Different common fixed point theorems of integral type for pairs of subcompatible mappings

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Abstract. In this paper, a general common fixed point theorem for two pairs of subcompatible mappings satisfying integral type implicit relations is obtained in a metric space. Our result improves several results especially the result of Pathak et al. [6]. Also, another common fixed point theorem of Greguš type for four mappings satisfying a contractive condition of integral type in a metric space using the concept of subcompatibility is established which generalizes the result of Djoudi and Aliouche [1] and others. Again a third common fixed point theorem for two pairs of near-contractive subcompatible mappings is given which enlarges the result of Mbarki [5] and references therein.

1. Introduction

Let \((X, d)\) be a metric space and let \(f, g\) be two mappings from \(X\) into itself. \(f\) and \(g\) commute if \(fgx = gfx\) for all \(x \in X\).

This commutativity was weakened in 1982 by Sessa [7] with the notion of weakly commuting mappings. \(f\) and \(g\) above are weakly commuting if \(d(fgx, gfx) \leq d(gx, fx)\) for all \(x\) in \(X\).

Later on, Jungck [3] enlarged the class of commuting and weakly commuting mappings by compatible mappings which asserts that the above mappings \(f\) and \(g\) are compatible if \(\lim_{n \to \infty} d(fgx_n, gfx_n) = 0\) whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t\) for some \(t \in X\).

This concept was further improved by Jungck [4] with the notion of weakly compatible mappings. \(f\) and \(g\) are weakly compatible if \(ft = gt\) for some \(t \in X\) implies that \(fgt = gft\).

Recently in 2007, Pathak et al. [6] stated and proved a general common fixed point theorem of integral type for two pairs of weakly compatible mappings satisfying integral type implicit relations in a symmetric space.

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Our aim here is to improve and extend the result of [6] by using the new concept of mappings called subcompatibility which enlarges the concept of weakly compatible mappings.

We introduce the notion of subcompatible mappings as follows: Let \( f \) and \( g \) be two self-mappings of a metric space \((X, d)\). \( f \) and \( g \) are subcompatible if and only if there exists a sequence \( \{x_n\} \) in \( X \) such that
\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \text{ for some } t \in X \text{ and } \lim_{n \to \infty} d(fgx_n,gfx_n) = 0.
\]
It is clear to see that weakly compatible mappings are subcompatible, however the implication is not reversible.

**Example 1.1.** Let \( X = [0, \infty) \) with the usual metric \( d \). Define \( f, g : X \to X \) as follows
\[
fx = x^2 \text{ and } gx = \begin{cases} x + 12, & \text{if } x \in [0, 16] \cup (25, \infty), \\ x + 240, & \text{if } x \in (16, 25].
\end{cases}
\]
Let \( \{x_n\} \) be a sequence in \( X \) defined by \( x_n = 4 + \frac{1}{n} \) for \( n \in \mathbb{N}^* = \{1, 2, \ldots\} \). Then, we have
\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} x_n^2 = 16 = \lim_{n \to \infty} gx_n = \lim_{n \to \infty} (x_n + 12)
\]
and
\[
fgx_n = f(x_n + 12) = (x_n + 12)^2 \to 256 \text{ as } n \to \infty,
\]
\[
gfx_n = g(x_n^2) = x_n^2 + 240 \to 256 \text{ as } n \to \infty.
\]
Therefore, \( \lim_{n \to \infty} d(fgx_n, gfx_n) = 0 \). Hence, \( f \) and \( g \) are subcompatible mappings.

On the other hand, we have \( fx = gx \) if and only if \( x = 4 \) but
\[
fg(4) = f(16) = 256 \neq 28 = gf(4) = g(16).
\]
Thus, \( f \) and \( g \) are not weakly compatible.

For our first main result we need the following implicit relations.

2. **Implicit relations**

Let \( \mathbb{R}_+ \) be the set of all nonnegative real numbers, \( \Psi \) be the family of all \( \psi : \mathbb{R}_+ \to \mathbb{R} \) Lebesgue-integrable and summable mappings and \( \Phi \) be the set of all real continuous functions \( \varphi : \mathbb{R}_+^6 \to \mathbb{R} \) satisfying the following conditions:
\[
(\varphi_1) \text{ for all } u, v \geq 0, \text{ if }
(\varphi_\theta) \int_0^{\varphi(u,v,u+0,u+v)} \psi(t) \, dt \leq 0 \text{ or }
(\varphi_b) \int_0^{\varphi(u,v,u,u+v,0)} \psi(t) \, dt \leq 0,
\]
we have \( u \leq v \),
(\varphi_2) \int_0^{\varphi(u,u,0,0,u,u)} \psi(t) \, dt > 0, \text{ for } u > 0.

**Example 2.1.** Let \(\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - k \max \{t_2,t_3,t_4,\frac{t_5+t_6}{2}\}\), where \(k \in (0,1)\) and \(\psi(t) = t\). Then \(\varphi\) is continuous and \(\psi\) is a Lebesgue-integrable mapping which is summable. We have

\((\varphi_1)\) Let \(u > 0\) and \(v \geq 0\). If \(u > v\) then

\[
\varphi(u,v,v,u,0,u+v) = \varphi(u,v,u,v,u+v,0) = u - k \max \left\{u,v,\frac{u+v}{2}\right\} = u(1-k),
\]

then

\[
\int_0^{u(1-k)} t \, dt = \frac{1}{2} u^2 (1-k)^2 \leq 0
\]

impossible, hence \(u \leq v\). If \(u = 0\), then \(u \leq v\).

\((\varphi_2)\) \(\varphi(u,u,0,0,u,u) = u(1-k)\), so

\[
\int_0^{u(1-k)} t \, dt = \frac{1}{2} u^2 (1-k)^2 > 0,
\]

for \(u > 0\).

**Example 2.2.** \(\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = (1 + \alpha t_2) t_1 - \alpha \max \{t_3 t_4,t_5 t_6\} - \beta \max \{t_2,t_3,t_4,\frac{1}{2} (t_5 + t_6)\}\), where \(\alpha \geq 0\) and \(0 < \beta < 1\) and \(\psi(t) = 1\).

\((\varphi_1)\) Let \(u > 0\) and \(v \geq 0\). Suppose that \(u > v\), then

\[
\varphi(u,v,v,u,0,u+v) = \varphi(u,v,u,v,u+v,0) = (1 + \alpha v) u - \alpha \max \{uv,0\} - \beta \max \left\{v,u,\frac{u+v}{2}\right\}
\]

\[
= u(1-\beta),
\]

then

\[
\int_0^{u(1-\beta)} d t = u(1-\beta) \leq 0,
\]

which is impossible. Thus, \(u \leq v\). If \(u = 0\), then \(u \leq v\).

\((\varphi_2)\) \(\varphi(u,u,0,0,u,u) = u(1-\beta)\), then

\[
\int_0^{u(1-\beta)} d t = u(1-\beta) > 0, \quad \text{for all } u > 0.
\]

Now, we state and prove our main results. We begin by the first one.
3. Main results

**Theorem 3.1.** Let $f$, $g$, $h$ and $k$ be four mappings of a metric space $(X, d)$ into itself such that

\[
\int_0^\infty \varphi(d(fx, gy), d(hx, ky), d(fx, hx), d(gy, ky), d(ky, fx), d(hx, gy)) \psi(t) \, dt \leq 0,
\]

for all $x, y$ in $X$, where $\varphi \in \Phi$ and $\psi \in \Psi$. Suppose that $(f, h)$ and $(g, k)$ are subcompatible and $h$ and $k$ are continuous, then, $f$, $g$, $h$ and $k$ have a unique common fixed point.

**Proof.** Since the pairs $(f, h)$ and $(g, k)$ are subcompatible, then, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} hx_n = t$ for some $t \in X$ and $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} fy_n = 0$; $\lim_{n \to \infty} gy_n = \lim_{n \to \infty} ky_n = z$ for some $z \in X$ and $\lim_{n \to \infty} gky_n = 0$.

First we prove that $z = t$. Indeed, by inequality (1) we get

\[
\int_0^\infty \varphi(d(fx_n, gy_n), d(hx_n, ky_n), d(fx_n, hx_n), d(gy_n, ky_n), d(ky_n, fx_n), d(hx_n, gy_n)) \psi(t) \, dt \leq 0.
\]

Since $\varphi$ is continuous, we obtain at infinity

\[
\int_0^\infty \varphi(d(tz, 0, 0, 0, 0, 0)) \psi(t) \, dt \leq 0,
\]

which contradicts $(\varphi_2)$ if $d(t, z) > 0$. Then, $z = t$.

Since $h$ is continuous, then $h^2x_n \to ht$, $hfx_n \to ht$. Also we have

\[
d(fhx_n, ht) \leq d(fhx_n, hfx_n) + d(hfx_n, ht).
\]

Since $f$ and $h$ are subcompatible, taking the limit as $n \to \infty$ in the above inequality we have $\lim_{n \to \infty} fhx_n = ht$. The use of condition (1) gives

\[
\int_0^\infty \varphi(d(fh_n, gy_n), d(h^2x_n, ky_n), d(fh_n, h^2x_n), d(gy_n, ky_n), d(ky_n, fh_n), d(h^2x_n, gy_n)) \psi(t) \, dt \leq 0.
\]

At infinity we obtain

\[
\int_0^\infty \varphi(d(tz, 0, 0, 0, 0, 0)) \psi(t) \, dt \leq 0,
\]

which contradicts $(\varphi_2)$. Hence $ht = t$.

Again using (1) we get

\[
\int_0^\infty \varphi(d(ft, gy_n), d(ht, ky_n), d(ft, ht), d(gy_n, ky_n), d(ky_n, ft), d(ht, gy_n)) \psi(t) \, dt \leq 0.
\]

Taking the limit as $n \to \infty$, we get

\[
\int_0^\infty \varphi(d(ft, t), 0, 0, 0, 0, 0) \psi(t) \, dt \leq 0,
\]
which implies \( d(ft, t) = 0 \) by using condition \((\varphi_b)\). Thus, \( ft = t \).

Now, since \( k \) is continuous we have \( \lim_{n \to \infty} k^2 y_n = \lim_{n \to \infty} k g y_n = kt \). Also we have
\[
d(gky_n, kt) \leq d(gky_n, kgy_n) + d(kgy_n, kt).
\]
Since the pair \((g, k)\) is subcompatible we obtain at infinity \( \lim_{n \to \infty} gky_n = kt \).

Using condition \((1)\) we have
\[
\int_0 \varphi(d(ft, gky_n), d(ht, k^2 y_n), d(ft, ht), d(gky_n, k^2 y_n), d(k^2 y_n, ft), d(ht, gky_n)) \psi(t) \, dt \leq 0.
\]
When \( n \) tends to infinity, we get
\[
\int_0 \varphi(d(t, kt), d(t, kt), 0, 0, d(kt, t), d(kt, kt)) \psi(t) \, dt \leq 0,
\]
which contradicts \((\varphi_2)\) when \( d(t, kt) > 0 \). Hence, \( kt = t \).

If \( gt \neq t \), using inequality \((1)\) we have
\[
\int_0 \varphi(d(ft, gt), d(ht, kt), d(ft, ht), d(gt, kt), d(kt, ft), d(ht, gt)) \psi(t) \, dt \leq 0,
\]
i.e.,
\[
\int_0 \varphi(d(t, gt), 0, 0, d(gt, t), 0, d(t, gt)) \psi(t) \, dt \leq 0,
\]
which implies \( d(t, gt) = 0 \) by using condition \((\varphi_a)\). Thus, \( gt = t \).

For the uniqueness of common fixed point \( t \), let \( z \neq t \) be another common fixed point of \( f, g, h \) and \( k \). Then using \((1)\) we obtain
\[
\int_0 \varphi(d(ft, gz), d(ht, kz), d(ft, ht), d(gz, kz), d(kz, ft), d(ht, gz)) \psi(t) \, dt \leq 0,
\]
that is,
\[
\int_0 \varphi(d(t, z), d(t, z), 0, 0, d(z, t), d(t, z)) \psi(t) \, dt \leq 0,
\]
which is a contradiction of \((\varphi_2)\). Therefore \( z = t \).

\[\square\]

**Corollary 3.1.** Let \((\mathcal{X}, d)\) be a metric space and let \( f \) and \( h \) be two mappings from \( \mathcal{X} \) into itself satisfying the condition
\[
\int_0 \varphi(d(fx, fy), d(hx, hy), d(fx, hx), d(fy, hy), d(hy, fx), d(hx, fy)) \psi(t) \, dt \leq 0,
\]
for all \( x, y \) in \( \mathcal{X} \), where \( \varphi \in \Phi \) and \( \psi \in \Psi \). If \( h \) is continuous and the pair \((f, h)\) is subcompatible, then, \( f \) and \( h \) have a unique common fixed point.

**Corollary 3.2.** Let \((\mathcal{X}, d)\) be a metric space and let \( f, g \) and \( h \) be three self-mappings of \( \mathcal{X} \) such that

(i) \( h \) is continuous,

(ii) the pairs \((f, h)\) and \((g, h)\) are subcompatible and
(iii) the inequality
\[ \int_0^\varphi(d(fx,gy),d(hx,hy),d(fx,hx),d(gy,hy),d(hy,fx),d(hx,gy)) \psi(t) \, dt \leq 0, \]
holds for all \( x, y \in X \), where \( \varphi \in \Phi \) and \( \psi \in \Psi \), then, \( f, g \) and \( h \) have a unique common fixed point.

Now, we give a generalization of Theorem 3.1.

**Theorem 3.2.** Let \( h, k \) and \( \{f_n\}_{n \in \mathbb{N}^*} \) be mappings from a metric space \((X, d)\) into itself such that

(i) the pairs \((f_n, h)\) and \((f_{n+1}, k)\) are subcompatible,

(ii) the inequality
\[ \int_0^{\varphi(d(f_nx,f_{n+1}y),d(hx,ky),d(f_nx,hx),d(f_{n+1}y,ky),d(ky,f_nx),d(hx,f_{n+1}y)))} \psi(t) \, dt \leq 0 \]
holds for all \( x, y \in X \), each \( n \in \mathbb{N}^* \), \( \varphi \in \Phi \) and \( \psi \in \Psi \). If \( h \) and \( k \) are continuous, then, \( h, k \) and \( \{f_n\}_{n \in \mathbb{N}^*} \) have a unique common fixed point.

Now, let \( F \) be the family of mappings \( F : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that each \( F \) is upper semi-continuous and \( F(t) < t \) for all \( t > 0 \) and let \( \Omega \) be the family of \( \omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that every \( \omega \) is a Lebesgue-integrable mapping which is summable and \( \int_0^\epsilon \omega(t) \, dt > 0 \) for each \( \epsilon > 0 \).

In their paper [1], Djoudi and Aliouche proved a common fixed point theorem of Greguš type for four mappings satisfying a contractive condition of integral type in a metric space using the concept of weak compatibility.

Our objective here is to improve, extend and generalize the result of [1] by using the notion of subcompatibility.

**Theorem 3.3.** Let \( f, g, h \) and \( k \) be mappings from a metric space \((X, d)\) into itself satisfying inequality
\[
\left( \int_0^{d(fx,gy)} \omega(t) \, dt \right)^p \leq F \left[ a \left( \int_0^{d(hx,ky)} \omega(t) \, dt \right)^p + (1 - a) \max \left\{ \int_0^{d(fx,hx)} \omega(t) \, dt, \int_0^{d(gy,ky)} \omega(t) \, dt \right\}^{\frac{1}{2}} \left( \int_0^{d(fx,gy)} \omega(t) \, dt \right)^{\frac{1}{2}} \right],
\]
for all \( x, y \in X \), where \( 0 < a < 1 \), \( p \) is an integer such that \( p \geq 1 \), \( F \in \mathcal{F} \) and \( \omega \in \Omega \). If \( h \) and \( k \) are continuous and the pairs \((f, h)\) and \((g, k)\) are subcompatible, then, \( f, g, h \) and \( k \) have a unique common fixed point.
Proof. Since the pair \((f, h)\) as well as \((g, k)\) is subcompatible, then, there are two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(\mathcal{X}\) such that \(\lim_{n \to \infty} hx_n = \lim_{n \to \infty} fx_n = t\) for some \(t \in \mathcal{X}\) and \(\lim_{n \to \infty} d(fh x_n, h f x_n) = 0\); \(\lim_{n \to \infty} gy_n = \lim_{n \to \infty} k y_n = z\) for some \(z \in \mathcal{X}\) and \(\lim_{n \to \infty} d(g k y_n, k g y_n) = 0\).

First, we prove that \(z = t\). If \(t \neq z\), using inequality (2) we get

\[
\left( \int_0^{d(f x_n, g y_n)} \omega(t) \, dt \right)^p \leq F \left[ a \left( \int_0^{d(h x_n, k y_n)} \omega(t) \, dt \right)^p + (1 - a) \max \left\{ \int_0^{d(f x_n, h x_n)} \omega(t) \, dt, \int_0^{d(g y_n, k y_n)} \omega(t) \, dt \right\} \right].
\]

Letting \(n \to \infty\), we obtain

\[
\left( \int_0^{d(t, z)} \omega(t) \, dt \right)^p \leq F \left[ a \left( \int_0^{d(t, z)} \omega(t) \, dt \right)^p + (1 - a) \left( \int_0^{d(t, z)} \omega(t) \, dt \right)^p \right]
= F \left[ \left( \int_0^{d(t, z)} \omega(t) \, dt \right)^p \right] < \left( \int_0^{d(t, z)} \omega(t) \, dt \right)^p,
\]

which is a contradiction, then \(\int_0^{d(t, z)} \omega(t) \, dt = 0\), hence \(z = t\).

Since \(h\) is continuous, then we have \(h^2 x_n \to ht\), \(h f x_n \to ht\). Also, we have

\(d(fh x_n, ht) \leq d(fh x_n, h f x_n) + d(h f x_n, ht)\).

As \(f\) and \(h\) are subcompatible, letting \(n\) tends to infinity in the above inequality, we obtain \(\lim_{n \to \infty} fh x_n = ht\). The use of condition (2) gives

\[
\left( \int_0^{d(fh x_n, g y_n)} \omega(t) \, dt \right)^p \leq F \left[ a \left( \int_0^{d(h^2 x_n, k y_n)} \omega(t) \, dt \right)^p + (1 - a) \max \left\{ \int_0^{d(fh x_n, h^2 x_n)} \omega(t) \, dt, \right\} \right].
\]
\[
\int_{0}^{d(g_{n},k_{n})} \omega(t) \, dt, \left( \int_{0}^{d(f_{n},h_{1}^{2}x_{n})} \omega(t) \, dt \right)^{\frac{1}{2}}, \left( \int_{0}^{d(f_{n},k_{n})} \omega(t) \, dt \right)^{\frac{1}{2}}, \\
\left( \int_{0}^{d(h_{2}^{2}x_{n},g_{n})} \omega(t) \, dt \right)^{\frac{1}{2}}, \left( \int_{0}^{d(f_{n},k_{n})} \omega(t) \, dt \right)^{\frac{1}{2}} \right].
\]

We obtain at infinity

\[
\left( \int_{0}^{d(h_{n},t)} \omega(t) \, dt \right)^{p}
\leq F \left[ a \left( \int_{0}^{d(h_{n},t)} \omega(t) \, dt \right)^{p} + (1 - a) \left( \int_{0}^{d(h_{n},t)} \omega(t) \, dt \right)^{p} \right]
= F \left[ \left( \int_{0}^{d(h_{n},t)} \omega(t) \, dt \right)^{p} \right] < \left( \int_{0}^{d(h_{n},t)} \omega(t) \, dt \right)^{p},
\]

which is a contradiction, therefore \(h_{n} = t\).

Again by inequality (2) we have

\[
\left( \int_{0}^{d(f_{n},g_{n})} \omega(t) \, dt \right)^{p}
\leq F \left[ a \left( \int_{0}^{d(h_{n},t)} \omega(t) \, dt \right)^{p} + (1 - a) \max \left\{ \int_{0}^{d(f_{n},ht)} \omega(t) \, dt, \int_{0}^{d(g_{n},k_{n})} \omega(t) \, dt, \left( \int_{0}^{d(f_{n},ht)} \omega(t) \, dt \right)^{\frac{1}{2}}, \left( \int_{0}^{d(f_{n},k_{n})} \omega(t) \, dt \right)^{\frac{1}{2}} \right\} \right].
\]

At infinity we obtain

\[
\left( \int_{0}^{d(f_{n},t)} \omega(t) \, dt \right)^{p} \leq F \left[ (1 - a) \left( \int_{0}^{d(f_{n},t)} \omega(t) \, dt \right)^{p} \right]
< (1 - a) \left( \int_{0}^{d(f_{n},t)} \omega(t) \, dt \right)^{p}
< \left( \int_{0}^{d(f_{n},t)} \omega(t) \, dt \right)^{p},
\]

which is a contradiction. Hence \(f_{n}t = t\).
Now, since $k$ is continuous, then, we have $k^2y_n \to kt$ and $kgy_n \to kt$ and
\[
 d(gky_n, kt) \leq d(gky_n, kgy_n) + d(kgy_n, kt).
\]
Since the pair $(g, k)$ is subcompatible, we get at infinity $\lim_{n \to \infty} gky_n = kt$.

Using (2) we have
\[
\left( \int_0^{d(ft, gky_n)} \omega(t) \, dt \right)^p \leq F \left[ a \left( \int_0^{d(ht, k^2y_n)} \omega(t) \, dt \right)^p + (1 - a) \max \left\{ \int_0^{d(ft, ht)} \omega(t) \, dt, \int_0^{d(gky_n, k^2y_n)} \omega(t) \, dt, \left( \int_0^{d(ft, k^2y_n)} \omega(t) \, dt \right)^{\frac{1}{2}}, \left( \int_0^{d(ht, gky_n)} \omega(t) \, dt \right)^{\frac{1}{2}} \right\} \right].
\]

We get at infinity
\[
\left( \int_0^{d(t, kt)} \omega(t) \, dt \right)^p \leq F \left[ a \left( \int_0^{d(t, kt)} \omega(t) \, dt \right)^p + (1 - a) \left( \int_0^{d(t, kt)} \omega(t) \, dt \right)^p \right] = F \left[ \left( \int_0^{d(t, kt)} \omega(t) \, dt \right)^p \right] < \left( \int_0^{d(t, kt)} \omega(t) \, dt \right)^p.
\]

This contradiction implies that $kt = t$.

Suppose that $gt \neq t$, the use of inequality (2) gives
\[
\left( \int_0^{d(ft, gt)} \omega(t) \, dt \right)^p \leq F \left[ a \left( \int_0^{d(ht, kt)} \omega(t) \, dt \right)^p + (1 - a) \max \left\{ \int_0^{d(ft, ht)} \omega(t) \, dt, \int_0^{d(gt, kt)} \omega(t) \, dt, \left( \int_0^{d(ft, ht)} \omega(t) \, dt \right)^{\frac{1}{2}}, \left( \int_0^{d(gt, kt)} \omega(t) \, dt \right)^{\frac{1}{2}} \right\} \right].
\]
i.e.,
\[
\left( \int_0^{d(t,gt)} \omega(t) \, dt \right)^p \leq F \left[ (1 - a) \left( \int_0^{d(t,gt)} \omega(t) \, dt \right)^p \right]
\]
\[
< (1 - a) \left( \int_0^{d(t,gt)} \omega(t) \, dt \right)^p
\]
\[
< \left( \int_0^{d(t,gt)} \omega(t) \, dt \right)^p,
\]
which is a contradiction. Hence \( gt = t \). Therefore \( t = z \) is a common fixed point of both \( f, g, h \) and \( k \).

Suppose that \( f, g, h \) and \( k \) have another common fixed point \( z \neq t \). Then, by inequality (2) we get
\[
\left( \int_0^{d(ft,gz)} \omega(t) \, dt \right)^p
\]
\[
\leq F \left[ a \left( \int_0^{d(ht,kz)} \omega(t) \, dt \right)^p + (1 - a) \max \left\{ \int_0^{d(ft,ht)} \omega(t) \, dt, \right. \right.
\]
\[
\left. \int_0^{d(gz,kz)} \omega(t) \, dt \right. \left. \right\} \left. \frac{1}{2} \left( \int_0^{d(ft,kz)} \omega(t) \, dt \right)^\frac{1}{2} \right],
\]
that is
\[
\left( \int_0^{d(t,z)} \omega(t) \, dt \right)^p \leq F \left[ \left( \int_0^{d(t,z)} \omega(t) \, dt \right)^p \right]
\]
\[
< \left( \int_0^{d(t,z)} \omega(t) \, dt \right)^p.
\]
This contradiction implies that \( z = t \). \( \square \)

If \( f = g \) and \( h = k \) in Theorem 3.3, we get the next result:

**Corollary 3.3.** Let \( f \) and \( h \) be two self-mappings of a metric space \((X,d)\) such that
\[
\left( \int_0^{d(fx,fx)} \omega(t) \, dt \right)^p
\
\[
\leq F \left[ a \left( \int_0^{d(hx,hy)} \omega(t) \, dt \right)^p + (1 - a) \max \left\{ \int_0^{d(fx,fx)} \omega(t) \, dt, \right. \right.
\]
\[
\left. \int_0^{d(hx,hx)} \omega(t) \, dt \right. \left. \right\} \right].
\]
\[
\int_0^{d(fy, hy)} \omega(t) \, dt, \left( \int_0^{d(fx, hx)} \omega(t) \, dt \right)^{\frac{1}{2}} \left( \int_0^{d(fx, hy)} \omega(t) \, dt \right)^{\frac{1}{2}}
\]

for all \( x, y \) in \( X \), where \( 0 < a < 1 \), \( p \) is an integer such that \( p \geq 1 \), \( F \in F \) and \( \omega \in \Omega \). If \( h \) is continuous and the pair \((f, h)\) is subcompatible, then, \( f \) and \( h \) have a unique common fixed point.

If we let in Theorem 3.3 \( h = k \), then we get the following corollary:

**Corollary 3.4.** Let \( f, g \) and \( h \) be three self-mappings of a metric space \((X, d)\) such that

\[
\left( \int_0^{d(fx, gy)} \omega(t) \, dt \right)^{p} \leq F \left[ a \left( \int_0^{d(hx, hy)} \omega(t) \, dt \right)^{p} + (1 - a) \max \left\{ \int_0^{d(fx, hx)} \omega(t) \, dt, \right. \right.
\]

\[
\int_0^{d(gy, hy)} \omega(t) \, dt, \left( \int_0^{d(fx, hx)} \omega(t) \, dt \right)^{\frac{1}{2}} \left( \int_0^{d(fx, hy)} \omega(t) \, dt \right)^{\frac{1}{2}}
\]

for all \( x, y \) in \( X \), where \( 0 < a < 1 \), \( p \) is an integer such that \( p \geq 1 \), \( F \in F \) and \( \omega \in \Omega \). If \( h \) is continuous and the pairs \((f, h)\) and \((g, h)\) are subcompatible, then, \( f \), \( g \) and \( h \) have a unique common fixed point.

The next result is a generalization of Theorem 3.3.

**Theorem 3.4.** Let \( h, k \) and \( \{f_n\}_{n \in \mathbb{N}^*} \) be self-mappings of a metric space \((X, d)\) satisfying the inequality

\[
\left( \int_0^{d(f_nx, f_{n+1}y)} \omega(t) \, dt \right)^{p} \leq F \left[ a \left( \int_0^{d(hx, ky)} \omega(t) \, dt \right)^{p} + (1 - a) \max \left\{ \int_0^{d(f_nx, hx)} \omega(t) \, dt, \right. \right.
\]

\[
\int_0^{d(f_{n+1}y, ky)} \omega(t) \, dt, \left( \int_0^{d(f_nx, hx)} \omega(t) \, dt \right)^{\frac{1}{2}} \left( \int_0^{d(f_nx, ky)} \omega(t) \, dt \right)^{\frac{1}{2}}
\]
\[
\left( \int_0^{d(hx,f_{n+1}y)} \omega(t) \, dt \right)^{\frac{1}{2}} \left( \int_0^{d(f_n,ky)} \omega(t) \, dt \right)^{\frac{1}{2}} \right\}^p,
\]
for all \( x, y \in \mathcal{X} \), where \( 0 < a < 1 \), \( p \) is an integer such that \( p \geq 1 \), \( F \in \mathcal{F} \) and \( \omega \in \mathcal{Q} \). If \( h \) and \( k \) are continuous and the pairs \((f_n, h)\) and \((f_{n+1}, k)\) are subcompatible, then \( h, k \) and \( \{f_n\}_{n \in \mathbb{N}^*} \) have a unique common fixed point.

We end our work by establishing another result which improves, extends and generalizes especially the result of [5].

**Theorem 3.5.** Let \((\mathcal{X}, d)\) be a metric space, \( f, g, h \) and \( k \) be mappings from \( \mathcal{X} \) into itself and \( F \) be an upper semi-continuous function of \([0, \infty)\) into itself such that \( F(t) = 0 \) if and only if \( t = 0 \) and satisfying inequality

\[
(3) \quad \int_0^{F(d(fx,gy))} \omega(t) \, dt 
\leq a(d(hx,ky)) \int_0^{F(d(hx,ky))} \omega(t) \, dt 
+ b(d(hx,ky)) \int_0^{F(d(hx,fx)+F(d(ky,gy)))} \omega(t) \, dt 
+ c(d(hx,ky)) \int_0^{\min\{F(d(hx,gy)),F(d(ky,fx))\}} \omega(t) \, dt,
\]
for all \( x, y \in \mathcal{X} \), where \( \omega \in \mathcal{Q} \) and \( a, b, c : [0, \infty) \to [0, 1) \) are upper semi-continuous and satisfying the condition

\[ a(t) + c(t) < 1, \quad t > 0. \]

If the pairs \((f, h)\) and \((g, k)\) are subcompatible and \( h \) and \( k \) are continuous, then, \( f, g, h \) and \( k \) have a unique common fixed point.

**Proof.** Since the pairs \((f, h)\) and \((g, k)\) are subcompatible, then, there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( \mathcal{X} \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} hx_n = t \) for some \( t \in \mathcal{X} \) and \( \lim_{n \to \infty} d(fhx_n, hf_x) = 0; \lim_{n \to \infty} gyn = \lim_{n \to \infty} kyn = z \) for some \( z \in \mathcal{X} \) and \( \lim_{n \to \infty} d(gkyn, kgyn) = 0 \).

First, we prove that \( z = t \). Suppose that \( F(d(t, z)) > 0 \), using inequality (3) we get

\[
\int_0^{F(d(fx,gy))} \omega(t) \, dt \leq a(d(hx,ky)) \int_0^{F(d(hx,ky))} \omega(t) \, dt 
+ b(d(hx,ky)) \int_0^{F(d(hx,fx)+F(d(ky,gy)))} \omega(t) \, dt 
+ c(d(hx,ky)) \int_0^{\min\{F(d(hx,gy)),F(d(ky,fx))\}} \omega(t) \, dt.
\]
Taking the limit as \( n \to \infty \), we obtain
\[
\int_{0}^{\int_{0}^{F(d(t,z))} \omega(t) \, dt} \omega(t) \, dt \leq [a(d(t,z)) + c(d(t,z))] \int_{0}^{F(d(t,z))} \omega(t) \, dt
\]
\[
< \int_{0}^{F(d(t,z))} \omega(t) \, dt,
\]
which is a contradiction. Hence \( F(d(t,z)) = 0 \) which implies that \( d(t,z) = 0 \), thus \( t = z \).

Since \( h \) is continuous, then, we have \( h^2x_n \to ht, \ hfx_n \to ht \). Also, we have
\[
d(fhx_n, ht) \leq d(fhx_n, hfx_n) + d(hfx_n, ht).
\]
As \( f \) and \( h \) are subcompatible, letting \( n \) tends to infinity in the above inequality, we obtain \( \lim_{n \to \infty} fhx_n = ht \). If \( F(d(ht,t)) > 0 \), the use of condition (3) gives
\[
\int_{0}^{\int_{0}^{F(d(ht,t))} \omega(t) \, dt} \omega(t) \, dt \leq [a(d(ht,t)) + c(d(ht,t))] \int_{0}^{F(d(ht,t))} \omega(t) \, dt
\]
\[
< \int_{0}^{F(d(ht,t))} \omega(t) \, dt.
\]
This contradiction implies that \( F(d(ht,t)) = 0 \) and hence \( ht = t \).

Suppose that \( F(d(ft,t)) > 0 \), using condition (3) we get
\[
\int_{0}^{\int_{0}^{F(d(ft,t))} \omega(t) \, dt} \omega(t) \, dt \leq a(d(ht, ky_n)) \int_{0}^{F(d(ht,ky_n))} \omega(t) \, dt
\]
\[
+ b(d(ht, ky_n)) \int_{0}^{F(d(ht,ft)) + F(d(ky_n,gy_n))} \omega(t) \, dt
\]
\[
+ c(d(ht, ky_n)) \int_{0}^{\min\{F(d(ht,gy_n)),F(d(ky_n,ft))\}} \omega(t) \, dt.
\]
We obtain at infinity
\[
\int_{0}^{\int_{0}^{F(d(ft,t))} \omega(t) \, dt} \omega(t) \, dt \leq b(0) \int_{0}^{F(d(t,ft))} \omega(t) \, dt < \int_{0}^{F(d(ft,t))} \omega(t) \, dt,
\]
which is a contradiction, hence \( F(d(ft,t)) = 0 \) which implies that \( ft = t \).
Now, since $k$ is continuous, then, we have $k^2 y_n \to kt$, $kgy_n \to kt$ and $d(gky_n, kt) \leq d(gky_n, kgy_n) + d(kgy_n, kt)$.

Since the pair $(g, k)$ is subcompatible, we get at infinity $\lim_{n \to \infty} gky_n = kt$. We claim that $kt = t$, if not, then by (3) we have
\[
\int_0^{F(d(t, gky_n))} \omega(t) \, dt \leq a(d(ht, k^2 y_n)) \int_0^{F(d(ht, k^2 y_n))} \omega(t) \, dt
\]
\[
+ b(d(ht, k^2 y_n)) \int_0^{F(d(ht, ft) + F(d(k^2 y_n, gky_n))} \omega(t) \, dt
\]
\[
+ c(d(ht, k^2 y_n)) \int_0^{\min \{F(d(ht, gky_n), F(d(k^2 y_n, ft))\}} \omega(t) \, dt.
\]

Taking the limit when $n \to \infty$ we have
\[
\int_0^{F(d(t, kt))} \omega(t) \, dt \leq [a(d(t, kt)) + c(d(t, kt))] \int_0^{F(d(t, kt))} \omega(t) \, dt
\]
\[
< \int_0^{F(d(t, kt))} \omega(t) \, dt,
\]
\[
\Phi(d(t, kt)) \leq [a(d(t, kt)) + c(d(t, kt))] \Phi(d(t, kt))
\]
\[
< \Phi(d(t, kt)),
\]
which is a contradiction, thus $kt = t$.

Suppose that $F(d(t, gt)) > 0$, then the use of inequality (3) yields
\[
\int_0^{F(d(t, gt))} \omega(t) \, dt = \int_0^{F(d(t, gt))} \omega(t) \, dt
\]
\[
\leq a(d(ht, kt)) \int_0^{F(d(ht, kt))} \omega(t) \, dt
\]
\[
+ b(d(ht, kt)) \int_0^{F(d(ht, ft) + F(d(k^2 y_n, gky_n))} \omega(t) \, dt
\]
\[
+ c(d(ht, kt)) \int_0^{\min \{F(d(ht, gky_n), F(d(k^2 y_n, ft))\}} \omega(t) \, dt
\]
\[
= b(0) \int_0^{F(d(t, gt))} \omega(t) \, dt < \int_0^{F(d(t, gt))} \omega(t) \, dt,
\]
which is a contradiction, thus $F(d(t, gt)) = 0$ which implies that $d(t, gt) = 0$ i.e. $gt = t$.

Now, assume that there exists another common fixed point $z$ of $f, g, h$ and $k$ such that $z \neq t$. By inequality (3) we obtain
\[
\int_0^{F(d(t, z))} \omega(t) \, dt = \int_0^{F(d(ft, gz))} \omega(t) \, dt
\]

\[
\begin{align*}
\leq & \ a(d(ht, kz)) \int_0^F(d(ht, kz)) \omega(t) \, dt \\
& + \ b(d(ht, kz)) \int_0^{F(d(ht, ft)) + F(d(kz, gz))} \omega(t) \, dt \\
& + \ c(d(ht, kz)) \int_0^{\min\{F(d(ht, gz)), F(d(kz, ft))\}} \omega(t) \, dt \\
& = \ [a(d(t, z)) + c(d(t, z))] \int_0^{F(d(t, z))} \omega(t) \, dt \\
& < \int_0^{F(d(t, z))} \omega(t) \, dt.
\end{align*}
\]

This contradiction implies that \( F(d(t, z)) = 0 \iff d(t, z) = 0 \), hence \( z = t \). \( \square \)

**Remark 3.1.** Theorem 3.5 remains valid if we replace inequality (3) by the following one

\[
\int_0^{F(d(fx, gy))} \omega(t) \, dt \leq a(d(hx, ky)) \int_0^{F(d(hx, ky))} \omega(t) \, dt \\
+ \ b(d(hx, ky)) \int_0^{\frac{F(d(hx, fx)) + F(d(ky, gy))}{2}} \omega(t) \, dt \\
+ \ c(d(hx, ky)) \int_0^{\frac{F(d(hx, gy)) + F(d(ky, fx))}{2}} \omega(t) \, dt.
\]

**Corollary 3.5.** Let \( f \) and \( h \) be self-mappings of a metric space \((X, d)\). Assume that \( h \) is continuous, the pair \((f, h)\) is subcompatible and satisfies the inequality

\[
\int_0^{F(d(fx, fy))} \omega(t) \, dt \leq a(d(hx, hy)) \int_0^{F(d(hx, hy))} \omega(t) \, dt \\
+ \ b(d(hx, hy)) \int_0^{F(d(hx, fx)) + F(d(hy, fy))} \omega(t) \, dt \\
+ \ c(d(hx, hy)) \int_0^{\min\{F(d(hx, fy)), F(d(hy, fx))\}} \omega(t) \, dt,
\]

for all \( x, y \) in \( X \), where \( F, \omega, a, b \) and \( c \) are as in Theorem 3.5. Then, \( f \) and \( h \) have a unique common fixed point.

**Corollary 3.6.** Let \( f, g, h : X \to X \) be mappings satisfying the following inequality

\[
\int_0^{F(d(fx, gy))} \omega(t) \, dt \leq a(d(hx, hy)) \int_0^{F(d(hx, hy))} \omega(t) \, dt
\]
Different common fixed point theorems of integral type for ...

\[ + b(d(hx, hy)) \int_0^\infty \omega(t) \, dt \]
\[ + c(d(hx, hy)) \int_0^{\min\{F(d(hx, hy)), F(d(hy, fx))\}} \omega(t) \, dt, \]

for all \( x, y \) in \( X \), where \( F, \omega, a, b \) and \( c \) are as in Theorem 3.5. If \( h \) is continuous and the pairs \((f, h)\) and \((g, h)\) are subcompatible, then, \( f, g \) and \( h \) have a unique common fixed point.

Now, we give a generalization of Theorem 3.5.

**Theorem 3.6.** Let \((X, d)\) be a metric space, \( h, k, \{f_n\}_{n \in \mathbb{N}^*}\) be mappings from \( X \) into itself and \( F \) be an upper semi-continuous function of \([0, \infty)\) into itself such that \( F(0) = 0 \) if and only if \( t = 0 \) and satisfying the inequality

\[
\int_0^F(d(f_nx, f_{n+1}y)) \omega(t) \, dt \leq \int_0^F(d(hx, ky)) \omega(t) \, dt \\
+ b(d(hx, ky)) \int_0^{F(d(hx, f_{n+1}y) + F(d(ky, f_nx)))} \omega(t) \, dt \\
+ c(d(hx, ky)) \int_0^{\min\{F(d(hx, f_{n+1}y)), F(d(ky, f_nx))\}} \omega(t) \, dt,
\]

for all \( x, y \) in \( X \), where \( \omega \in \Omega, a, b, c : [0, \infty) \to [0, 1) \) are upper semi-continuous and satisfying the condition

\[ a(t) + c(t) < 1, \quad t > 0. \]

If the pairs \((f_n, h)\) and \((f_{n+1}, k)\) are subcompatible and \( h \) and \( k \) are continuous, then \( h, k \) and \( \{f_n\}_{n \in \mathbb{N}^*} \) have a unique common fixed point.

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