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Strong commutativity preserving derivations on Lie ideals of prime Γ-rings

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Abstract. Let M be a Γ-ring and $S \subseteq M$. A mapping $f : M \to M$ is called *strong commutativity preserving* on S if $[f(x), f(y)]_\alpha = [x, y]_\alpha$, for all $x, y \in S$, $\alpha \in \Gamma$. In the present paper, we investigate the commutativity of the prime Γ-ring M of characteristic not 2 with center $Z(M) \neq (0)$ admitting a derivation which is strong commutativity preserving on a nonzero square closed Lie ideal U of M. Moreover, we also obtain a related result when a mapping d is assumed to be a derivation on U satisfying the condition $d(u) \circ_\alpha d(v) = u \circ_\alpha v$, for all $u, v \in U$, $\alpha \in \Gamma$.

1. Introduction

Nobusawa [13] developed the concept of a gamma ring and then Barnes [1] weakened slightly the defining conditions for a gamma ring. After these definitions a number of mathematicians have studied on gamma rings in the sense of Barnes and Nobusawa and get results parallel to the ring theory (see for example [1], [11], [9]).

Let R be any ring. The symbol $[a, b]$ denotes $ab - ba$ for $a, b \in R$. R is called *prime* if $aRb = (0)$ implies either $a = 0$ or $b = 0$, and R is called *semiprime* if $aRa = (0)$ implies $a = 0$. An additive mapping d is called a *derivation* on R if

$$d(ab) = d(a)b + ad(b)$$

holds for all $a, b \in R$.

A mapping f is said to be *commutativity preserving* on R if $[f(a), f(b)] = 0$ whenever $[a, b] = 0$, for all $a, b \in R$. In 1976, Watkins [14] obtained the first result on commutativity preserving maps for a $n \times n$ matrix algebra when $n \geq 4$ and f is a monomorphism on R. Recently, the study of commutativity preserving maps has become an active research area in ring theory (see for example [4], [6], [8], [12] and references therein).

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Let S be a subset of R. A map f is called *strong commutativity preserving* (SCP) on S if $[f(a), f(b)] = [a, b]$, for all $a, b \in S$. Clearly, a map that is strong commutativity preserving on a set S is also commutativity preserving on S, but the inverse is not true in general. The notion of a strong commutativity preserving map was first introduced by H.E. Bell and G. Mason [3]. Later, H.E. Bell and M.N. Daif [2] proved that if a semiprime ring R admits a nonzero derivation which is strong commutativity preserving on a right ideal ρ of R, then $\rho \subseteq Z(R)$ where $Z(R)$ is the center of R. In particular, R is commutative if $\rho = R$. M. Brešar and C.R. Miers [5] characterized SCP additive maps on a semiprime ring. In [10], Brešar and Miers's result was extended to Lie ideals of prime rings by J.-S. Lin and C.-K. Liu. Later, Q. Deng and M. Ashraf [7] proved that if there exists a derivation d of a semiprime ring R and a mapping $f : I \to R$ defined on a nonzero ideal I of R such that $[f(a), d(b)] = [a, b]$, for all $a, b \in I$, then R contains a nonzero central ideal. They also showed that R is commutative when $I = R$. There are lots of generalizations similar to these results can be found in the literature.

Recently, X. Xu, J. Ma and Y. Zhou [15] proved that a semiprime Γ-ring with a strong commutativity preserving derivation on itself must be commutative and that a strong commutativity preserving endomorphism σ on a semiprime Γ-ring M must have the form $\sigma(a) = a + \xi(a)$ $(a \in M)$ where ξ is a map from M into its center, which extends some results by Bell and Daif to semiprime Γ-rings.

Motivated by all these results, in the present paper, we study strong commutativity preserving derivations on a nonzero square closed Lie ideal of prime Γ-rings and prove that if M is a prime Γ-ring of characteristic not 2 such that its center $Z(M) \neq (0)$ and d is a SCP derivation on a nonzero square closed Lie ideal U of M, then $U \subseteq Z(M)$. In particular, M is commutative if $U = M$. Moreover, we also obtain the same result when a mapping d is assumed to be a derivation on U satisfying the condition $d(u) \circ_\alpha d(v) = u \circ_\alpha v$, for all $u, v \in U$, $\alpha \in \Gamma$.

2. Preliminaries

Before giving our results, we first present some preliminary definitions. In this paper, M will represent a Γ-ring in the sense of Barnes [1] unless otherwise stated.

An additive subgroup K of a Γ-ring M is called a *left (resp. right) ideal* of M if $M\Gamma K \subseteq K$ (resp. $K\Gamma M \subseteq K$). A left ideal K of a Γ-ring M is called an *ideal* of M if it is also a right ideal of M. The set of all elements a satisfying $a\alpha b = b\alpha a$ for all $b \in M$ and $\alpha \in \Gamma$ is called the *center* of M.

A Γ-ring M is said to be *prime* if $a\Gamma M\Gamma b = (0)$ for $a, b \in M$ implies that $a = 0$ or $b = 0$. An additive mapping d is called a *derivation* on M if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$.

Let M be a Γ-ring and $a, b \in M$, $\alpha \in \Gamma$. The commutator of a and b with respect to α is defined as the element $a\alpha b - b\alpha a$ and denoted by $[a, b]_\alpha$. According to this definition we have the following equations,

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha[b, c]_\beta + a\alpha c\beta b - a\beta c\alpha b, \tag{1}$$

$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha[a, c]_\beta + b\beta a\alpha c - b\alpha a\beta c, \tag{2}$$

where $a, b, c \in M$, $\alpha, \beta \in \Gamma$. Similarly, the anti-commutator of a and b with respect to α is defined as the element $a\alpha b + b\alpha a$ and denoted by $a \circ_\alpha b$. According to this definition we have the following equations,

$$\begin{aligned} (a\alpha b) \circ_\beta c &= a\alpha(b \circ_\beta c) - [a, c]_\beta \alpha b + a\alpha c\beta b - a\beta c\alpha b \\ &= (a \circ_\beta c)\alpha b + a\alpha[b, c]_\beta + a\beta c\alpha b - a\alpha c\beta b, \end{aligned}$$

$$\begin{aligned} a \circ_\beta (b\alpha c) &= (a \circ_\beta b)\alpha c - b\alpha[a, c]_\beta + b\beta a\alpha c - b\alpha a\beta c \\ &= b\alpha(a \circ_\beta c) + [a, b]_\beta \alpha c + b\alpha a\beta c - b\beta a\alpha c, \end{aligned}$$

where $a, b, c \in M$, $\alpha, \beta \in \Gamma$.

An additive subgroup U of a Γ-ring M is called a *Lie ideal* if $[u, m]_\alpha \in U$, for all $u \in U$, $m \in M$ and $\alpha \in \Gamma$. A Lie ideal U of M is said to be a *square closed Lie ideal* of M, if $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. Clearly, $u\alpha v + v\alpha u \in U$, for all $u, v \in U$, $\alpha \in \Gamma$. Similarly, we have $u\alpha v - v\alpha u \in U$. Moreover, by using these relations, we get $2u\alpha v \in U$ which will be used in the whole paper frequently.

A map f from a Γ-ring M into itself is called *strong commutativity preserving* (SCP) on a subset S of M if $[f(a), f(b)]_\alpha = [a, b]_\alpha$ holds for all $a, b \in S$ and $\alpha \in \Gamma$.

3. The Results

First, we work on SCP derivations on Lie ideals of prime Γ-rings. The following lemma will play an crucial role in the proofs of our main theorems.

Lemma 3.1. *Let M be a prime Γ-ring and $Z(M) \neq (0)$. Then the equations*

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha[b, c]_\beta,$$

$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha[a, c]_\beta$$

hold for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$.

Proof. For any $c \in M$, $\alpha, \beta \in \Gamma$, the symbol $[\alpha, \beta]_c$ denotes $\alpha c\beta - \beta c\alpha$. Then, the commutator formulas in (1) and (2) become

$$[a\alpha b, c]_\beta = [a, c]_\beta \alpha b + a\alpha[b, c]_\beta + a[\alpha, \beta]_c b \tag{3}$$

and

$$[a, b\alpha c]_\beta = [a, b]_\beta \alpha c + b\alpha[a, c]_\beta + b[\beta, \alpha]_a c,$$

for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$.

Since $Z(M) \neq (0)$, there exists a nonzero element x in $Z(M)$. Thus,

$$\begin{aligned} x\gamma y\delta a\alpha c\beta b &= y\gamma x\delta a\alpha c\beta b = y\gamma a\delta x\alpha c\beta b \\ &= y\gamma a\delta c\alpha x\beta b = y\gamma a\delta c\alpha b\beta x \\ &= y\gamma a\delta x\beta c\alpha b = y\gamma x\delta a\beta c\alpha b \\ &= x\gamma y\delta a\beta c\alpha b, \end{aligned}$$

for all $a, b, c, y \in M$, $\alpha, \beta, \gamma, \delta \in \Gamma$. Then we have that

$$x\gamma y\delta a[\alpha, \beta]_c b = 0, \tag{4}$$

for all $a, b, c, y \in M$, $\alpha, \beta, \gamma, \delta \in \Gamma$. Multiplying the two sides of (3) by $x\gamma y\delta$ from the left hand side, and then comparing with (4) we get for all $a, b, c, y \in M$, $\alpha, \beta, \gamma, \delta \in \Gamma$

$$x\gamma y\delta[a\alpha b, c]_\beta = x\gamma y\delta[a, c]_\beta\alpha b + x\gamma y\delta a\alpha[b, c]_\beta.$$

That is $x\Gamma M\Gamma([a\alpha b, c]_\beta - [a, c]_\beta\alpha b - a\alpha[b, c]_\beta) = 0$, for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$. Since M is prime and x is nonzero, we have

$$[a\alpha b, c]_\beta - [a, c]_\beta\alpha b - a\alpha[b, c]_\beta = 0,$$

for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$. For the second equation, one can use the same method above, and this completes the proof. □

Now, we can give a similar result for the anti-commutator formulas of Γ-rings.

Lemma 3.2. *Let M be a prime Γ-ring in the sense of Barnes and $Z(M) \neq (0)$. Then the equations*

$$\begin{aligned} (a\alpha b) \circ_\beta c &= a\alpha(b \circ_\beta c) - [a, c]_\beta\alpha b \\ &= (a \circ_\beta c)\alpha b + a\alpha[b, c]_\beta, \end{aligned}$$

$$\begin{aligned} a \circ_\beta (b\alpha c) &= (a \circ_\beta b)\alpha c - b\alpha[a, c]_\beta \\ &= b\alpha(a \circ_\beta c) + [a, b]_\beta\alpha c \end{aligned}$$

hold for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$.

Proof. It can be proved by using the techniques of Lemma 3.1. □

We need the following results to prove our main theorems.

Lemma 3.3. *Let M be a prime Γ-ring of characteristic not 2 with the center $Z(M) \neq (0)$ and U be a Lie ideal of M. If $U \nsubseteq Z(M)$, then there exists an ideal K of M such that $[K, M]_\Gamma \subseteq U$ but $[K, M]_\Gamma \nsubseteq Z(M)$.*

Proof. First, we show that the Lie product of U by itself is different from zero. Suppose that $[U, U]_\Gamma = (0)$. Then we have $[a, [a, m]_\alpha]_\beta = 0$, for all $a \in U$, $m \in M$ and $\alpha, \beta \in \Gamma$. Replacing m by $m\gamma x$ for $\gamma \in \Gamma$ and $x \in M$, we get

$$[a, m]_\beta\,\gamma\,[a, x]_\alpha + [a, m]_\alpha\,\gamma\,[a, x]_\beta = 0. \tag{5}$$

Now, replacing β by α in (5) we have $[a,m]_\alpha \gamma [a,x]_\alpha = 0$, for all $a \in U$, $m,x \in M$ and $\alpha,\gamma \in \Gamma$. Replacing x by $y\delta x$ for $y \in M$ and $\delta \in \Gamma$ in the last equation, we get $[a,m]_\alpha \Gamma M \Gamma [a,x]_\alpha = (0)$, for all $a \in U$, $m,x \in M$ and $\alpha \in \Gamma$. Therefore, we have $U \subseteq Z(M)$ since M is prime. But this contradicts with the hypothesis of the theorem. Hence, there exist $u,v \in U$ and $\beta \in \Gamma$ such that $[u,v]_\beta \neq 0$.

Let $K := M\Gamma [u,v]_\beta \Gamma M$ and $T(U) := \{x \in M \mid [x,M]_\Gamma \subseteq U\}$. Then, it is clear that $K \neq (0)$ is an ideal of M; $T(U)$ is a Lie ideal and a subring of M. Moreover, $U \subseteq T(U)$. Since $[u, v\gamma m]_\beta = [u,v]_\beta \gamma m + v\gamma [u,m]_\beta$ for all $m \in M$ and $\gamma \in \Gamma$, we get $[u,v]_\beta \Gamma M \subseteq T(U)$. Hence,

$$\Big[[u,v]_\beta \alpha m, n\Big]_\gamma \in T(U),$$

for all $n,m \in M$ and $\alpha,\gamma \in \Gamma$. Expanding this we get $n\gamma [u,v]_\beta \alpha m \in T(U)$ for all $n,m \in M$ and $\alpha,\gamma \in \Gamma$. Then, we have $M\Gamma [u,v]_\beta \Gamma M = K \subseteq T(U)$ which yields to $[K,M]_\Gamma \subseteq U$.

Now, suppose $[K,M]_\Gamma \subseteq Z(M)$. Therefore, we have $[K,[K,M]_\Gamma]_\Gamma = (0)$ and using the same argument above we get $K \subseteq Z(M)$. Let $x \in M$. Then $n\alpha k\gamma m \in K$ for all $n,m \in M$, $k \in K$ and $\alpha,\gamma \in \Gamma$. Since $K \subseteq Z(M)$ we have $[x, n\alpha k\gamma m]_\delta = 0$. Expanding this we get $K\Gamma M\Gamma [x,M]_\Gamma = (0)$. Therefore, $x \in Z(M)$ since M is prime and $K \neq (0)$. But this contradicts with $U \nsubseteq Z(M)$. This completes the proof. □

Lemma 3.4. *Let M be a prime Γ-ring of characteristic not 2 with the center $Z(M) \neq (0)$ and U be a Lie ideal of M. If $U \nsubseteq Z(M)$ and $a,b \in M$ such that $a\Gamma U\Gamma b = (0)$, then either $a = 0$ or $b = 0$.*

Proof. By Lemma 3.3, there exists an ideal K of M such that $[K,M]_\Gamma \subseteq U$ but $[K,M]_\Gamma \nsubseteq Z(M)$. Let $u \in U$, $k \in K$, $m \in M$ and $\alpha,\beta,\gamma \in \Gamma$. Then, we have

$$[k\alpha a\beta u, m]_\gamma \in [K,M]_\Gamma \subseteq U.$$

It follows from that

$$\begin{aligned} 0 &= a\lambda[k\alpha a\beta u, m]_\gamma \epsilon b = a\lambda k\alpha a\beta[u,m]_\gamma \epsilon b + a\lambda[k\alpha a, m]_\gamma \beta u\epsilon b \\ &= a\lambda k\alpha a\gamma m\beta u\epsilon b - a\lambda m\gamma k\alpha a\beta u\epsilon b \\ &= a\lambda k\alpha a\gamma m\beta u\epsilon b, \end{aligned}$$

for all $u \in U$, $k \in K$, $m \in M$ and $\alpha,\beta,\gamma,\lambda,\epsilon \in \Gamma$. Therefore, we get $a\Gamma K\Gamma a = (0)$ or $U\Gamma b = (0)$ since M is prime. In the first case, we see that a must be zero by using the primeness of M. In the second case, we get

$$[u,m]_\alpha \gamma b = 0,$$

for all $u \in U$, $m \in M$ and $\alpha,\gamma \in \Gamma$. Expanding this we have

$$[u\gamma b, m]_\alpha - u\gamma[b,m]_\alpha = 0,$$

that is $u\gamma m\alpha b = 0$, for all $u \in U$, $m \in M$ and $\alpha, \gamma \in \Gamma$. Therefore, $b = 0$ since M is prime and $U \neq (0)$. □

Lemma 3.5. *Let M be a prime Γ-ring with the center $Z(M) \neq (0)$ and $x \in M$. If $a \in Z(M)$ and $a\gamma x \in Z(M)$ for all $\gamma \in \Gamma$, then $a = 0$ or $x \in Z(M)$.*

Proof. Suppose that $a \neq 0$. Since $a\gamma x \in Z(M)$, we have $[a\gamma x, m]_\delta = 0$ for all $m \in M$ and $\delta, \gamma \in \Gamma$. Expanding this we get $a\gamma[x, m]_\delta = 0$. Replacing m by $m\beta n$ for $n \in M$ and $\beta \in \Gamma$ we conclude that $x \in Z(M)$ since M is prime. This completes the proof. □

Lemma 3.6. *Let M be a prime Γ-ring of characteristic not 2 with the center $Z(M) \neq (0)$ and U be a Lie ideal of M. If $[U, U]_\Gamma \subseteq Z(M)$, then $U \subseteq Z(M)$.*

Proof. By hypothesis we have $[u, [u, x]_\alpha]_\beta \in Z(M)$ for all $u \in U$, $x \in M$ and $\alpha, \beta \in \Gamma$. Since

$$[u, [u, x]_\alpha]_\beta \gamma u = [u, [u, x]_\alpha \gamma u]_\beta = [u, [u, x\gamma u]_\alpha]_\beta$$

and $[u, [u, x\gamma u]_\alpha]_\beta \in [U, U]_\Gamma$, we have $[u, [u, x]_\alpha]_\beta \gamma u \in Z(M)$. Therefore, we get $[u, [u, x]_\alpha]_\beta = 0$ or $u \in Z(M)$ by Lemma 3.5. Now, let $[u, [u, x]_\alpha]_\beta = 0$ for all $x \in M$, $\alpha, \beta \in \Gamma$ and for some $u \in U$. Replacing x by $x\gamma m$ we get

$$[u, x]_\beta \gamma [u, m]_\alpha + [u, x]_\alpha \gamma [u, m]_\beta = 0, \tag{6}$$

for all $x, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Replacing β by α in the equation (6) we get $[u, x]_\alpha \gamma [u, m]_\alpha = 0$ since M is a Γ-ring of characteristic not 2. Replacing m by $m\delta n$ for $n \in M$, $\delta \in \Gamma$ in the last equation, we conclude that $u \in Z(M)$ since M is prime. Consequently, we see that U must be a subset of $Z(M)$. □

Theorem 3.1. *Let M be a prime Γ-ring of characteristic not 2 and U be a nonzero square closed Lie ideal of M. If d is a SCP derivation on U, then $U \subseteq Z(M)$ or $Z(M) = (0)$.*

Proof. Suppose that $Z(M) \neq (0)$. We have $[d(x), d(y)]_\alpha = [x, y]_\alpha$ for all $x, y \in U$ and $\alpha \in \Gamma$ by hypothesis. Replacing y by $2y\beta z$ for $z \in U$ and $\beta \in \Gamma$, we get

$$[d(x), d(2y\beta z)]_\alpha = [x, 2y\beta z]_\alpha,$$

for all $x, y, z \in U$ and $\alpha, \beta \in \Gamma$. By applying Lemma 3.1, we expand the last equation and we get

$$d(y)\beta[d(x), z]_\alpha + [d(x), y]_\alpha \beta d(z) = 0, \tag{7}$$

since M is a Γ-ring of characteristic not 2. Replacing z by $2z\gamma t$ for $z, t \in U$ and $\gamma \in \Gamma$ in the equation (7) we obtain that

$$\begin{aligned} d(y)\beta[d(x), z]_\alpha \gamma t + d(y)\beta z\gamma[d(x), t]_\alpha + [d(x), y]_\alpha \beta d(z)\gamma t \\ +[d(x), y]_\alpha \beta z\gamma d(t) = 0, \end{aligned} \tag{8}$$

since M is a Γ-ring of characteristic not 2. Multiplying the two sides of (7) by γt from the right hand side, we have

$$d(y)\beta[d(x),z]_\alpha\gamma t+[d(x),y]_\alpha\beta d(z)\gamma t=0, \tag{9}$$

for all $x,y,z,t\in U$ and $\alpha,\beta,\gamma\in\Gamma$. Comparing (9) with (8), we have that

$$d(y)\beta z\gamma[d(x),t]_\alpha+[d(x),y]_\alpha\beta z\gamma d(t)=0,$$

for all $x,y,z,t\in U$ and $\alpha,\beta,\gamma\in\Gamma$. Since U is a nonzero square closed Lie ideal of M, we have $[U,U]_\Gamma$ is a nonzero square closed Lie ideal of M, too. Writing $t=d(x)$ for $x\in[U,U]_\Gamma$, we obtain that

$$[d(x),y]_\alpha\beta z\gamma d^2(x)=0, \tag{10}$$

for all $y,z\in U$, $x\in[U,U]_\Gamma$ and $\alpha,\beta,\gamma\in\Gamma$. If we replace y by $d(y)$ for $y\in[U,U]_\Gamma$ in the equation (10), we obtain $[x,y]_\alpha\Gamma U\Gamma d^2(x)=(0)$ for all $x,y\in[U,U]_\Gamma$, and $\alpha\in\Gamma$ since d is SCP on U. Therefore,

$$[x,y]_\alpha\beta 2[m,z]_\alpha\Gamma U\Gamma[x,y]_\alpha\beta 2[m,z]_\alpha=(0),$$

since

$$[x,y]_\alpha\Gamma U\Gamma[d^2(x),d^2(y)]_\alpha\beta 2[m,z]_\alpha=(0),$$

for all $x,y\in[U,U]_\Gamma$, $m\in M$, $z\in U$ and $\alpha,\beta\in\Gamma$. Since M is a Γ-ring of characteristic not 2, we have $[x,y]_\alpha\beta[m,z]_\alpha=0$ by Lemma 3.4. Replacing m by $m\gamma t$ for $t\in M$ and $\gamma\in\Gamma$ we get

$$[x,y]_\alpha\beta m\gamma[t,z]_\alpha=0,$$

for all $x,y\in[U,U]_\Gamma$, $m,t\in M$, $z\in U$ and $\alpha,\beta,\gamma\in\Gamma$. By the primeness of the Γ-ring M, we get either $[x,y]_\alpha=0$ or $[t,z]_\alpha=0$, for all $x,y\in[U,U]_\Gamma$, $z\in U$, $t\in M$ and $\alpha\in\Gamma$. In the second case, we see that $z\in Z(M)$ that is $U\subseteq Z(M)$. In the first case, using Lemma 3.6, we have $[U,U]_\Gamma\subseteq Z(M)$. Consequently, applying Lemma 3.6 again, we get that $U\subseteq Z(M)$ which completes the proof. □

In particular, if we take $U=M$, then Theorem 3.1 gives a commutativity criterion as follows.

Corollary 3.1. *Let M be a prime Γ-ring of characteristic not 2 and d be a derivation of M. If $Z(M)\neq(0)$ and d is SCP on M, then M is commutative.*

Since we can use the similar techniques of Theorem 3.1, we can obtain the following theorems which partially generalize the result of Bell and Daif to prime Γ-rings.

Theorem 3.2. *Let M be a prime Γ-ring of characteristic not 2 and U be a nonzero square closed Lie ideal of M. If $[d(x),d(y)]_\alpha=-[x,y]_\alpha$ for all $x,y\in U$ and $\alpha\in\Gamma$, then $U\subseteq Z(M)$ or $Z(M)=(0)$.*

Proof. It can be proved easily by using the same method in Theorem 3.1. □

Corollary 3.2. *Let M be a prime Γ-ring of characteristic not 2 and d be a derivation of M. If $Z(M) \neq (0)$ and $[d(x), d(y)]_\alpha = -[x, y]_\alpha$ for all $x, y \in M$, $\alpha \in \Gamma$, then M is commutative.*

Theorem 3.3. *Let M be a prime Γ-ring of characteristic not 2 and U be a nonzero square closed Lie ideal of M. If d is a derivation of M such that $d(x) \circ_\alpha d(y) = x \circ_\alpha y$ for all $x, y \in U$ and $\alpha \in \Gamma$, then $U \subseteq Z(M)$ or $Z(M) = (0)$.*

Proof. Suppose that $Z(M) \neq (0)$. By the hypothesis we obtain that

$$d(x) \circ_\alpha d(y) - x \circ_\alpha y = 0, \tag{11}$$

for all $x, y \in U$ and $\alpha \in \Gamma$. Replacing x by $2x\beta z$ for $z \in U$, $\beta \in \Gamma$ in the equation (11) we get

$$d(x)\beta[z, d(y)]_\alpha - [x, d(y)]_\alpha\beta d(z) + 2x\beta y\alpha z = 0, \tag{12}$$

since M is a Γ-ring of characteristic not 2. Taking $2z\gamma x$ for z in the equation (12) we have

$$\begin{aligned} d(x)\beta[z, d(y)]_\alpha\gamma x + d(x)\beta z\gamma[x, d(y)]_\alpha - [x, d(y)]_\alpha\beta d(z)\gamma x \\ -[x, d(y)]_\alpha\beta z\gamma d(x) + 2x\beta y\alpha z\gamma x = 0, \end{aligned} \tag{13}$$

for all $x, y, z \in U$, $\alpha, \beta, \gamma \in \Gamma$. Multiplying the two sides of (12) by γx from the right hand side, we get

$$d(x)\beta[z, d(y)]_\alpha\gamma x - [x, d(y)]_\alpha\beta d(z)\gamma x + 2x\beta y\alpha z\gamma x = 0, \tag{14}$$

for all $x, y, z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. If we compare (13) and (14), we have that

$$d(x)\beta z\gamma[x, d(y)]_\alpha - [x, d(y)]_\alpha\beta z\gamma d(x) = 0, \tag{15}$$

for all $x, y, z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. Replacing z by $2z\sigma[x, d(y)]_\alpha$ for $y \in [U, U]_\Gamma$ and $\sigma \in \Gamma$ in the equation (15) we get

$$d(x)\beta z\sigma[x, d(y)]_\alpha\gamma[x, d(y)]_\alpha - [x, d(y)]_\alpha\beta z\sigma[x, d(y)]_\alpha\gamma d(x) = 0, \tag{16}$$

since M is a Γ-ring of characteristic not 2. Taking σ for γ in (15) we have

$$d(x)\beta z\sigma[x, d(y)]_\alpha = [x, d(y)]_\alpha\beta z\sigma d(x). \tag{17}$$

If we use the equation (17) in the equation (16) we get

$$[x, d(y)]_\alpha\beta z\sigma d(x)\gamma[x, d(y)]_\alpha = [x, d(y)]_\alpha\beta z\sigma[x, d(y)]_\alpha\gamma d(x)$$

and so

$$[x, d(y)]_\alpha\beta z\sigma[d(x), [x, d(y)]_\alpha]_\gamma = 0, \tag{18}$$

for all $x, z \in U$, $y \in [U, U]_\Gamma$ and $\alpha, \beta, \gamma, \sigma \in \Gamma$. Taking $\beta = \gamma$ in (18), we get

$$[x, d(y)]_\alpha\gamma z\sigma[d(x), [x, d(y)]_\alpha]_\gamma = 0, \tag{19}$$

for all $x, z \in U$, $y \in [U,U]_\Gamma$ and $\alpha, \gamma, \sigma \in \Gamma$. Multiplying the equation (19) on the left by $d(x)\gamma$ for $x \in [U,U]_\Gamma$, we have

$$d(x)\gamma[x,d(y)]_\alpha\gamma z\sigma[d(x),[x,d(y)]_\alpha]_\gamma = 0. \tag{20}$$

Taking $2d(x)\gamma z$ for z in (19) we obtain that

$$[x,d(y)]_\alpha\gamma d(x)\gamma z\sigma[d(x),[x,d(y)]_\alpha]_\gamma = 0, \tag{21}$$

for all $z \in U$, $x, y \in [U,U]_\Gamma$ and $\alpha, \gamma, \sigma \in \Gamma$ since M is a Γ-ring of characteristic not 2. Subtracting (21) from (20) we see that

$$[d(x),[x,d(y)]_\alpha]_\gamma\gamma z\sigma[d(x),[x,d(y)]_\alpha]_\gamma = 0,$$

for all $z \in U$, $x, y \in [U,U]_\Gamma$ and $\alpha, \gamma, \sigma \in \Gamma$. Therefore, by Lemma 3.4 we have that

$$[d(x),[x,d(y)]_\alpha]_\gamma = 0, \tag{22}$$

for all $x, y \in [U,U]_\Gamma$ and $\alpha, \gamma \in \Gamma$. Replacing z by x for $x \in [U,U]_\Gamma$ and $\beta = \gamma$ in (12) and using the equation (22) we conclude that $x\Gamma[U,U]_\Gamma\Gamma x = (0)$ for all $x \in [U,U]_\Gamma$ since M is a Γ-ring of characteristic not 2. We know that $[U,U]_\Gamma$ is a nonzero square closed Lie ideal of M. So by using Lemma 3.4 we get either $x = 0$ for all $x \in [U,U]_\Gamma$ or $[U,U]_\Gamma \subseteq Z(M)$. The first case contradicts with the hypothesis $[U,U]_\Gamma \neq (0)$. Then we have that $[U,U]_\Gamma \subseteq Z(M)$. Hence, applying Lemma 3.6 we obtain that $U \subseteq Z(M)$. This completes the proof. □

Corollary 3.3. *Let d be a derivation of a prime Γ-ring M of characteristic not 2. If $d(x) \circ_\alpha d(y) = x \circ_\alpha y$ for all $x, y \in M$, $\alpha \in \Gamma$ and $Z(M) \neq (0)$, then M is commutative.*

Theorem 3.4. *Let M be a prime Γ-ring of characteristic not 2 and U be a nonzero square closed Lie ideal of M. If d is a derivation of M such that $d(x) \circ_\alpha d(y) = -(x \circ_\alpha y)$ for all $x, y \in U$, $\alpha \in \Gamma$, then $U \subseteq Z(M)$ or $Z(M) = (0)$.*

Proof. Suppose that $Z(M) \neq (0)$. By the hypothesis we have that

$$d(x) \circ_\alpha d(y) + x \circ_\alpha y = 0, \tag{23}$$

for all $x, y \in U$ and $\alpha \in \Gamma$. Replacing x by $2x\beta z$ for $z \in U$, $\beta \in \Gamma$ in the equation (23) we get

$$d(x)\beta[z,d(y)]_\alpha - [x,d(y)]_\alpha\beta d(z) + 2x\beta z\alpha y = 0, \tag{24}$$

since M is a Γ-ring of characteristic not 2. Taking $2z\gamma x$ for x in the equation (24) we have

$$\begin{aligned} d(z)\gamma x\beta[z,d(y)]_\alpha + z\gamma d(x)\beta[z,d(y)]_\alpha - [z,d(y)]_\alpha\gamma x\beta d(z)\\ -z\gamma[x,d(y)]_\alpha\beta d(z) + 2z\gamma x\beta z\alpha y = 0. \end{aligned} \tag{25}$$

Multiplying the two sides of (24) by $z\gamma$ from the left hand side, we get

$$z\gamma d(x)\beta[z,d(y)]_\alpha - z\gamma[x,d(y)]_\alpha\beta d(z) + 2z\gamma x\beta z\alpha y = 0, \tag{26}$$

for all $x, y, z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. If we compare (25) and (26), we have that

$$d(z)\gamma x\beta[z, d(y)]_\alpha - [z, d(y)]_\alpha\gamma x\beta d(z) = 0, \tag{27}$$

for all $x, y, z \in U$ and $\alpha, \beta, \gamma \in \Gamma$. Replacing x by $2x\sigma[z, d(y)]_\alpha$ for $y \in [U, U]_\Gamma$ and $\sigma \in \Gamma$ in the equation (27) we get

$$d(z)\gamma x\sigma[z, d(y)]_\alpha\beta[z, d(y)]_\alpha - [z, d(y)]_\alpha\gamma x\sigma[z, d(y)]_\alpha\beta d(z) = 0, \tag{28}$$

since M is a Γ-ring of characteristic not 2. Taking σ for β in (27) we have

$$d(z)\gamma x\sigma[z, d(y)]_\alpha = [z, d(y)]_\alpha\gamma x\sigma d(z). \tag{29}$$

If we use the equation (29) in the equation (28) we get

$$[z, d(y)]_\alpha\gamma x\sigma d(z)\beta[z, d(y)]_\alpha = [z, d(y)]_\alpha\gamma x\sigma[z, d(y)]_\alpha\beta d(z)$$

and so

$$[z, d(y)]_\alpha\gamma x\sigma[d(z), [z, d(y)]_\alpha]_\beta = 0, \tag{30}$$

for all $x, z \in U$, $y \in [U, U]_\Gamma$ and $\alpha, \beta, \gamma, \sigma \in \Gamma$. Taking $\beta = \gamma$ in (30), we get

$$[z, d(y)]_\alpha\gamma x\sigma[d(z), [z, d(y)]_\alpha]_\gamma = 0, \tag{31}$$

for all $x, z \in U$, $y \in [U, U]_\Gamma$ and $\alpha, \gamma, \sigma \in \Gamma$. Multiplying the equation (31) on the left by $d(z)\gamma$ for $z \in [U, U]_\Gamma$, we have

$$d(z)\gamma[z, d(y)]_\alpha\gamma x\sigma[d(z), [z, d(y)]_\alpha]_\gamma = 0. \tag{32}$$

Taking $2d(z)\gamma x$ for x in (31) we obtain that

$$[z, d(y)]_\alpha\gamma d(z)\gamma x\sigma[d(z), [z, d(y)]_\alpha]_\gamma = 0, \tag{33}$$

for all $x \in U$, $y, z \in [U, U]_\Gamma$ and $\alpha, \gamma, \sigma \in \Gamma$ since M is a Γ-ring of characteristic not 2. Subtracting (33) from (32) we see that

$$[d(z), [z, d(y)]_\alpha]_\gamma\gamma x\sigma[d(z), [z, d(y)]_\alpha]_\gamma = 0,$$

for all $x \in U$, $y, z \in [U, U]_\Gamma$ and $\alpha, \gamma, \sigma \in \Gamma$. Therefore, by Lemma 3.4 we have that

$$[d(z), [z, d(y)]_\alpha]_\gamma = 0,$$

for all $y, z \in [U, U]_\Gamma$ and $\alpha, \gamma \in \Gamma$. Then, the proof is completed by using the similar steps in the equation (22) in Theorem 3.3. □

Corollary 3.4. *Let d be a derivation of a prime Γ-ring M of characteristic not 2. If $d(x) \circ_\alpha d(y) = -(x \circ_\alpha y)$ for all $x, y \in M$, $\alpha \in \Gamma$ and $Z(M) \neq (0)$, then M is commutative.*

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