta(b,a)}} \lambda^{qst} dt = \frac{\lambda^{qs \frac{x-a}{\eta(b,a)}} - 1}{qs \ln \lambda},$$

$$\int_{\frac{x-a}{\eta(b,a)}}^1 \lambda^{qst} dt = \frac{\lambda^{qs} - \lambda^{qs \frac{x-a}{\eta(b,a)}}}{qs \ln \lambda}.$$

In the case where $\lambda = 1$, (11) becomes

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right|$$

$$\leq \begin{cases} \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}} |f'(a)|^s \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right), & \text{if } |f'(a)| \leq 1; \\ \frac{\eta(b,a) N_{(s,q,\lambda)}^{\frac{1}{q}}}{(p+1)^{\frac{1}{p}}} |f'(a)|^{2-s} \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right), & \text{if } |f'(a)| > 1. \end{cases}$$

From (12) and (13) we get the desired result. $\square$

Corollary 3.4. *In Theorem 3.2, if we choose $x = \frac{2a+\eta(b,a)}{2}$, then we obtain the following midpoint inequality*

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right|$$

$$\leq \frac{\eta(b,a)}{(p+1)^{\frac{1}{p}}} \times \begin{cases} \frac{N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1+\frac{1}{p}}} \left(\left(\frac{\lambda^{\frac{qs}{2}} - 1}{qs \ln \lambda} \right)^{\frac{1}{q}} + \left(\frac{\lambda^{qs} - \lambda^{\frac{qs}{2}}}{qs \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1, \\ \frac{1}{2} |f'(a)|^s, & \text{if } |f'(a)| \leq 1 = \lambda, \\ \frac{1}{2} |f'(a)|^{2-s}, & \text{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Remark 3.3. Theorem 3.2 will be reduces to Theorem 11 from [8], and Corollary 3.4 will be reduces to Corollary 12 from [8] and Theorem 4.2 from [14] if we put $s = 1$.

Corollary 3.5. *In Theorem 3.2, if we choose $\eta(b,a) = b-a$, then we obtain the following inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}}$$

$$\times \begin{cases} N_{(s,q,\lambda)}^{\frac{1}{q}} \left(\left(\frac{x-a}{b-a} \right)^{1+\frac{1}{p}} \left(\frac{\lambda^{qs} \frac{x-a}{b-a} - 1}{qs \ln \lambda} \right)^{\frac{1}{q}} \right. \\ \quad \left. + \left(\frac{b-x}{b-a} \right)^{1+\frac{1}{p}} \left(\frac{\lambda^{qs} - \lambda^{qs} \frac{x-a}{b-a}}{qs \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1; \\ |f'(a)|^s \left(\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right), & \text{if } |f'(a)| \leq 1 = \lambda; \\ |f'(a)|^{2-s} \left(\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right), & \text{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Moreover if we choose $s = 1$ we get the following inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}} |f'(a)| \\ \times \begin{cases} \left(\left(\frac{x-a}{b-a} \right)^{1+\frac{1}{p}} \left(\frac{\lambda^q \frac{x-a}{b-a} - 1}{q \ln \lambda} \right)^{\frac{1}{q}} + \left(\frac{b-x}{b-a} \right)^{1+\frac{1}{p}} \left(\frac{\lambda^q - \lambda^q \frac{x-a}{b-a}}{q \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1; \\ \left(\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right), & \text{if } 1 = \lambda. \end{cases}$$

Corollary 3.6. In Theorem 3.2, if we choose $\eta(b, a) = b - a$, and $x = \frac{a+b}{2}$ we have the following inequality

$$\left| f\left(\frac{b+a}{2}\right) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}} \\ \times \begin{cases} \frac{N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1+\frac{1}{p}}} \left(\left(\frac{\lambda^{\frac{qs}{2}} - 1}{qs \ln \lambda} \right)^{\frac{1}{q}} + \left(\frac{\lambda^{qs} - \lambda^{\frac{qs}{2}}}{qs \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1; \\ \frac{1}{2} |f'(a)|^s, & \text{if } |f'(a)| \leq 1 = \lambda; \\ \frac{1}{2} |f'(a)|^{2-s}, & \text{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Moreover if we choose $s = 1$ we get the following inequality

$$\left| f\left(\frac{b+a}{2}\right) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} |f'(a)| \\ \times \begin{cases} \frac{1}{2^{\frac{1}{p}}} \left(\left(\frac{\lambda^{\frac{q}{2}} - 1}{q \ln \lambda} \right)^{\frac{1}{q}} + \left(\frac{\lambda^q - \lambda^{\frac{q}{2}}}{q \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1; \\ 1, & \text{if } 1 = \lambda. \end{cases}$$

Remark 3.4. The case $\lambda \neq 1$ in the last inequality of Corollary 3.6 represent Corollary 4.4 from [14].

Theorem 3.3. Let $f : K \rightarrow (0, \infty)$ be a differentiable function such that $f' \in L([a, a + \eta(b, a)])$ and let $q > 1$. If $|f'|^q$ is s -logarithmically preinvex

function in the second sense for some fixed $s \in (0, 1]$ with $|f'(a)| \neq 0$, then we have the following inequality

$$\begin{aligned} & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du \right| \\ & \leq \begin{cases} \frac{\eta(b,a) N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1-\frac{1}{q}}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{2-\frac{2}{q}} \left(\frac{\frac{x-a}{\eta(b,a)} \lambda^{qs} \frac{x-a}{\eta(b,a)}}{qs \ln \lambda} + \frac{1-\lambda^{qs} \frac{x-a}{\eta(b,a)}}{(qs \ln \lambda)^2} \right)^{\frac{1}{q}} \right. \\ \quad \left. + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{2-\frac{2}{q}} \left(\frac{\lambda^{qs} - \lambda^{qs} \frac{x-a}{\eta(b,a)}}{(qs \ln \lambda)^2} - \frac{\left(1 - \frac{x-a}{\eta(b,a)} \right) \lambda^{qs} \frac{x-a}{\eta(b,a)}}{qs \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1, \\ \frac{\eta(b,a) |f'(a)|^s}{2} \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right), & \text{if } |f'(a)| \leq 1 = \lambda, \\ \frac{\eta(b,a) |f'(a)|^{2-s}}{2} \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right), & \text{if } |f'(a)| > 1 = \lambda, \end{cases} \end{aligned}$$

where λ and $N_{(s,q,\lambda)}$ are defined as in (1) and (10) respectively.

Proof. From Lemma 2.2, property of modulus, power mean inequality, s -logarithmically preinvexity of $|f'|^q$, and Lemma 1, we get

$$\begin{aligned} & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) \, du \right| \\ & \leq \eta(b,a) \left(\left(\int_0^{\frac{x-a}{\eta(b,a)}} t \, dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} t |f'(a + t\eta(b,a))|^q \, dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) \, dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) |f'(a + t\eta(b,a))|^q \, dt \right)^{\frac{1}{q}} \right) \\ & = \frac{\eta(b,a)}{2^{1-\frac{1}{q}}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{2(1-\frac{1}{q})} \left(\int_0^{\frac{x-a}{\eta(b,a)}} t |f'(a + t\eta(b,a))|^q \, dt \right)^{\frac{1}{q}} \right. \end{aligned}$$

$$\begin{aligned}
 & + \left(1 - \frac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) |f'(a+t\eta(b,a))|^q dt \right)^{\frac{1}{q}} \Bigg) \\
 & \leq \frac{\eta(b,a)}{2^{1-\frac{1}{q}}} \left(\left(\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} t |f'(a)|^{q(1-t)^s} |f'(b)|^{qt^s} dt \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left(1 - \frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) |f'(a)|^{q(1-t)^s} |f'(b)|^{qt^s} dt \right)^{\frac{1}{q}} \right) \\
 & \leq \frac{\eta(b,a) N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1-\frac{1}{q}}} \left(\left(\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} t \lambda^{qst} dt \right)^{\frac{1}{q}} \right. \\
 (13) \quad & \left. + \left(1 - \frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) \lambda^{qst} dt \right)^{\frac{1}{q}} \right),
 \end{aligned}$$

where λ and $N_{(s,q,\lambda)}$ are defined as in (1) and (10) respectively.

For $\lambda \neq 1$, (13), becomes

$$\begin{aligned}
 & \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \\
 & \leq \frac{\eta(b,a) N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1-\frac{1}{q}}} \left(\left(\frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}} \left(\frac{\frac{x-a}{\eta(b,a)} \lambda^{qs \frac{x-a}{\eta(b,a)}}}{qs \ln \lambda} + \frac{1-\lambda^{qs \frac{x-a}{\eta(b,a)}}}{(qs \ln \lambda)^2} \right)^{\frac{1}{q}} \right. \\
 (14) \quad & \left. + \left(1 - \frac{x-a}{\eta(b,a)}\right)^{2-\frac{2}{q}} \left(\frac{\lambda^{qs} - \lambda^{qs \frac{x-a}{\eta(b,a)}}}{(qs \ln \lambda)^2} - \frac{\left(1 - \frac{x-a}{\eta(b,a)}\right) \lambda^{qs \frac{x-a}{\eta(b,a)}}}{qs \ln \lambda} \right)^{\frac{1}{q}} \right),
 \end{aligned}$$

where we have used the fact that

$$\int_0^{\frac{x-a}{\eta(b,a)}} t \lambda^{qst} dt = \frac{\frac{x-a}{\eta(b,a)} \lambda^{qs \frac{x-a}{\eta(b,a)}}}{qs \ln \lambda} + \frac{1 - \lambda^{qs \frac{x-a}{\eta(b,a)}}}{(qs \ln \lambda)^2}$$

and

$$\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t) \lambda^{qst} dt = \frac{\lambda^{qs} - \lambda^{qs \frac{x-a}{\eta(b,a)}}}{(qs \ln \lambda)^2} - \frac{\left(1 - \frac{x-a}{\eta(b,a)}\right) \lambda^{qs \frac{x-a}{\eta(b,a)}}}{qs \ln \lambda}.$$

In the case where $\lambda = 1$, (13) gives

$$(15) \quad \left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \begin{cases} \frac{\eta(b,a)|f'(a)|^s}{2} \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right), & \text{if } |f'(a)| \leq 1 = \lambda; \\ \frac{\eta(b,a)|f'(a)|^{2-s}}{2} \left(\left(\frac{x-a}{\eta(b,a)} \right)^2 + \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right), & \text{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Thus, from (14) and (15) we get the desired result. $\square$

Corollary 3.7. *In Theorem 3.3, if we choose $x = \frac{2a+\eta(b,a)}{2}$, then we obtain the following midpoint inequality where*

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u) du \right| \leq \begin{cases} \frac{\eta(b,a)N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{3-\frac{3}{q}}} \left(\left(\frac{\lambda^{\frac{qs}{2}}}{2qs \ln \lambda} + \frac{1-\lambda^{\frac{qs}{2}}}{(qs \ln \lambda)^2} \right)^{\frac{1}{q}} + \left(\frac{\lambda^{qs} - \lambda^{\frac{qs}{2}}}{(qs \ln \lambda)^2} - \frac{\lambda^{\frac{qs}{2}}}{2qs \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1; \\ \frac{\eta(b,a)|f'(a)|^s}{4}, & \text{if } |f'(a)| \leq 1 = \lambda; \\ \frac{\eta(b,a)|f'(a)|^{2-s}}{4}, & \text{if } |f'(a)| > 1 = \lambda. \end{cases}$$

Remark 3.5. Theorem 3.3 and Corollary 3.7 will be reduces to Theorem 15 and Corollary 16 from [8] if we put $s = 1$.

Corollary 3.8. *In Theorem 3.3, if we choose $\eta(b,a) = b-a$, then we obtain the following inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right|$$

$$\leq \left\{ \begin{array}{ll} \frac{(b-a)N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{1-\frac{1}{q}}} \left(\left(\frac{x-a}{b-a} \right)^{2-\frac{2}{q}} \left(\frac{\frac{x-a}{b-a} \lambda^{qs} \frac{x-a}{b-a}}{qs \ln \lambda} + \frac{1-\lambda^{qs} \frac{x-a}{b-a}}{(qs \ln \lambda)^2} \right)^{\frac{1}{q}} \right. \\ \quad \left. + \left(\frac{b-x}{b-a} \right)^{2-\frac{2}{q}} \left(\frac{\lambda^{qs} - \lambda^{qs} \frac{x-a}{b-a}}{(qs \ln \lambda)^2} - \frac{(\frac{b-x}{b-a}) \lambda^{qs} \frac{x-a}{b-a}}{qs \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1; \\ \frac{(b-a)|f'(a)|^s}{2} \left(\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right), & \text{if } |f'(a)| \leq 1 = \lambda; \\ \frac{(b-a)|f'(a)|^{2-s}}{2} \left(\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right), & \text{if } |f'(a)| > 1 = \lambda. \end{array} \right.$$

Moreover if we choose $s = 1$ we get the following inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \left\{ \begin{array}{ll} \frac{(b-a)|f'(a)|}{2^{1-\frac{1}{q}}} \left(\left(\frac{x-a}{b-a} \right)^{2-\frac{2}{q}} \left(\frac{\frac{x-a}{b-a} \lambda^q \frac{x-a}{b-a}}{q \ln \lambda} + \frac{1-\lambda^q \frac{x-a}{b-a}}{(q \ln \lambda)^2} \right)^{\frac{1}{q}} \right. \\ \quad \left. + \left(\frac{b-x}{b-a} \right)^{2-\frac{2}{q}} \left(\frac{\lambda^q - \lambda^q \frac{x-a}{b-a}}{(q \ln \lambda)^2} - \frac{(\frac{b-x}{b-a}) \lambda^q \frac{x-a}{b-a}}{q \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1, \\ \frac{(b-a)|f'(a)|}{2} \left(\left(\frac{x-a}{b-a} \right)^2 + \left(\frac{b-x}{b-a} \right)^2 \right), & \text{if } 1 = \lambda. \end{array} \right.$$

Corollary 3.9. In Theorem 3.3, if we choose $\eta(b, a) = b - a$, and $x = \frac{a+b}{2}$ we have the following inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \left\{ \begin{array}{ll} \frac{(b-a)N_{(s,q,\lambda)}^{\frac{1}{q}}}{2^{3-\frac{3}{q}}} \left(\left(\frac{\lambda^{\frac{qs}{2}}}{2qs \ln \lambda} + \frac{1-\lambda^{\frac{qs}{2}}}{(qs \ln \lambda)^2} \right)^{\frac{1}{q}} + \left(\frac{\lambda^{qs} - \lambda^{\frac{qs}{2}}}{(qs \ln \lambda)^2} - \frac{\lambda^{\frac{qs}{2}}}{2qs \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1; \\ \frac{(b-a)|f'(a)|^s}{4}, & \text{if } |f'(a)| \leq 1 = \lambda; \\ \frac{(b-a)|f'(a)|^{2-s}}{4}, & \text{if } |f'(a)| > 1 = \lambda. \end{array} \right.$$

Moreover if we choose $s = 1$ we get the following inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u) \, du \right|$$

$$\leq \begin{cases} \frac{(b-a)|f'(a)|}{2^{3-\frac{3}{q}}} \left(\left(\frac{\lambda^{\frac{q}{2}}}{2q \ln \lambda} + \frac{1-\lambda^{\frac{q}{2}}}{(q \ln \lambda)^2} \right)^{\frac{1}{q}} + \left(\frac{\lambda^q - \lambda^{\frac{q}{2}}}{(q \ln \lambda)^2} - \frac{\lambda^{\frac{q}{2}}}{2q \ln \lambda} \right)^{\frac{1}{q}} \right), & \text{if } \lambda \neq 1, \\ \frac{(b-a)|f'(a)|}{4}, & \text{if } 1 = \lambda. \end{cases}$$

REFERENCES

- [1] F. Bai, F. Qi, B. Xi, *Hermite-Hadamard type inequalities for the m - and (α, m) -logarithmically convex functions*, Filomat, 27 (1) (2013), 1–7.
- [2] A. Ben-Israel, B. Mond, *What is invexity?*, J. Austral. Math. Soc. Ser. B, 28 (1) (1986), 1–9.
- [3] M. A. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl. 80 (2) (1981), 545–550.
- [4] I. İşcan, *Ostrowski type inequalities for functions whose derivatives are preinvex*, Bull. Iranian Math. Soc., 40 (2) (2014), 373–386.
- [5] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput., 147 (1) (2004), 137–146.
- [6] U. S. Kirmaci, M. E. Özdemir, *On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput., 153 (2) (2004), 361–368.
- [7] B. Meftah, *Some new Ostrowski's inequalities for functions whose n th derivatives are r -convex*, Int. J. Anal., (2016), Article ID: 6749213, 7 pages.
- [8] B. Meftah, *Ostrowski inequalities for functions whose first derivatives are logarithmically preinvex*, Chin. J. Math. (N.Y.), (2016), Article ID: 5292603, 10 pages.
- [9] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Classical and new inequalities in analysis*, Mathematics and its Applications (East European Series), 61. Kluwer Academic Publishers Group, Dordrecht, 1993.
- [10] M. A. Noor, *Variational-like inequalities*, Optimization, 30 (4) (1994), 323–330.
- [11] M. A. Noor, *Invex equilibrium problems*, J. Math. Anal. Appl., 302 (2) (2005), 463–475.
- [12] J. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.
- [13] R. Pini, *Invexity and generalized convexity*, Optimization, 22 (4) (1991), 513–525.
- [14] M. Z. Sarıkaya, N. Alp, H. Bozkurt, *On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions*, Contemporary Analysis and Applied Mathematics, 1 (2) (2013), 237–252.
- [15] E. Set, M. E. Özdemir, M. Z. Sarıkaya, *New inequalities of Ostrowski's type for s -convex functions in the second sense with applications*, Facta Univ. Ser. Math. Inform., 27 (1) (2012), 67–82.

- [16] S. Wang, X. Liu, *New Hermite-Hadamard type inequalities for n -times differentiable and s -logarithmically preinvex functions*, Abstr. Appl. Anal., (2014), Article ID: 725987.
- [17] T. Weir, B. Mond, *Pre-invex functions in multiple objective optimization*, J. Math. Anal. Appl., 136 (1) (1988), 29–38.
- [18] X. -M. Yang, D. Li, *On properties of preinvex functions*, J. Math. Anal. Appl., 256 (1) (2001), 229–241.

BADREDDINE MEFTAH

LABORATOIRE DES TÉLÉCOMMUNICATIONS

FACULTÉ DES SCIENCES ET DE LA TECHNOLOGIE

UNIVERSITY OF 8 MAY 1945 GUELMA

P.O. Box 401, 24000 GUELMA

ALGERIA

E-mail address: `badrimeftah@yahoo.fr`