Mathematica Moravica
Vol. **22**, No. **2** (2018), 117–130 doi: 10.5937/MatMor1802117M

Derivation $d_{a,\beta}$ of ordered Γ-semirings

Marapureddy Murali Krishna Rao, Kona Rajendr Kumar, Bolineni Venkateswarlu, Bandaru Ravi Kumar

Abstract. In this paper, we introduce the concept of derivation $d_{a,\beta}$ of ordered Γ-semiring. We study some of the properties of derivation $d_{a,\beta}$ of ordered Γ-semirings. We prove that if a derivation $d_{a,\beta}$ is nonzero on an integral Γ-semiring M then it is non-zero on any non-zero ideal of M and we characterize k-ideal and $m-k$ ideal using derivation $d_{a,\beta}$ of ordered Γ-semiring.

1. Introduction

In 1995, Murali Krishna Rao [12, 13, 16] introduced the notion of Γ- semiring as a generalization of Γ- ring, ternary semiring and semiring. Semiring is an algebraic structure as, a common generalization of ring and distributive lattice.Semiring was first introduced by American mathematician Vandiver [19] in 1934 but non trivial examples of semirings had appeared in the earlier studies on the theory of commutative ideals of rings by German mathematician Richard Dedekind in 19th century. Semiring is a universal algebra with two associative binary operations called addition and multiplication where one of them is distributive over the other. Bounded distributive lattices are commutative semirings which are both additively idempotent and multiplicatively idempotent. A natural example of semiring is the set of all natural numbers under usual addition and multiplication of numbers. In particular, if I is the unit interval on the real line then $(I, \max, \min)$, in which 0 is the additive identity and 1 is the multiplicative identity, is a semiring. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. Additive and multiplicative structures of a semiring play an important role in determining the structure of a semiring. Semirings are used in the areas of theoretical computer science as well as in the solutions of graph theory and optimization theory and in particular for studying automata, coding theory and formal

2000 *Mathematics Subject Classification.* 16Y60, 06F35, 08A30, 03G25.

Key words and phrases. Derivation, Γ-semiring, ordered Γ-semiring, integral ordered Γ-semiring.

Full paper. Received 19 April 2018, revised 17 May 2018, accepted 24 May 2018, available online 15 December 2018.

languages as the basic algebraic structure [6, 7]. Semiring theory has many applications in other branches [3, 4].

The notion of $\Gamma-$ring was introduced by Nobusawa [10] as a generalization of ring in 1964. Sen [18] introduced the notion of $\Gamma-$ semigroup in 1981. The notion of ternary algebraic system was introduced by Lehmer [8] in 1932. Lister [9] introduced the notion of ternary ring. Dutta & Kar introduced the notion of ternary semiring which is a generalization of ternary ring and semiring. After the paper [12] was published, many mathematicians obtained interesting results on Γ-semirings.

Over the last few decades several authors have investigated the relationship between the commutativity of ring R and the existence of certain specified derivations of R. The first result in this direction is due to Posner [11] in 1957. In the year 1990, Bresar and Vukman [2] established that a prime ring must be a commutative if it admits a non zero left derivation. The notion of derivation of ring is useful for characterization of rings [1, 17, 5]. Murali Krishna Rao and Venkateswarlu [14, 15] introduced the concept of generalized right derivation of $\Gamma-$incline and the concept of right derivation of ordered Γ-semirings. In this paper, we introduce the concept of derivation $d_{a,\beta}$ of ordered Γ-semirings. We study some of the properties of derivation $d_{a,\beta}$ of ordered Γ-semirings and we prove that if a derivation $d_{a,\beta}$ is non-zero on an integral ordered Γ-semiring M, then it is non-zero on any non-zero ideal of M.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions necessary for this paper.

Definition 2.1. Let $(M,+)$ and $(\Gamma,+)$ be commutative semigroups. Then we call M as a $\Gamma-$semiring,if there exists a mapping $M \times \Gamma \times M \to M$ is written (x, α, y) as $x\alpha y$ such that it satisfies the following axioms

(i) $x\alpha(y+z) = x\alpha y + x\alpha z$,
(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$,
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Every semiring R is a $\Gamma-$semiring with $\Gamma = R$ and ternary operation $x\gamma y$ as the usual semiring multiplication.

We illustrate the definition of $\Gamma-$semiring by the following example

Example 2.1. Let S be a semiring and $M_{p,q}(S)$ denote the additive abelian semigroup of all $p \times q$ matrices with identity element whose entries are from S. Then $M_{p,q}(S)$ is a $\Gamma-$semiring with $\Gamma = M_{p,q}(S)$ ternary operation is defined by $x\alpha z = x(\alpha^t)z$ as the usual matrix multiplication, where α^t denote the transpose of the matrix α; for all x, y and $\alpha \in M_{p,q}(S)$.

Definition 2.2. A $\Gamma-$semiring M is called an ordered $\Gamma-$semiring if it admits a compatible relation $\leq$. i.e.$\leq$ is a partial ordering on M satisfies the following conditions. If $a \leq b$ and $c \leq d$ then

(i) $a + c \leq b + d$,
(ii) $a\alpha c \leq b\alpha d$,
(iii) $c\alpha a \leq d\alpha b$, for all $a, b, c, d \in M, \alpha \in \Gamma$.

Definition 2.3. A non-empty subset A of ordered $\Gamma-$semiring M is called a $\Gamma-$subsemiring M if $(A, +)$ is a subsemigroup of $(M, +)$ and $a\alpha b \in A$ for all $a, b \in A$ and $\alpha \in \Gamma$.

Definition 2.4. A non-empty subset A of an ordered $\Gamma-$semiring. M is called a left (right) ideal of ordered $\Gamma-$semiring M if A satisfies the following conditions.

(i) A is closed under addition,
(ii) $M\Gamma A \subseteq A$ $(A\Gamma M \subseteq A)$,
(iii) if for any $a \in M,\ b \in$A, $a \leq b$ implies $a \in$A.

A is called an ideal of M if it is both a left ideal and a right ideal of M.

Definition 2.5. A non-empty subset A of an ordered $\Gamma-$semiring M is called a $k-$ideal if A is an ideal and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.

Definition 2.6. An ordered $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M, \alpha \in \Gamma$.

Definition 2.7. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.8. An element $a \in M$ is said to be idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α idempotent.

Definition 2.9. A semigroup $(M, +)$ is said to be band if $a+a = a,$ for all $a \in M$.

Definition 2.10. Every element of an ordered $\Gamma-$semiring M is an idempotent of M then M is said to be idempotent ordered $\Gamma-$semiring M.

Definition 2.11. A non zero element a in an ordered $\Gamma-$semiring M is said to be zero divisor if there exits non zero element $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 0$.

Definition 2.12. An ordered $\Gamma-$semiring M with unity 1 and zero element 0 is called an integral ordered $\Gamma-$semiring if it has no zero divisors.

Definition 2.13. An ordered $\Gamma-$semiring M is said to be totally ordered $\Gamma-$semiring M if any two elements of M are comparable.

Definition 2.14. In an ordered $\Gamma-$semiring M

(i) the semigroup $(M,+)$ is said to be positively ordered , if $a \leq a+b$ and $b \leq a+b,$ for all $a,b \in M$.
(ii) the semigroup $(M,+)$ is said to be negatively ordered, if $a+b \leq a$ and $a+b \leq b,$ for all $a,b \in M$.
(iii) the $\Gamma-$semigroup M is said to be positively ordered, if $a \leq a\alpha b$ and $b \leq a\alpha b,$ for all $\alpha \in \Gamma, a,b \in M$.
(iv) $\Gamma-$semigroup M is said to be negatively ordered if $a\alpha b \leq a$ and $a\alpha b \leq b$ for all $\alpha \in \Gamma, a,b \in M$.

Definition 2.15. Let M and N be ordered $\Gamma-$ semirings. A mapping $f : M \to N$ is called a homomorphism if

(i) $f(a+b) = f(a) + f(b)$,
(ii) $f(a\alpha b) = f(a)\alpha f(b)$, for all $a,b \in M, \alpha \in \Gamma$.

Definition 2.16. Let M be an ordered Γ-semiring. A mapping $d : M \to M$ is called a derivation if it satisfies the following conditions.

(i) $d(x+y) = d(x) + d(y)$,
(ii) $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all $x,y \in M$ and $\alpha \in \Gamma$.

3. DERIVATION $d_{a,\beta}$ OF ORDERED Γ-SEMIRINGS

In this section, we introduce the notion of derivation of the form $d_{a,\beta}$ of ordered Γ-semirings. We study some of the properties of derivation $d_{a,\beta}$ of ordered Γ-semirings.

Definition 3.1. Let M be an ordered Γ-semiring. Then for any $a \in M$ and $\beta \in \Gamma$, we define a mapping $d : M \to M$ by $d(x) = x\beta a$, for all $x \in M$.This function d is denoted by $d_{a,\beta}$.

Definition 3.2. Let M be an ordered Γ-semiring and $d_{a,\beta}$ be a function. Then $d_{a,\beta}$ is said to be derivation of M if

(i) $d_{a,\beta}(x+y) = d_{a,\beta}(x) + d_{a,\beta}(y)$,
(ii) $d_{a,\beta}(x\alpha y) = d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y)$, for all $x,y \in M, \alpha \in \Gamma$.

Example 3.1. Let $M = \{0,a,b,1\}$ and $\Gamma = \{\alpha,\beta\}$. If we define the the additive operations on M and Γ, by

+	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

+	α	β
α	α	β
β	β	α

ternary operation, by

α	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

β	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

and $x \leq y$ if and only if $x + y = y$, for all $x, y \in M$ then M is an ordered $\Gamma-$semiring .

Let $a \in M$ and $\beta \in \Gamma$. Define $d_{a,\beta} = x\beta a$, for all $x \in M$.
Obviously $d_{a,\beta}$ is a derivation of M

Example 3.2. Let $M = [0, 1]$ and $\Gamma =$ set of all natural numbers. Define the binary operations $+$ on M and Γ,by $a + b = \max\{a, b\}$, $\alpha + \beta = \max\{\alpha, \beta\}$, and ternary operation by $a\alpha b = \min\{a, \alpha, b\}$, for all $a, b \in M$, $\alpha \in \Gamma$
and $a \leq b$ if and only if $a + b = b$, for all $a, b \in M.\alpha, \beta \in \Gamma$ Then M is an ordered $\Gamma-$semiring .

Let $a \in M$ and $\beta \in \Gamma$. Define $d_{a,\beta}(x) = x\beta a$, for all $x \in M$.

Obviously $d_{a,\beta}$ is a derivation of M.

Theorem 3.1. *Let M be an ordered commutative Γ-semiring in which semigroup $(M, +)$ is a band .Then $d_{a,\beta}$ is a derivation of M.*

Proof. Let M be an ordered commutative Γ-semiring in which semigroup $(M, +)$ is a band , $x, y \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} d_{a,\beta}(x\alpha y) &= (x\alpha y)\beta a \\ d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) &= (x\beta a)\alpha y + x\alpha y\beta a \\ &= y\alpha(x\beta a) + x\alpha y\beta a \\ &= (y\alpha x)\beta a + x\alpha y\beta a \\ &= x\alpha y\beta a + x\alpha y\beta a \\ &= x\alpha y\beta a. \end{aligned}$$

Hence $d_{a,\beta}$ is a derivation of M. □

Theorem 3.2. *Let M be an ordered commutative Γ-semiring in which semigroup $(M, +)$ is a band with unity element 1. Then for each $x \in M$,there exists a derivation $d_{1,\beta}$, $\beta \in \Gamma$ such that $d_{1,\beta}(x) = x$.*

Proof. Let $x \in M$. Then there exists $\beta \in \Gamma$ such that $x\beta 1 = x$. By Theorem 3.1 $d_{1,\beta}$ is a derivation and $d_{1,\beta}(x) = x\beta 1 = x$ □

Theorem 3.3. *Let M be an ordered Γ-semiring in which semigroup $(M, +)$ is is a band and positively ordered, Γ-semigroup M is negatively ordered and $d_{a,\beta}$ be a derivation.Then*

(i) $d_{a,\beta}(x\alpha y) \leq d_{a,\beta}(x) + d_{a,\beta}(y)$,

(ii) $d_{a,\beta}(x) \leq x$,

(iii) *if* $x \leq y$ *then* $d_{a,\beta}(x\alpha y) \leq y$.

Proof. (i) Let $x, y \in M, \alpha \in \Gamma$ then

$$\begin{aligned} d_{a,\beta}(x\alpha y) &= (x\alpha y)\beta a \\ &= (x+x)\alpha y\beta a \\ &= x\alpha y\beta a + x\alpha y\beta a \\ &\leq x\beta a + y\beta a \\ &= d_{a,\beta}(x) + d_{a,\beta}(y). \end{aligned}$$

(ii) $d_{a,\beta}(x) = x\beta a \leq x$.
(iii) Suppose $x \leq y$. Then

$$\begin{aligned} &x + y \leq y + y, \\ &x + y \leq y \leq x + y, \\ &x + y = y. \end{aligned}$$

$$\begin{aligned} d_{a,\beta}(x\alpha y) &\leq d_{a,\beta}(x) + d_{a,\beta}(y) \\ &\leq x + y = y. \end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let* $d_{a,\beta}$ *be a derivation of an ordered* Γ*-semiring* M*.*
Then $d_{a,\beta}(0) = 0$.

Proof. By Definition 3.1, $d_{a,\beta}(x) = x\beta a$, for all $x \in M$. Then $d_{a,\beta}(0) = 0\beta a = 0$.
Therefore $d_{a,\beta}(0) = 0$. □

Theorem 3.5. *Let* $d_{a,\beta}$ *be a derivation of an idempotent ordered* $\Gamma-$*semiring* M *in which* $\Gamma-$*semigroup* M *is negatively ordered , semigroup* $(M, +)$ *is a band.Then* $d_{a,\beta}(x) \leq x$*, for all* $x \in M$*.*

Proof. Let $d_{a,\beta}$ be a derivation of an idempotent ordered $\Gamma-$semiring M in which $\Gamma-$semigroup M is negatively ordered. Suppose $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x\alpha x = x$.

$$\begin{aligned} d_{a,\beta}(x) &= d_{a,\beta}(x\alpha x) \\ &= d_{a,\beta}(x)\alpha x + x\alpha d_{a,\beta}(x) \\ &\leq x + x. \end{aligned}$$

Therefore $d_{a,\beta}(x) \leq x$.
This completes the proof. □

Theorem 3.6. *Let* M *be an ordered* $\Gamma-$*semiring in which* $\Gamma-$*semigroup* M *is negatively ordered. Then* $d_{a,\beta}(x\alpha y) \leq d_{a,\beta}(x+y)$*, for all* $x, y \in M, \alpha \in \Gamma$*.*

Proof. Let M be an ordered $\Gamma-$semiring in which $\Gamma-$semigroup M is negatively ordered. Suppose $x, y \in M, \alpha \in \Gamma$. Then

$$d_{a,\beta}(x)\alpha y \leq d_{a,\beta}(x), \quad x\alpha d_{a,\beta}(y) \leq d_{a,\beta}(y).$$

Therefore

$$\begin{aligned} d_{a,\beta}(x\alpha y) &= d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) \\ &\leq d_{a,\beta}(x) + d_{a,\beta}(y) \\ &= d_{a,\beta}(x+y). \end{aligned}$$

This completes the proof. □

Theorem 3.7. *Let M be an idempotent ordered Γ-semiring in which Γ-semigroup M is negatively ordered and semigroup $(M,+)$ is a band. Then the following hold for all $x, y \in M, \alpha \in \Gamma$.*

(i) $d_{a,\beta}(x\alpha y) \leq d_{a,\beta}(x) + d_{a,\beta}(y)$,
(ii) *If $x \leq y$ then $d_{a,\beta}(x\alpha y) \leq d_{a,\beta}(y)$,*
(iii) $d_{a,\beta}(x) \leq x$.

Proof. (i) The following holds

$$\begin{aligned} d_{a,\beta}(x\alpha y) &= d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) \\ &\leq d_{a,\beta}(x) + d_{a,\beta}(y). \end{aligned}$$

(ii) Suppose $x \leq y$, then

$$\begin{aligned} x\alpha d_{a,\beta}(y) &\leq y\alpha d_{a,\beta}(y) \leq y, \\ d_{a,\beta}(x)\alpha y &\leq y, \\ d_{a,\beta}(x\alpha y) &= d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) \\ &\leq y + y = y. \end{aligned}$$

(iii) Let $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x\alpha x = x$.

$$\begin{aligned} d_{a,\beta}(x) &= d_{a,\beta}(x\alpha x) \\ &= d_{a,\beta}(x)\alpha x + x\alpha d_{a,\beta}(x) \\ &\leq x + x = x. \end{aligned}$$

This completes the proof. □

Theorem 3.8. *Let M be an idempotent ordered Γ-semiring with unity 1 in which semigroup $(M,+)$ is a band and positively ordered, Γ-semigroup M is negatively ordered and $d_{a,\beta}$ be a derivation of M. Then the following hold for all $x \in M$,*

(i) $x\alpha d_{a,\beta}(1) \leq d_{a,\beta}(x), \alpha \in \Gamma$,
(ii) *If $d_{a,\beta}(1) = 1$ then $d_{a,\beta}(x) = x$, for all $x \in M$.*

Proof. (i) Let $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x\alpha 1 = x$.
Therefore

$$\begin{aligned} d_{a,\beta}(x\alpha 1) &= d_{a,\beta}(x), \\ d_{a,\beta}(x)\alpha 1 + x\alpha d_{a,\beta}(1) &= d_{a,\beta}(x), \\ x\alpha d_{a,\beta}(1) &\leq d_{a,\beta}(x). \end{aligned}$$

(ii) Suppose $d_{a,\beta}(1) = 1$. We have

$$x\alpha d_{a,\beta}(1) \leq d_{a,\beta}(x),$$
$$x\alpha 1 \leq d_{a,\beta}(x).$$

Therefore $x \leq d_{a,\beta}(x)$.

By Theorem 3.7 , $d_{a,\beta}(x) \leq x$. Hence $d_{a,\beta}(x) = x$, for all $x \in M$. □

Theorem 3.9. *Let M be an ordered Γ-semiring with unity 1 in which semigroup $(M, +)$ is is a band and positively ordered, Γ-semigroup M is negatively ordered and $d_{a,\beta}$ be a derivation of M. If $x \in M$ then there exists $\alpha \in \Gamma$ such that*

(i) $x\alpha d_{a,\beta}(1) \leq d_{a,\beta}(x)$,
(ii) *If $d_{a,\beta}(1) = 1$ then $x \leq d_{a,\beta}(x)$.*

Proof. (i) Let M be an ordered $\Gamma-$semiring with unity 1, $d_{a,\beta}$ be a derivation of M and $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x\alpha 1 = x$.

$$\begin{aligned} d_{a,\beta}(x) &= d_{a,\beta}(x\alpha 1) \\ &= d_{a,\beta}(x)\alpha 1 + x\alpha d_{a,\beta}(1), \\ x\alpha d_{a,\beta}(1) &\leq d_{a,\beta}(x)\alpha 1 + x\alpha d_{a,\beta}(1) = d_{a,\beta}(x). \end{aligned}$$

(ii) Suppose $d_{a,\beta}(1) = 1$ and $x\alpha d_{a,\beta}(1) \leq d_{a,\beta}(x)$. Then $\Rightarrow x\alpha 1 \leq d_{a,\beta}(x)$ and $x \leq d_{a,\beta}(x)$.

This completes the proof. □

Theorem 3.10. *Let M be an idempotent ordered $\Gamma-$semiring in which $\Gamma-$semigroup M is negatively ordered and semigroup $(M, +)$ is a band. If $d^2_{a,\beta}(x) = d_{a,\beta}(d_{a,\beta}(x)) = d_{a,\beta}(x)$ then $d_{a,\beta}(x\alpha d_{a,\beta}(x)) \leq d_{a,\beta}(x)$, for all $x \in M$.*

Proof. Let M be an idempotent ordered $\Gamma-$semiring and

$$d^2_{a,\beta}(x) = d_{a,\beta}(d_{a,\beta}(x)) = d_{a,\beta}(x),$$

for all $x \in M$. Then

$$\begin{aligned} d_{a,\beta}(x\alpha d_{a,\beta}(x)) &= d_{a,\beta}(x)\alpha d_{a,\beta}(x) + x\alpha d(d_{a,\beta}(x)) \\ &= d_{a,\beta}(x) + x\alpha d_{a,\beta}(x) \\ &\leq d_{a,\beta}(x) + d_{a,\beta}(x) \\ &= d_{a,\beta}(x). \end{aligned}$$

Therefore $d_{a,\beta}(x\alpha d_{a,\beta}(x)) \leq d_{a,\beta}(x)$.

This completes the proof. □

Theorem 3.11. *Let $d_{a,\beta}$ be a derivation of an ordered integral-Γ-semiring M with unity and $a \in M$. If $a\alpha d_{a,\beta}(x) = 0$ for all $x \in M$, $\alpha \in \Gamma$ then either $a = 0$ or $d_{a,\beta} = 0$.*

Proof. Suppose $a\alpha d_{a,\beta}(x) = 0$, for all $x \in M$, $\alpha \in \Gamma$.

Let $y \in M$ and $\gamma \in \Gamma$. If we replace x by $x\gamma y$, then we obtain $a\alpha d_{a,\beta}(x\gamma y) = 0$ and

$$a\alpha[d_{a,\beta}(x)\gamma y + x\beta d_{a,\gamma}(y)] = 0,$$
$$a\alpha x\gamma d_{a,\beta}(y) = 0, \quad \gamma \in \Gamma,$$
$$a\alpha 1\gamma d_{a,\beta}(y) = 0,$$
$$a\alpha d_{a,\beta}(y) = 0.$$

Therefore $a = 0$ or $d_{a,\beta}(y) = 0$ since M has no zero divisors.

This completes the proof. □

Definition 3.3. An ideal I of an ordered Γ-semiring M is said to be $m - k$ ideal if $x\alpha y \in I$, $x \in I$, $1 \neq y \in M$ and $\alpha \in \Gamma$ then $y \in I$.

Definition 3.4. Let $d_{a,\beta}$ be a derivation of an ordered Γ-semiring M. Derivation $d_{a,\beta}$ is called an isotone derivation if $x \leq y$ then $d_{a,\beta}(x) \leq d_{a,\beta}(y)$ for all $x, y \in M$.

Theorem 3.12. *Let $d_{a,\beta}$ be an isotone derivation of an ordered Γ-semiring M. Define $ker\ d_{a,\beta} = \{x \in M/d_{a,\beta}(x) = 0\}$. Then $ker\ d_{a,\beta}$ is a $k-$ideal of an ordered Γ-semiring M.*

Proof. Let $x, y \in ker\ d_{a,\beta}$ and $\alpha \in \Gamma$. Then

$$x\beta a = y\beta a = 0,$$
$$d_{a,\beta}(x + y) = (x + y)\beta a = 0.$$

Therefore $x + y \in ker\ d_{a,\beta}$.

$$\begin{aligned} d_{a,\beta}(x\alpha y) &= d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) \\ &= (x\beta a)\alpha y + x\alpha(y\beta a) \\ &= 0\alpha y + x\alpha 0 = 0. \end{aligned}$$

Therefore $x\alpha y \in ker\ d_{a,\beta}$.

Suppose $y \in ker\ d_{a,\beta}, x \in M$ and $x \leq y$. Then

$$d_{a,\beta}(x) \leq d_{a,\beta}(y),$$
$$x\beta a \leq y\beta a = 0,$$
$$x\beta a = 0,$$
$$x \in ker\ d_{a,\beta}.$$

Hence $ker\ d_{a,\beta}$ is an ideal.

Suppose $x + y \in ker\ d_{a,\beta}$ and $y \in ker\ d_{a,\beta}$. Then

$$d_{a,\beta}(x + y) = 0,$$
$$d_{a,\beta}(x) + d_{a,\beta}(y) = 0,$$
$$d_{a,\beta}(x) = 0,$$

$$x \in ker\ d_{a,\beta}.$$

This completes the proof. □

Theorem 3.13. *Let $d_{a,\beta}$ be an isotone derivation of an integral ordered Γ-semiring M. Then $ker\ d_{a,\beta}$ is a $m-k$ ideal of M.*

Proof. By Theorem 3.12, $ker\ d_{a,\beta}$ is an ideal of an ordered Γ-semiring M.

Let $0 \neq y \in ker\ d_{a,\beta}$, $x \in M$ $\alpha \in \Gamma$ and $x\alpha y \in ker\ d_{a,\beta}$. Then

$$d_{a,\beta}(x\alpha y) = 0,$$
$$d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) = 0,$$
$$d_{a,\beta}(x)\alpha y = 0,$$
$$d_{a,\beta}(x) = 0,$$

since M is an integral ordered Γ-semiring.

Therefore $ker\ d_{a,\beta}$ is a $m-k$ ideal of M. □

Theorem 3.14. *Let $d_{a,\beta}$ be a derivation of multiplicatively cancellative commutative idempotent ordered Γ-semiring M where $(M,+)$ is positively ordered and band, $\Gamma-$semigroup M is negatively ordered and $d_{a,\beta}(1) = 1$. Define a set $Fixd_{a,\beta}(M) = \{x \in M \mid d_{a,\beta}(x) = x\}$. Then $Fixd_{a,\beta}(M)$ is a $m-k$ ideal.*

Proof. Obviously $Fixd_{a,\beta}(M) = \{x \in M/d_{a,\beta}(x) = x\}$ is an ideal of M. Suppose $x\alpha y \in Fixd_{a,\beta}(M)$, $x \in Fixd_{a,\beta}(M)$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
d_{a,\beta}(x\alpha y) &= x\alpha y,\\
d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) &= x\alpha y,\\
x\alpha y + x\alpha d_{a,\beta}(y) &= x\alpha y,\\
x\alpha[y + d_{a,\beta}(y)] &= x\alpha y,\\
y + d_{a,\beta}(y) &= y,\\
d_{a,\beta}(y) \leq y + d_{a,\beta}(y) &= y.
\end{aligned}$$

By Theorem 3.9, we have $y \leq d_{a,\beta}(y)$. Hence $d_{a,\beta}(y) = y$, $y \in Fixd_{a,\beta}(M)$.

Therefore, $Fixd_{a,\beta}(M)$ is a $m-k-$ideal of M. □

Theorem 3.15. *Let M be an ordered commutative Γ-semiring and $d_{a,\beta}$, $d_{b,\gamma}$ be derivations of M. If $d_{a,\beta}\ d_{b,\gamma} = 0$ then $d_{b,\gamma}\ d_{a,\beta}$ is a derivation of M.*

Proof. Let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
0 &= d_{a,\beta}\ d_{b,\gamma}\ (x\alpha y)\\
&= d_{a,\beta}[\ d_{b,\gamma}\ (x)\alpha y + x\alpha\ d_{b,\gamma}\ (y)]\\
&= d_{a,\beta}(\ d_{b,\gamma}\ (x))\alpha y +\ d_{b,\gamma}\ (x)\alpha d_{a,\beta}(y) + d_{a,\beta}(x)\alpha\ d_{b,\gamma}\ (y) + x\alpha d_{a,\beta}(\ d_{b,\gamma}\ (y)),\\
0 &= d_{b,\gamma}\ (x)\alpha d_{a,\beta}(y) + d_{a,\beta}(x)\alpha\ d_{b,\gamma}\ (y).
\end{aligned}$$

$$\begin{aligned} d_{b,\gamma}\ d_{a,\beta}(x+y) &= \ d_{b,\gamma}\ [d_{a,\beta}(x)+d_{a,\beta}(y)] \\ &= \ d_{b,\gamma}\ (d_{a,\beta}(x))+\ d_{b,\gamma}\ (d_{a,\beta}(y)) \end{aligned}$$

and

$$\begin{aligned} d_{b,\gamma}\ d_{a,\beta}(x\alpha y) &= d_{b,\gamma}\ [d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y)] \\ &= d_{b,\gamma}\ d_{a,\beta}(x)\alpha y + d_{a,\beta}(x)\alpha\ d_{b,\gamma}\ (y) +\ d_{b,\gamma}\ (x)\alpha d_{a,\beta}(y) \\ &\quad + x\alpha\ d_{b,\gamma}\ (d_{a,\beta}(y)) \\ &= d_{b,\gamma}\ (d_{a,\beta}(x))\alpha y + x\alpha\ d_{b,\gamma}\ (d_{a,\beta}(y)). \end{aligned}$$

Hence $d_{b,\gamma}\ d_{a,\beta}$ is a derivation of M. □

Theorem 3.16. *Let $d_{a,\beta}$ be a derivation of an ordered integral Γ-semiring M with unity and $b \in M, \alpha \in \Gamma$. Then if $b\alpha d_{a,\beta}(x) = 0$, for all $x \in M, \alpha \in \Gamma$. Then either $b = 0$ or $d_{a,\beta}$ is zero.*

Proof. Suppose $b\alpha d_{a,\beta}(x) = 0$, for all $x \in M$. Let $y \in M$. Replace x by $x\alpha y$, then

$$\begin{aligned} b\alpha d_{a,\beta}(x\alpha y) &= 0, \\ b\alpha[d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y)] &= 0, \\ b\alpha x\alpha d_{a,\beta}(y) &= 0, \quad \alpha \in \Gamma. \end{aligned}$$

Since $b \in M$, there exists $\gamma \in \Gamma$ such that $b\gamma 1 = b$, then

$$\begin{aligned} b\gamma d_{a,\beta}(y) &= 0, \\ b = 0 \text{ or } d_{a,\beta}(y) &= 0, \\ b = 0 \text{ or } d_{a,\beta} &= 0. \end{aligned}$$

This completes the proof. □

Theorem 3.17. *Let M be an ordered Γ-semiring in which $(M, +)$ is positively ordered and band,$\Gamma-$semigroup M is negatively ordered and $d_{a,\beta}$ be a derivation of M. Then the following hold,*

(i) $d_{a,\beta}(x\alpha y) \le d_{a,\beta}(x)$,
(ii) $d_{a,\beta}(x\alpha y) \le d_{a,\beta}(y)$,
(iii) $x \le y$ *then* $d_{a,\beta}(x) \le d_{a,\beta}(y)$, *for all* $x, y \in M, \alpha \in \Gamma$.

Proof. (i) Let $x, y \in M, \alpha \in \Gamma$. Then

$$d_{a,\beta}(x\alpha y) = (x\alpha y)\beta a \le x\beta a = d_{a,\beta}(x).$$

(ii) Similarly we can prove $d_{a,\beta}(x\alpha y) \le d_{a,\beta}(y)$.
(iii) Suppose $x \le y$. Then

$$\begin{aligned} x + y &\le y + y, \\ x + y \le y &\le x + y, \\ x + y &= y, \\ d_{a,\beta}(x + y) &= d_{a,\beta}(y), \end{aligned}$$

$$d_{a,\beta}(x) + d_{a,\beta}(y) = d_{a,\beta}(y).$$

Therefore $d_{a,\beta}(x) \leq d_{a,\beta}(y)$. $\square$

Theorem 3.18. *Let $d_{a,\beta}$ be a derivation of an integral ordered Γ-semiring M in which semigroup $(M,+)$ is a band. Define $d^2_{a,\beta}(x) = d_{a,\beta}(d_{a,\beta}(x))$, for all $x \in M$. If $d^2_{a,\beta} = 0$ then $d_{a,\beta} = 0$.*

Proof. Let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
&d^2_{a,\beta}(x\alpha y) = 0,\\
&d_{a,\beta}[d_{a,\beta}(x\alpha y)] = 0,\\
&d_{a,\beta}[d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y)] = 0,\\
&d^2_{a,\beta}(x)\alpha y + d_{a,\beta}(x)\alpha d_{a,\beta}(y) + d_{a,\beta}(x)\alpha d_{a,\beta}(y) + x\alpha d^2_{a,\beta}(y) = 0,\\
&d_{a,\beta}(x)\alpha d_{a,\beta}(y) + d_{a,\beta}(x)\alpha d_{a,\beta}(y) = 0,\\
&d_{a,\beta}(x)\alpha d_{a,\beta}(y) = 0,\\
&d_{a,\beta}(x) = 0 \text{ } \mathit{or} \text{ } d_{a,\beta}(y) = 0,
\end{aligned}$$

for all $x, y \in M$. Therefore in both the cases we have $d_{a,\beta} = 0$.

This completes the proof. $\square$

Theorem 3.19. *Let I be a non-zero ideal of an integral ordered Γ-semiring M in which $\Gamma-$semigroup M is negatively ordered. If $d_{a,\beta}$ is a non-zero derivation of M then $d_{a,\beta}$ is non-zero on I.*

Proof. Suppose $d_{a,\beta}$ is a non-zero derivation of M and $d_{a,\beta}(x) = 0$ for all $x \in I$. Let $y \in M, \alpha \in \Gamma$ and $x \in I$. Then $x\alpha y \leq x$. Therefore $x\alpha y \in I$, and

$$\begin{aligned}
d_{a,\beta}(x\alpha y) &= 0,\\
d_{a,\beta}(x)\alpha y + x\alpha d_{a,\beta}(y) &= 0,\\
x\alpha d_{a,\beta}(y) &= 0.
\end{aligned}$$

Since M has no zero divisors, we have $x = 0$ or $d_{a,\beta}(y) = 0$, for all $y \in M$.

Further, since I is a non-zero ideal, we get $d_{a,\beta}(y) = 0$, for all $y \in M$, which is a contradiction to $d_{a,\beta} \neq 0$ on M. Hence $d_{a,\beta}$ is non-zero derivation on I.

This completes the proof. $\square$

Theorem 3.20. *Let $d_{a,\beta}$ be a non-zero derivation of an integral ordered Γ-semiring M. If I is a non-zero ideal of M and $t \in M$ such that $t\alpha d_{a,\beta}(I) = 0$ then $t = 0$.*

Proof. By Theorem 3.19, there exists $x \in I$ such that $d_{a,\beta}(x) \neq 0$.

Suppose $t\alpha d_{a,\beta}(I) = 0$. Then

$$\begin{aligned}
t\alpha d_{a,\beta}(x\alpha x) &= 0,\\
t\alpha[d_{a,\beta}(x)\alpha x + x\alpha d_{a,\beta}(x)] &= 0,
\end{aligned}$$

$$t\alpha d_{a,\beta}(x)\alpha x + t\alpha(x\alpha d_{a,\beta}(x)) = 0,$$
$$t\alpha(x\alpha d_{a,\beta}(x)) = 0.$$

Therefore $t = 0$.
This completes the proof. □

4. Conclusion

In this paper, we introduced the concept of derivation $d_{a,\beta}$ of ordered Γ-semiring. We studied some of the properties of derivation $d_{a,\beta}$ of ordered Γ-semirings. We proved that if a derivation $d_{a,\beta}$ is non-zero on an integral Γ-semiring M, then it is non-zero on any non-zero ideal of M and we characterized k-ideal and $m-k$ ideal using derivation $d_{a,\beta}$ of ordered Γ-semiring.

References

[1] M. J. Atteya, G. Haetinger, D. I. Rasen, (σ,τ)*-derivations of semiprime rings*, Kragujevac Journal of Mathematics, 43 (2) (2019), 239–246.

[2] M. Bresar, J. Vukman, *On the left derivation and related mappings,* Proc. Amer. Math. Soc., 10 (1990), 7–16.

[3] N. Damljanović, M. Ćirić, J. Ignjatović, *Bisimulations for weighted automata over an additively idempotent semiring*, Theoretical computer science, 534 (2014), 86–100.

[4] N. Damljanović, M. Ćirić, S. Bogdanović, *Congruence openings of additive Green's relation on a semiring*, Semigroup Forum, 82 (3) (2011), 437–454.

[5] B. Dhara, *Generalized derivations with left annihilator conditions in prime and semiprime rings*, Discussiones Mathematicae General Algebra and Applications, 37 (2) (2017), 161–175.

[6] K. Glazek, *A Guide to the Literature on Semirings and Their Applications in Mathematics and Information Sciences*, Kluwer Academic Publishers, Dordrechnt, 2002.

[7] W. Kuich, A. Salomaa, *Semirings, Automata, Languages*, EATCS Monographs in Theoretical Computer Science, Springer, 1986.

[8] H. Lehmer, *A ternary analogue of abelian groups,* American J. of Math., 59(1932), 329– 338.

[9] W. G. Lister, *Ternary rings,* Tran. Of American Math. Society, 154 (1971), 37–55.

[10] N. Nobusawa, *On a generalization of the ring theory,* Osaka. J. Math., 1 (1964), 81–89.

[11] E. C. Posner, *Derivations in prime rings,* Proc. Amer. Math. Soc., 8 (1957), 1093–1100.

[12] M. M. K. Rao, Γ*-semirings-I,* Southeast Asian Bulletin of Mathematics, 19(1)(1995), 49–54.

[13] M. M. K. Rao, Γ*-semirings-II,* Southeast Asian Bulletin of Mathematics, 21(3)(1997), 281–287.

[14] M. M. K. Rao, B. Venkateswarlu, *Right Derivation of ordered* $\Gamma-$*semirings,* Discussiones Mathematicae General Algebra and Applications, 36 (2016), 209–221.

[15] M. M. K. Rao, B. Venkateswarlu, *On generalized right derivations of* $\Gamma-$ *incline,* Journal Of The International Mathematical Virtual Institute, 6 (2016), 31–47.

[16] M. M. K. Rao, *Ideals in ordered* $\Gamma-$*semirings,* Discussiones Mathematicae General Algebra and Applications, 38 (2018) 47–68.

[17] M. M. K. Rao, B. Venkakteswarlu, B. R. Kumar, K. R. Kumar, (f,g)*-derivation of ordered* Γ*-semirings*, Mathematica Moravica, 22 (1) (2018), 107–121.

[18] M. K. Sen, *On* Γ*-semigroup,* Proc. of International Conference of algebra and its application (1981), Decker Publication, New York, 301–308.

[19] H. S. Vandiver, *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc. (N.S.), 40 (1934), 914–920.

MARAPUREDDY MURALI KRISHNA RAO
DEPARTMENT OF MATHEMATICS
GIT, GITAM UNIVERSITY
VISAKHAPATNAM – 530 045
INDIA
E-mail address: mmarapureddy@gmail.com

KONA RAJENDR KUMAR
DEPARTMENT OF MATHEMATICS
GIT, GITAM UNIVERSITY
VISAKHAPATNAM – 530 045
INDIA
E-mail address: rkkona72@rediffmail.com

BOLINENI VENKATESWARLU
DEPARTMENT OF MATHEMATICS
GST, GITAM UNIVERSITY
DODDABALLAPURA – 561 203
BENGALURU RUAL
INDIA
E-mail address: bvlmaths@gmail.com

BANDARU RAVI KUMAR
DEPARTMENT OF MATHEMATICS
GST, GITAM UNIVERSITY
MEDAK DISTRICT – 502329
HYDERBAD
INDIA
E-mail address: ravimaths83@gmail.com