Different common fixed point theorems of integral type for pairs of subcompatible mappings

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ABSTRACT. In this paper, a general common fixed point theorem for two pairs of subcompatible mappings satisfying integral type implicit relations is obtained in a metric space. Our result improves several results especially the result of Pathak et al. [6]. Also, another common fixed point theorem of Greguš type for four mappings satisfying a contractive condition of integral type in a metric space using the concept of subcompatibility is established which generalizes the result of Djoudi and Aliouche [1] and others. Again a third common fixed point theorem for two pairs of near-contractive subcompatible mappings is given which enlarges the result of Mbarki [5] and references therein.

1. INTRODUCTION

Let $(\mathcal{X}, d)$ be a metric space and let f, g be two mappings from $\mathcal{X}$ into itself. f and g commute if $fgx = gfx$ for all $x \in \mathcal{X}$.

This commutativity was weakened in 1982 by Sessa [7] with the notion of weakly commuting mappings. f and g above are weakly commuting if $d(fgx, gfx) \leq d(gx, fx)$ for all x in $\mathcal{X}$.

Later on, Jungck [3] enlarged the class of commuting and weakly commuting mappings by compatible mappings which asserts that the above mappings f and g are compatible if $\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$ for some $t \in \mathcal{X}$.

This concept was further improved by Jungck [4] with the notion of weakly compatible mappings. f and g are weakly compatible if $ft = gt$ for some $t \in \mathcal{X}$ implies that $fgt = gft$.

Recently in 2007, Pathak et al. [6] stated and proved a general common fixed point theorem of integral type for two pairs of weakly compatible mappings satisfying integral type implicit relations in a symmetric space.

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Our aim here is to improve and extend the result of [6] by using the new concept of mappings called subcompatibility which enlarges the concept of weakly compatible mappings.

We introduce the notion of subcompatible mappings as follows: Let f and g be two self-mappings of a metric space $(\mathcal{X}, d)$. f and g are subcompatible if and only if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = t$ for some $t \in \mathcal{X}$ and $\lim_{n \rightarrow \infty} d(f g x_n, g f x_n) = 0$.

It is clear to see that weakly compatible mappings are subcompatible, however the implication is not reversible.

Example 1.1. Let $\mathcal{X} = [0, \infty)$ with the usual metric d . Define $f, g : \mathcal{X} \rightarrow \mathcal{X}$ as follows

$$f x = x^2 \text{ and } g x = \begin{cases} x + 12, & \text{if } x \in [0, 16] \cup (25, \infty), \\ x + 240, & \text{if } x \in (16, 25]. \end{cases}$$

Let $\{x_n\}$ be a sequence in $\mathcal{X}$ defined by $x_n = 4 + \frac{1}{n}$ for $n \in \mathbb{N}^* = \{1, 2, \dots\}$. Then, we have

$$\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} x_n^2 = 16 = \lim_{n \rightarrow \infty} g x_n = \lim_{n \rightarrow \infty} (x_n + 12)$$

and

$$\begin{aligned} f g x_n &= f(x_n + 12) = (x_n + 12)^2 \rightarrow 256 \text{ as } n \rightarrow \infty, \\ g f x_n &= g(x_n^2) = x_n^2 + 240 \rightarrow 256 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} d(f g x_n, g f x_n) = 0$. Hence, f and g are subcompatible mappings.

On the other hand, we have $f x = g x$ if and only if $x = 4$ but

$$f g(4) = f(16) = 256 \neq 28 = g f(4) = g(16).$$

Thus, f and g are not weakly compatible.

For our first main result we need the following implicit relations.

2. IMPLICIT RELATIONS

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers, Ψ be the family of all $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ Lebesgue-integrable and summable mappings and Φ be the set of all real continuous functions $\varphi : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned} (\varphi_1) & \text{ for all } u, v \geq 0, \text{ if} \\ (\varphi_a) & \int_0^{\varphi(u, v, v, u, 0, u+v)} \psi(t) \, dt \leq 0 \text{ or} \\ (\varphi_b) & \int_0^{\varphi(u, v, u, v, u+v, 0)} \psi(t) \, dt \leq 0, \\ & \text{we have } u \leq v, \end{aligned}$$

$$(\varphi_2) \int_0^{\varphi(u,u,0,0,u,u)} \psi(t) \, dt > 0, \text{ for } u > 0.$$

Example 2.1. Let $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max \left\{ t_2, t_3, t_4, \frac{t_5+t_6}{2} \right\}$, where $k \in (0, 1)$ and $\psi(t) = t$. Then φ is continuous and ψ is a Lebesgue-integrable mapping which is summable. We have

(φ_1) Let $u > 0$ and $v \geq 0$. If $u > v$ then

$$\begin{aligned} \varphi(u, v, v, u, 0, u+v) &= \varphi(u, v, u, v, u+v, 0) \\ &= u - k \max \left\{ u, v, \frac{u+v}{2} \right\} \\ &= u(1-k), \end{aligned}$$

then

$$\int_0^{u(1-k)} t \, dt = \frac{1}{2} u^2 (1-k)^2 \leq 0$$

impossible, hence $u \leq v$. If $u = 0$, then $u \leq v$.

(φ_2) $\varphi(u, u, 0, 0, u, u) = u(1-k)$, so

$$\int_0^{u(1-k)} t \, dt = \frac{1}{2} u^2 (1-k)^2 > 0,$$

for $u > 0$.

Example 2.2. $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = (1 + \alpha t_2)t_1 - \alpha \max \{t_3 t_4, t_5 t_6\} - \beta \max \left\{ t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6) \right\}$, where $\alpha \geq 0$ and $0 < \beta < 1$ and $\psi(t) = 1$.

(φ_1) Let $u > 0$ and $v \geq 0$. Suppose that $u > v$, then

$$\begin{aligned} \varphi(u, v, v, u, 0, u+v) &= \varphi(u, v, u, v, u+v, 0) \\ &= (1 + \alpha v)u - \alpha \max \{uv, 0\} - \beta \max \left\{ v, u, \frac{u+v}{2} \right\} \\ &= u(1-\beta), \end{aligned}$$

then

$$\int_0^{u(1-\beta)} dt = u(1-\beta) \leq 0,$$

which is impossible. Thus, $u \leq v$. If $u = 0$, then $u \leq v$.

(φ_2) $\varphi(u, u, 0, 0, u, u) = u(1-\beta)$, then

$$\int_0^{u(1-\beta)} dt = u(1-\beta) > 0, \quad \text{for all } u > 0.$$

Now, we state and prove our main results. We begin by the first one.

3. MAIN RESULTS

Theorem 3.1. *Let f, g, h and k be four mappings of a metric space $(\mathcal{X}, d)$ into itself such that*

$$(1) \quad \int_0^{\varphi(d(fx,gy),d(hx,ky),d(fx,hx),d(gy,ky),d(ky,fx),d(hx,gy))} \psi(t) dt \leq 0,$$

for all x, y in $\mathcal{X}$, where $\varphi \in \Phi$ and $\psi \in \Psi$. Suppose that (f, h) and (g, k) are subcompatible and h and k are continuous, then, f, g, h and k have a unique common fixed point.

Proof. Since the pairs (f, h) and (g, k) are subcompatible, then, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} hx_n = t$ for some $t \in \mathcal{X}$ and $\lim_{n \rightarrow \infty} d(fhx_n, hfx_n) = 0$; $\lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} ky_n = z$ for some $z \in \mathcal{X}$ and $\lim_{n \rightarrow \infty} d(gky_n, kgyn) = 0$.

First we prove that $z = t$. Indeed, by inequality (1) we get

$$\int_0^{\varphi(d(fx_n,gy_n),d(hx_n,ky_n),d(fx_n,hx_n),d(gy_n,ky_n),d(ky_n,fx_n),d(hx_n,gy_n))} \psi(t) dt \leq 0.$$

Since φ is continuous, we obtain at infinity

$$\int_0^{\varphi(d(t,z),d(t,z),0,0,d(z,t),d(t,z))} \psi(t) dt \leq 0,$$

which contradicts (φ_2) if $d(t, z) > 0$. Then, $z = t$.

Since h is continuous, then $h^2x_n \rightarrow ht$, $hfx_n \rightarrow ht$. Also we have

$$d(fhx_n, ht) \leq d(fhx_n, hfx_n) + d(hfx_n, ht).$$

Since f and h are subcompatible, taking the limit as $n \rightarrow \infty$ in the above inequality we have $\lim_{n \rightarrow \infty} fhx_n = ht$. The use of condition (1) gives

$$\int_0^{\varphi(d(fhx_n,gy_n),d(h^2x_n,ky_n),d(fhx_n,h^2x_n),d(gy_n,ky_n),d(ky_n,fhx_n),d(h^2x_n,gy_n))} \psi(t) dt \leq 0.$$

At infinity we obtain

$$\int_0^{\varphi(d(t,z),d(t,z),0,0,d(z,t),d(t,z))} \psi(t) dt \leq 0,$$

which contradicts (φ_2) . Hence $ht = t$.

Again using (1) we get

$$\int_0^{\varphi(d(ft,gy_n),d(ht,ky_n),d(ft,ht),d(gy_n,ky_n),d(ky_n,ft),d(ht,gy_n))} \psi(t) dt \leq 0.$$

Taking the limit as $n \rightarrow \infty$, we get

$$\int_0^{\varphi(d(ft,t),0,d(ft,t),0,d(t,ft),0)} \psi(t) dt \leq 0,$$

which implies $d(ft, t) = 0$ by using condition (φ_b) . Thus, $ft = t$.

Now, since k is continuous we have $\lim_{n \rightarrow \infty} k^2 y_n = \lim_{n \rightarrow \infty} kgy_n = kt$. Also we have

$$d(gky_n, kt) \leq d(gky_n, kgy_n) + d(kgy_n, kt).$$

Since the pair (g, k) is subcompatible we obtain at infinity $\lim_{n \rightarrow \infty} gky_n = kt$.

Using condition (1) we have

$$\int_0^{\varphi(d(ft, gky_n), d(ht, k^2 y_n), d(ft, ht), d(gky_n, k^2 y_n), d(k^2 y_n, ft), d(ht, gky_n))} \psi(t) dt \leq 0.$$

When n tends to infinity, we get

$$\int_0^{\varphi(d(t, kt), d(t, kt), 0, 0, d(kt, t), d(t, kt))} \psi(t) dt \leq 0,$$

which contradicts (φ_2) when $d(t, kt) > 0$. Hence, $kt = t$.

If $gt \neq t$, using inequality (1) we have

$$\int_0^{\varphi(d(ft, gt), d(ht, kt), d(ft, ht), d(gt, kt), d(kt, ft), d(ht, gt))} \psi(t) dt \leq 0,$$

i.e.,

$$\int_0^{\varphi(d(t, gt), 0, 0, d(gt, t), 0, d(t, gt))} \psi(t) dt \leq 0,$$

which implies $d(t, gt) = 0$ by using condition (φ_a) . Thus, $gt = t$.

For the uniqueness of common fixed point t , let $z \neq t$ be another common fixed point of f, g, h and k . Then using (1) we obtain

$$\int_0^{\varphi(d(ft, gz), d(ht, kz), d(ft, ht), d(gz, kz), d(kz, ft), d(ht, gz))} \psi(t) dt \leq 0,$$

that is,

$$\int_0^{\varphi(d(t, z), d(t, z), 0, 0, d(z, t), d(t, z))} \psi(t) dt \leq 0,$$

which is a contradiction of (φ_2) . Therefore $z = t$. □

Corollary 3.1. *Let $(\mathcal{X}, d)$ be a metric space and let f and h be two mappings from $\mathcal{X}$ into itself satisfying the condition*

$$\int_0^{\varphi(d(fx, fy), d(hx, hy), d(fx, hx), d(fy, hy), d(hy, fx), d(hx, fy))} \psi(t) dt \leq 0,$$

for all x, y in $\mathcal{X}$, where $\varphi \in \Phi$ and $\psi \in \Psi$. If h is continuous and the pair (f, h) is subcompatible, then, f and h have a unique common fixed point.

Corollary 3.2. *Let $(\mathcal{X}, d)$ be a metric space and let f, g and h be three self-mappings of $\mathcal{X}$ such that*

- (i) h is continuous,
- (ii) the pairs (f, h) and (g, h) are subcompatible and

(iii) the inequality

$$\int_0^{\varphi(d(fx,gy),d(hx,hy),d(fx,hx),d(gy,hy),d(hy,fx),d(hx,gy))} \psi(t) dt \leq 0,$$

holds for all x, y in $\mathcal{X}$, where $\varphi \in \Phi$ and $\psi \in \Psi$, then, f, g and h have a unique common fixed point.

Now, we give a generalization of Theorem 3.1.

Theorem 3.2. Let h, k and $\{f_n\}_{n \in \mathbb{N}^*}$ be mappings from a metric space $(\mathcal{X}, d)$ into itself such that

- (i) the pairs (f_n, h) and (f_{n+1}, k) are subcompatible,
- (ii) the inequality

$$\int_0^{\varphi(d(f_n x, f_{n+1} y), d(hx, ky), d(f_n x, hx), d(f_{n+1} y, ky), d(ky, f_n x), d(hx, f_{n+1} y))} \psi(t) dt \leq 0$$

holds for all x, y in $\mathcal{X}$, each $n \in \mathbb{N}^*$, $\varphi \in \Phi$ and $\psi \in \Psi$. If h and k are continuous, then, h, k and $\{f_n\}_{n \in \mathbb{N}^*}$ have a unique common fixed point.

Now, let $\mathcal{F}$ be the family of mappings $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that each F is upper semi-continuous and $F(t) < t$ for all $t > 0$ and let Ω be the family of $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that every ω is a Lebesgue-integrable mapping which is summable and $\int_0^\epsilon \omega(t) dt > 0$ for each $\epsilon > 0$.

In their paper [1], Djoudi and Aliouche proved a common fixed point theorem of Greguš type for four mappings satisfying a contractive condition of integral type in a metric space using the concept of weak compatibility.

Our objective here is to improve, extend and generalize the result of [1] by using the notion of subcompatibility.

Theorem 3.3. Let f, g, h and k be mappings from a metric space $(\mathcal{X}, d)$ into itself satisfying inequality

$$\begin{aligned} (2) \quad & \left(\int_0^{d(fx,gy)} \omega(t) dt \right)^p \\ & \leq F \left[a \left(\int_0^{d(hx,ky)} \omega(t) dt \right)^p + (1-a) \max \left\{ \int_0^{d(fx,hx)} \omega(t) dt, \right. \right. \\ & \quad \int_0^{d(gy,ky)} \omega(t) dt, \left(\int_0^{d(fx,hx)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(fx,ky)} \omega(t) dt \right)^{\frac{1}{2}}, \\ & \quad \left. \left(\int_0^{d(hx,gy)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(fx,ky)} \omega(t) dt \right)^{\frac{1}{2}} \right\}^p \Big], \end{aligned}$$

for all x, y in $\mathcal{X}$, where $0 < a < 1$, p is an integer such that $p \geq 1$, $F \in \mathcal{F}$ and $\omega \in \Omega$. If h and k are continuous and the pairs (f, h) and (g, k) are subcompatible, then, f, g, h and k have a unique common fixed point.

Proof. Since the pair (f, h) as well as (g, k) is subcompatible, then, there are two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that $\lim_{n \rightarrow \infty} hx_n = \lim_{n \rightarrow \infty} fx_n = t$ for some $t \in \mathcal{X}$ and $\lim_{n \rightarrow \infty} d(fhx_n, hfx_n) = 0$; $\lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} ky_n = z$ for some $z \in \mathcal{X}$ and $\lim_{n \rightarrow \infty} d(gky_n, kgy_n) = 0$.

First, we prove that $z = t$. If $t \neq z$, using inequality (2) we get

$$\begin{aligned} & \left(\int_0^{d(fx_n, gy_n)} \omega(t) dt \right)^p \\ \leq & F \left[a \left(\int_0^{d(hx_n, ky_n)} \omega(t) dt \right)^p + (1-a) \max \left\{ \int_0^{d(fx_n, hx_n)} \omega(t) dt, \right. \right. \\ & \int_0^{d(gy_n, ky_n)} \omega(t) dt, \left(\int_0^{d(fx_n, hx_n)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(fx_n, ky_n)} \omega(t) dt \right)^{\frac{1}{2}}, \\ & \left. \left(\int_0^{d(hx_n, gy_n)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(fx_n, ky_n)} \omega(t) dt \right)^{\frac{1}{2}} \right\}^p \Big]. \end{aligned}$$

Letting $n \rightarrow \infty$, we obtain

$$\begin{aligned} & \left(\int_0^{d(t, z)} \omega(t) dt \right)^p \\ \leq & F \left[a \left(\int_0^{d(t, z)} \omega(t) dt \right)^p + (1-a) \left(\int_0^{d(t, z)} \omega(t) dt \right)^p \right] \\ = & F \left[\left(\int_0^{d(t, z)} \omega(t) dt \right)^p \right] < \left(\int_0^{d(t, z)} \omega(t) dt \right)^p, \end{aligned}$$

which is a contradiction, then $\int_0^{d(t, z)} \omega(t) dt = 0$, hence $z = t$.

Since h is continuous, then we have $h^2x_n \rightarrow ht$, $hfx_n \rightarrow ht$. Also, we have

$$d(fhx_n, ht) \leq d(fhx_n, hfx_n) + d(hfx_n, ht).$$

As f and h are subcompatible, letting n tends to infinity in the above inequality, we obtain $\lim_{n \rightarrow \infty} fhx_n = ht$. The use of condition (2) gives

$$\begin{aligned} & \left(\int_0^{d(fhx_n, gy_n)} \omega(t) dt \right)^p \\ \leq & F \left[a \left(\int_0^{d(h^2x_n, ky_n)} \omega(t) dt \right)^p + (1-a) \max \left\{ \int_0^{d(fhx_n, h^2x_n)} \omega(t) dt, \right. \right. \end{aligned}$$

$$\int_0^{d(gy_n, ky_n)} \omega(t) dt, \left(\int_0^{d(fhx_n, h^2x_n)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(fhx_n, ky_n)} \omega(t) dt \right)^{\frac{1}{2}}, \\ \left(\int_0^{d(h^2x_n, gy_n)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(fhx_n, ky_n)} \omega(t) dt \right)^{\frac{1}{2}} \Bigg\}^p \Bigg].$$

We obtain at infinity

$$\left(\int_0^{d(ht, t)} \omega(t) dt \right)^p \\ \leq F \left[a \left(\int_0^{d(ht, t)} \omega(t) dt \right)^p + (1-a) \left(\int_0^{d(ht, t)} \omega(t) dt \right)^p \right] \\ = F \left[\left(\int_0^{d(ht, t)} \omega(t) dt \right)^p \right] < \left(\int_0^{d(ht, t)} \omega(t) dt \right)^p,$$

which is a contradiction, therefore $ht = t$.

Again by inequality (2) we have

$$\left(\int_0^{d(ft, gy_n)} \omega(t) dt \right)^p \\ \leq F \left[a \left(\int_0^{d(ht, ky_n)} \omega(t) dt \right)^p + (1-a) \max \left\{ \int_0^{d(ft, ht)} \omega(t) dt, \right. \right. \\ \int_0^{d(gy_n, ky_n)} \omega(t) dt, \left(\int_0^{d(ft, ht)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(ft, ky_n)} \omega(t) dt \right)^{\frac{1}{2}}, \\ \left. \left. \left(\int_0^{d(ht, gy_n)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(ft, ky_n)} \omega(t) dt \right)^{\frac{1}{2}} \right\}^p \right].$$

At infinity we obtain

$$\left(\int_0^{d(ft, t)} \omega(t) dt \right)^p \leq F \left[(1-a) \left(\int_0^{d(ft, t)} \omega(t) dt \right)^p \right] \\ < (1-a) \left(\int_0^{d(ft, t)} \omega(t) dt \right)^p \\ < \left(\int_0^{d(ft, t)} \omega(t) dt \right)^p,$$

which is a contradiction. Hence $ft = t$.

Now, since k is continuous, then, we have $k^2 y_n \rightarrow kt$ and $ky_n \rightarrow kt$ and

$$d(gky_n, kt) \leq d(gky_n, ky_n) + d(ky_n, kt).$$

Since the pair (g, k) is subcompatible, we get at infinity $\lim_{n \rightarrow \infty} gky_n = kt$.

Using (2) we have

$$\begin{aligned} & \left(\int_0^{d(ft, gky_n)} \omega(t) \, dt \right)^p \\ & \leq F \left[a \left(\int_0^{d(ht, k^2 y_n)} \omega(t) \, dt \right)^p + (1-a) \max \left\{ \int_0^{d(ft, ht)} \omega(t) \, dt, \right. \right. \\ & \quad \int_0^{d(gky_n, k^2 y_n)} \omega(t) \, dt, \left(\int_0^{d(ft, ht)} \omega(t) \, dt \right)^{\frac{1}{2}} \left(\int_0^{d(ft, k^2 y_n)} \omega(t) \, dt \right)^{\frac{1}{2}}, \\ & \quad \left. \left(\int_0^{d(ht, gky_n)} \omega(t) \, dt \right)^{\frac{1}{2}} \left(\int_0^{d(ft, k^2 y_n)} \omega(t) \, dt \right)^{\frac{1}{2}} \right]^p \Bigg]. \end{aligned}$$

We get at infinity

$$\begin{aligned} & \left(\int_0^{d(t, kt)} \omega(t) \, dt \right)^p \\ & \leq F \left[a \left(\int_0^{d(t, kt)} \omega(t) \, dt \right)^p + (1-a) \left(\int_0^{d(t, kt)} \omega(t) \, dt \right)^p \right] \\ & = F \left[\left(\int_0^{d(t, kt)} \omega(t) \, dt \right)^p \right] < \left(\int_0^{d(t, kt)} \omega(t) \, dt \right)^p. \end{aligned}$$

This contradiction implies that $kt = t$.

Suppose that $gt \neq t$, the use of inequality (2) gives

$$\begin{aligned} & \left(\int_0^{d(ft, gt)} \omega(t) \, dt \right)^p \\ & \leq F \left[a \left(\int_0^{d(ht, kt)} \omega(t) \, dt \right)^p + (1-a) \max \left\{ \int_0^{d(ft, ht)} \omega(t) \, dt, \right. \right. \\ & \quad \int_0^{d(gt, kt)} \omega(t) \, dt, \left(\int_0^{d(ft, ht)} \omega(t) \, dt \right)^{\frac{1}{2}} \left(\int_0^{d(ft, kt)} \omega(t) \, dt \right)^{\frac{1}{2}}, \\ & \quad \left. \left(\int_0^{d(ht, gt)} \omega(t) \, dt \right)^{\frac{1}{2}} \left(\int_0^{d(ft, kt)} \omega(t) \, dt \right)^{\frac{1}{2}} \right]^p \Bigg], \end{aligned}$$

i.e.,

$$\begin{aligned}
 \left(\int_0^{d(t,gt)} \omega(t) \, dt \right)^p &\leq F \left[(1-a) \left(\int_0^{d(t,gt)} \omega(t) \, dt \right)^p \right] \\
 &< (1-a) \left(\int_0^{d(t,gt)} \omega(t) \, dt \right)^p \\
 &< \left(\int_0^{d(t,gt)} \omega(t) \, dt \right)^p,
 \end{aligned}$$

which is a contradiction. Hence $gt = t$. Therefore $t = z$ is a common fixed point of both f , g , h and k .

Suppose that f , g , h and k have another common fixed point $z \neq t$. Then, by inequality (2) we get

$$\begin{aligned}
 &\left(\int_0^{d(ft,gz)} \omega(t) \, dt \right)^p \\
 \leq &F \left[a \left(\int_0^{d(ht,kz)} \omega(t) \, dt \right)^p + (1-a) \max \left\{ \int_0^{d(ft,ht)} \omega(t) \, dt, \right. \right. \\
 &\int_0^{d(gz,kz)} \omega(t) \, dt, \left(\int_0^{d(ft,ht)} \omega(t) \, dt \right)^{\frac{1}{2}} \left(\int_0^{d(ft,kz)} \omega(t) \, dt \right)^{\frac{1}{2}}, \\
 &\left. \left. \left(\int_0^{d(ht,gz)} \omega(t) \, dt \right)^{\frac{1}{2}} \left(\int_0^{d(ft,kz)} \omega(t) \, dt \right)^{\frac{1}{2}} \right\}^p \right],
 \end{aligned}$$

that is

$$\begin{aligned}
 \left(\int_0^{d(t,z)} \omega(t) \, dt \right)^p &\leq F \left[\left(\int_0^{d(t,z)} \omega(t) \, dt \right)^p \right] \\
 &< \left(\int_0^{d(t,z)} \omega(t) \, dt \right)^p.
 \end{aligned}$$

This contradiction implies that $z = t$. □

If $f = g$ and $h = k$ in Theorem 3.3, we get the next result:

Corollary 3.3. *Let f and h be two self-mappings of a metric space $(\mathcal{X}, d)$ such that*

$$\begin{aligned}
 &\left(\int_0^{d(fx,fy)} \omega(t) \, dt \right)^p \\
 \leq &F \left[a \left(\int_0^{d(hx,hy)} \omega(t) \, dt \right)^p + (1-a) \max \left\{ \int_0^{d(fx,hx)} \omega(t) \, dt, \right. \right.
 \end{aligned}$$

$$\int_0^{d(fy,hy)} \omega(t) dt, \left(\int_0^{d(fx,hx)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(fx,hy)} \omega(t) dt \right)^{\frac{1}{2}},$$

$$\left(\int_0^{d(hx,fy)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(fx,hy)} \omega(t) dt \right)^{\frac{1}{2}} \Bigg\}^p \Bigg],$$

for all x, y in $\mathcal{X}$, where $0 < a < 1$, p is an integer such that $p \geq 1$, $F \in \mathcal{F}$ and $\omega \in \Omega$. If h is continuous and the pair (f, h) is subcompatible, then, f and h have a unique common fixed point.

If we let in Theorem 3.3 $h = k$, then we get the following corollary:

Corollary 3.4. *Let f, g and h be three self-mappings of a metric space $(\mathcal{X}, d)$ such that*

$$\left(\int_0^{d(fx,gy)} \omega(t) dt \right)^p$$

$$\leq F \left[a \left(\int_0^{d(hx,hy)} \omega(t) dt \right)^p + (1-a) \max \left\{ \int_0^{d(fx,hx)} \omega(t) dt, \right.$$

$$\int_0^{d(gy,hy)} \omega(t) dt, \left(\int_0^{d(fx,hx)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(fx,hy)} \omega(t) dt \right)^{\frac{1}{2}},$$

$$\left. \left(\int_0^{d(hx,gy)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(fx,hy)} \omega(t) dt \right)^{\frac{1}{2}} \right\}^p \Bigg],$$

for all x, y in $\mathcal{X}$, where $0 < a < 1$, p is an integer such that $p \geq 1$, $F \in \mathcal{F}$ and $\omega \in \Omega$. If h is continuous and the pairs (f, h) and (g, h) are subcompatible, then, f, g and h have a unique common fixed point.

The next result is a generalization of Theorem 3.3.

Theorem 3.4. *Let h, k and $\{f_n\}_{n \in \mathbb{N}^*}$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the inequality*

$$\left(\int_0^{d(f_n x, f_{n+1} y)} \omega(t) dt \right)^p$$

$$\leq F \left[a \left(\int_0^{d(hx, ky)} \omega(t) dt \right)^p + (1-a) \max \left\{ \int_0^{d(f_n x, hx)} \omega(t) dt, \right.$$

$$\int_0^{d(f_{n+1} y, ky)} \omega(t) dt, \left(\int_0^{d(f_n x, hx)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(f_n x, ky)} \omega(t) dt \right)^{\frac{1}{2}},$$

$$\left(\int_0^{d(hx, f_{n+1}y)} \omega(t) dt \right)^{\frac{1}{2}} \left(\int_0^{d(f_nx, ky)} \omega(t) dt \right)^{\frac{1}{2}} \Bigg\}^p \Bigg],$$

for all x, y in $\mathcal{X}$, where $0 < a < 1$, p is an integer such that $p \geq 1$, $F \in \mathcal{F}$ and $\omega \in \Omega$. If h and k are continuous and the pairs (f_n, h) and (f_{n+1}, k) are subcompatible, then h, k and $\{f_n\}_{n \in \mathbb{N}^*}$ have a unique common fixed point.

We end our work by establishing another result which improves, extends and generalizes especially the result of [5].

Theorem 3.5. Let $(\mathcal{X}, d)$ be a metric space, f, g, h and k be mappings from $\mathcal{X}$ into itself and F be an upper semi-continuous function of $[0, \infty)$ into itself such that $F(t) = 0$ if and only if $t = 0$ and satisfying inequality

$$\begin{aligned} (3) \quad & \int_0^{F(d(fx, gy))} \omega(t) dt \\ & \leq a(d(hx, ky)) \int_0^{F(d(hx, ky))} \omega(t) dt \\ & + b(d(hx, ky)) \int_0^{F(d(hx, fx)) + F(d(ky, gy))} \omega(t) dt \\ & + c(d(hx, ky)) \int_0^{\min\{F(d(hx, gy)), F(d(ky, fx))\}} \omega(t) dt, \end{aligned}$$

for all x, y in $\mathcal{X}$, where $\omega \in \Omega$ and $a, b, c : [0, \infty) \rightarrow [0, 1)$ are upper semi-continuous and satisfying the condition

$$a(t) + c(t) < 1, \quad t > 0.$$

If the pairs (f, h) and (g, k) are subcompatible and h and k are continuous, then, f, g, h and k have a unique common fixed point.

Proof. Since the pairs (f, h) and (g, k) are subcompatible, then, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} hx_n = t$ for some $t \in \mathcal{X}$ and $\lim_{n \rightarrow \infty} d(fhx_n, hfx_n) = 0$; $\lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} ky_n = z$ for some $z \in \mathcal{X}$ and $\lim_{n \rightarrow \infty} d(gky_n, kgy_n) = 0$.

First, we prove that $z = t$. Suppose that $F(d(t, z)) > 0$, using inequality (3) we get

$$\begin{aligned} \int_0^{F(d(fx_n, gy_n))} \omega(t) dt & \leq a(d(hx_n, ky_n)) \int_0^{F(d(hx_n, ky_n))} \omega(t) dt \\ & + b(d(hx_n, ky_n)) \int_0^{F(d(hx_n, fx_n)) + F(d(ky_n, gy_n))} \omega(t) dt \\ & + c(d(hx_n, ky_n)) \int_0^{\min\{F(d(hx_n, gy_n)), F(d(ky_n, fx_n))\}} \omega(t) dt. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$\begin{aligned} \int_0^{F(d(t,z))} \omega(t) \, dt &\leq [a(d(t,z)) + c(d(t,z))] \int_0^{F(d(t,z))} \omega(t) \, dt \\ &< \int_0^{F(d(t,z))} \omega(t) \, dt, \end{aligned}$$

which is a contradiction. Hence $F(d(t,z)) = 0$ which implies that $d(t,z) = 0$, thus $t = z$.

Since h is continuous, then, we have $h^2x_n \rightarrow ht$, $hfx_n \rightarrow ht$. Also, we have

$$d(fhx_n, ht) \leq d(fhx_n, hfx_n) + d(hfx_n, ht).$$

As f and h are subcompatible, letting n tends to infinity in the above inequality, we obtain $\lim_{n \rightarrow \infty} fhx_n = ht$. If $F(d(ht,t)) > 0$, the use of condition (3) gives

$$\begin{aligned} \int_0^{F(d(fhx_n, gy_n))} \omega(t) \, dt &\leq a(d(h^2x_n, ky_n)) \int_0^{F(d(h^2x_n, ky_n))} \omega(t) \, dt \\ &+ b(d(h^2x_n, ky_n)) \int_0^{F(d(h^2x_n, fhx_n)) + F(d(ky_n, gy_n))} \omega(t) \, dt \\ &+ c(d(h^2x_n, ky_n)) \int_0^{\min\{F(d(h^2x_n, gy_n)), F(d(ky_n, fhx_n))\}} \omega(t) \, dt. \end{aligned}$$

Letting $n \rightarrow \infty$ we obtain

$$\begin{aligned} \int_0^{F(d(ht,t))} \omega(t) \, dt &\leq [a(d(ht,t)) + c(d(ht,t))] \int_0^{F(d(ht,t))} \omega(t) \, dt \\ &< \int_0^{F(d(ht,t))} \omega(t) \, dt. \end{aligned}$$

This contradiction implies that $F(d(ht,t)) = 0$ and hence $ht = t$.

Suppose that $F(d(ft,t)) > 0$, using condition (3) we get

$$\begin{aligned} \int_0^{F(d(ft, gy_n))} \omega(t) \, dt &\leq a(d(ht, ky_n)) \int_0^{F(d(ht, ky_n))} \omega(t) \, dt \\ &+ b(d(ht, ky_n)) \int_0^{F(d(ht, ft)) + F(d(ky_n, gy_n))} \omega(t) \, dt \\ &+ c(d(ht, ky_n)) \int_0^{\min\{F(d(ht, gy_n)), F(d(ky_n, ft))\}} \omega(t) \, dt. \end{aligned}$$

We obtain at infinity

$$\int_0^{F(d(ft,t))} \omega(t) \, dt \leq b(0) \int_0^{F(d(t, ft))} \omega(t) \, dt < \int_0^{F(d(ft,t))} \omega(t) \, dt,$$

which is a contradiction, hence $F(d(ft,t)) = 0$ which implies that $ft = t$.

Now, since k is continuous, then, we have $k^2 y_n \rightarrow kt$, $ky_n \rightarrow kt$ and

$$d(gky_n, kt) \leq d(gky_n, kgy_n) + d(kgy_n, kt).$$

Since the pair (g, k) is subcompatible, we get at infinity $\lim_{n \rightarrow \infty} gky_n = kt$. We claim that $kt = t$, if not, then by (3) we have

$$\begin{aligned} \int_0^{F(d(ft, gky_n))} \omega(t) dt &\leq a(d(ht, k^2 y_n)) \int_0^{F(d(ht, k^2 y_n))} \omega(t) dt \\ &+ b(d(ht, k^2 y_n)) \int_0^{F(d(ht, ft)) + F(d(k^2 y_n, gky_n))} \omega(t) dt \\ &+ c(d(ht, k^2 y_n)) \int_0^{\min\{F(d(ht, gky_n)), F(d(k^2 y_n, ft))\}} \omega(t) dt. \end{aligned}$$

Taking the limit when $n \rightarrow \infty$ we have

$$\begin{aligned} \int_0^{F(d(t, kt))} \omega(t) dt &\leq [a(d(t, kt)) + c(d(t, kt))] \int_0^{F(d(t, kt))} \omega(t) dt \\ &< \int_0^{F(d(t, kt))} \omega(t) dt, \\ \Phi(d(t, kt)) &\leq [a(d(t, kt)) + c(d(t, kt))] \Phi(d(t, kt)) \\ &< \Phi(d(t, kt)), \end{aligned}$$

which is a contradiction, thus $kt = t$.

Suppose that $F(d(t, gt)) > 0$, then the use of inequality (3) yields

$$\begin{aligned} \int_0^{F(d(t, gt))} \omega(t) dt &= \int_0^{F(d(ft, gt))} \omega(t) dt \\ &\leq a(d(ht, kt)) \int_0^{F(d(ht, kt))} \omega(t) dt \\ &+ b(d(ht, kt)) \int_0^{F(d(ht, ft)) + F(d(kt, gt))} \omega(t) dt \\ &+ c(d(ht, kt)) \int_0^{\min\{F(d(ht, gt)), F(d(kt, ft))\}} \omega(t) dt \\ &= b(0) \int_0^{F(d(t, gt))} \omega(t) dt < \int_0^{F(d(t, gt))} \omega(t) dt, \end{aligned}$$

which is a contradiction, thus $F(d(t, gt)) = 0$ which implies that $d(t, gt) = 0$ i.e. $gt = t$.

Now, assume that there exists another common fixed point z of f, g, h and k such that $z \neq t$. By inequality (3) we obtain

$$\int_0^{F(d(t, z))} \omega(t) dt = \int_0^{F(d(ft, gz))} \omega(t) dt$$

$$\begin{aligned}
&\leq a(d(ht, kz)) \int_0^{F(d(ht, kz))} \omega(t) dt \\
&\quad + b(d(ht, kz)) \int_0^{F(d(ht, ft)) + F(d(kz, gz))} \omega(t) dt \\
&\quad + c(d(ht, kz)) \int_0^{\min\{F(d(ht, gz)), F(d(kz, ft))\}} \omega(t) dt \\
&= [a(d(t, z)) + c(d(t, z))] \int_0^{F(d(t, z))} \omega(t) dt \\
&< \int_0^{F(d(t, z))} \omega(t) dt.
\end{aligned}$$

This contradiction implies that $F(d(t, z)) = 0 \Leftrightarrow d(t, z) = 0$, hence $z = t$. $\square$

Remark 3.1. Theorem 3.5 remains valid if we replace inequality (3) by the following one

$$\begin{aligned}
\int_0^{F(d(fx, gy))} \omega(t) dt &\leq a(d(hx, ky)) \int_0^{F(d(hx, ky))} \omega(t) dt \\
&\quad + b(d(hx, ky)) \int_0^{\frac{F(d(hx, fx)) + F(d(ky, gy))}{2}} \omega(t) dt \\
&\quad + c(d(hx, ky)) \int_0^{\frac{F(d(hx, gy)) + F(d(ky, fx))}{2}} \omega(t) dt.
\end{aligned}$$

Corollary 3.5. *Let f and h be self-mappings of a metric space $(\mathcal{X}, d)$. Assume that h is continuous, the pair (f, h) is subcompatible and satisfies the inequality*

$$\begin{aligned}
\int_0^{F(d(fx, fy))} \omega(t) dt &\leq a(d(hx, hy)) \int_0^{F(d(hx, hy))} \omega(t) dt \\
&\quad + b(d(hx, hy)) \int_0^{F(d(hx, fx)) + F(d(hy, fy))} \omega(t) dt \\
&\quad + c(d(hx, hy)) \int_0^{\min\{F(d(hx, fy)), F(d(hy, fx))\}} \omega(t) dt,
\end{aligned}$$

for all x, y in $\mathcal{X}$, where F, ω, a, b and c are as in Theorem 3.5. Then, f and h have a unique common fixed point.

Corollary 3.6. *Let $f, g, h : \mathcal{X} \rightarrow \mathcal{X}$ be mappings satisfying the following inequality*

$$\int_0^{F(d(fx, gy))} \omega(t) dt \leq a(d(hx, hy)) \int_0^{F(d(hx, hy))} \omega(t) dt$$

$$\begin{aligned}
& + b(d(hx, hy)) \int_0^{F(d(hx, fx)) + F(d(hy, gy))} \omega(t) \, dt \\
& + c(d(hx, hy)) \int_0^{\min\{F(d(hx, gy)), F(d(hy, fx))\}} \omega(t) \, dt,
\end{aligned}$$

for all x, y in $\mathcal{X}$, where F, ω, a, b and c are as in Theorem 3.5. If h is continuous and the pairs (f, h) and (g, h) are subcompatible, then, f, g and h have a unique common fixed point.

Now, we give a generalization of Theorem 3.5.

Theorem 3.6. Let $(\mathcal{X}, d)$ be a metric space, $h, k, \{f_n\}_{n \in \mathbb{N}^*}$ be mappings from $\mathcal{X}$ into itself and F be an upper semi-continuous function of $[0, \infty)$ into itself such that $F(t) = 0$ if and only if $t = 0$ and satisfying the inequality

$$\begin{aligned}
\int_0^{F(d(f_n x, f_{n+1} y))} \omega(t) \, dt & \leq a(d(hx, ky)) \int_0^{F(d(hx, ky))} \omega(t) \, dt \\
& + b(d(hx, ky)) \int_0^{F(d(hx, f_n x)) + F(d(ky, f_{n+1} y))} \omega(t) \, dt \\
& + c(d(hx, ky)) \int_0^{\min\{F(d(hx, f_{n+1} y)), F(d(ky, f_n x))\}} \omega(t) \, dt,
\end{aligned}$$

for all x, y in $\mathcal{X}$, where $\omega \in \Omega$, $a, b, c : [0, \infty) \rightarrow [0, 1)$ are upper semi-continuous and satisfying the condition

$$a(t) + c(t) < 1, \quad t > 0.$$

If the pairs (f_n, h) and (f_{n+1}, k) are subcompatible and h and k are continuous, then h, k and $\{f_n\}_{n \in \mathbb{N}^*}$ have a unique common fixed point.

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