

On \mathcal{I} -Fréchet-Urysohn spaces and sequential \mathcal{I} -convergence groups

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ABSTRACT. In this paper, we introduce the concept of sequential \mathcal{I} -convergence spaces and \mathcal{I} -Fréchet-Urysohn space and study their properties. We give a sufficient condition for the product of two sequential \mathcal{I} -convergence spaces to be a sequential \mathcal{I} -convergence space.

Finally, we introduce sequential \mathcal{I} -convergence groups and obtain an \mathcal{I} -completion of these groups satisfying certain conditions.

1. INTRODUCTION

In [4], Hong introduced the notion of Fréchet spaces and sequential convergence groups. It has been discussed and developed by many authors [3, 5]. We try to extend this concept on ideal topological spaces. A non-empty collection \mathcal{I} of subsets of a set X is said to be an *ideal* on X [7] if it satisfies the following two conditions:

- (i) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$.
- (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.

A non-trivial ideal \mathcal{I} is called *admissible* [2] if and only if $\mathcal{I} \supset \{\{x\} \mid x \in X\}$. Several examples of nontrivial admissible ideals may be seen in [6]. Let (X, τ) be a topological space. A sequence (x_n) in X is said to be \mathcal{I} -convergent to $x_0 \in X$ [7] if for any non-empty open set U containing x_0 , $\{n \in \mathbb{N} \mid x_n \notin U\} \in \mathcal{I}$. It is denoted by $(x_n) \xrightarrow{\mathcal{I}} x_0$ and x_0 is called an \mathcal{I} -limit of the sequence (x_n) . A topological space (X, τ) is \mathcal{I} -Fréchet or \mathcal{I} -Fréchet-Urysohn space [9] if every point in the closure of a subset A of X is a \mathcal{I} -limit of a sequence of A . A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be *pseudo open* [2] if whenever $f^{-1}(y) \subset U$ with U open in X , $y \in \text{int}(f(U))$.

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Let $A \subset \mathbb{N}$. Put $A(n) = \{k \mid k \leq n, k \in A\}$. Then we call $\underline{\delta}(A) = \liminf_{n \rightarrow \infty} \frac{|A(n)|}{n}$ and $\bar{\delta}(A) = \limsup_{n \rightarrow \infty} \frac{|A(n)|}{n}$, the lower and upper asymptotic density [8] of A , respectively. If $\underline{\delta}(A) = \bar{\delta}(A)$, then $d(A) = \delta(A) = \lim_{n \rightarrow \infty} \frac{|A(n)|}{n}$ is called the asymptotic density (or natural density) of A .

This paper consists of four sections with new results. In Section 2, we introduce the concept of sequential \mathcal{I} -convergence spaces, \mathcal{I} -Fréchet-Urysohn spaces and study their properties. Recall that \mathcal{I} -Fréchet and \mathcal{I} -sequential spaces are generalizations of statistical versions of Fréchet-Urysohn spaces and sequential spaces considered in [1] and [11]. In Section 3, we give a sufficient condition for the product of two sequential \mathcal{I} -convergence spaces to be a sequential \mathcal{I} -convergence space. In Section 4, we introduce sequential \mathcal{I} -convergence groups and an \mathcal{I} -completion of these groups satisfying given condition (**). Throughout this paper, we consider only an admissible ideal. The following lemma will be useful in the sequel.

Lemma 1.1. [10] *Let $f : X \rightarrow Y$ be a quotient mapping and X be an \mathcal{I} -Fréchet-Urysohn space. Then Y is an \mathcal{I} -Fréchet-Urysohn space if and only if f is pseudo open.*

2. SEQUENTIAL \mathcal{I} -CONVERGENCE SPACES

Let X be a non-empty set and $S[X]$ be the set of all sequences in X . We use the notation $x_n \xrightarrow{\mathcal{I}} x$ for $((x_n), x) \in L_{\mathcal{I}}$. A non-empty subfamily $L_{\mathcal{I}}$ of $S[X] \times X$ is called a *sequential \mathcal{I} -convergence structure (SIC)* on X if it satisfies the following properties:

- (SIC1) For each $x \in X$, $((x), x) \in L_{\mathcal{I}}$, where (x) is the constant sequence whose n -th term is x for all indices $n \in \mathbb{N}$,
- (SIC2) If $((x_n), x) \in L_{\mathcal{I}}$, then $((x_{n_i}), x) \in L_{\mathcal{I}}$ for each subsequence (x_{n_i}) of (x_n) .
- (SIC3) Let $x \in X$ and $A \subset X$. If $((x_n), x) \notin L_{\mathcal{I}}$ for each $(x_n) \in S[A]$, then $((y_n), y) \notin L_{\mathcal{I}}$ for each $(y_n) \in S[\{y \in X \mid ((x_n), y) \in L_{\mathcal{I}} \text{ for some } (x_n) \in S[A]\}]$.

If a sequential \mathcal{I} -convergence structure $L_{\mathcal{I}}$ on X is given, the pair $(X, L_{\mathcal{I}})$ is called a *sequential \mathcal{I} -convergence space*. Hereafter, we use the notation $\text{SCI}[X]$ for the set of all sequential \mathcal{I} -convergence structures on X .

Let $(X, \tau_{\mathcal{I}})$ be a \mathcal{I} -Fréchet-Urysohn space and let $L_{\tau_{\mathcal{I}}}$ denote the set of all pairs $((x_n), x) \in S[X] \times X$ such that $(x_n) \xrightarrow{\mathcal{I}} x$ in $(X, \tau_{\mathcal{I}})$.

- (i) Suppose for each $x \in X$, then $((x), x) \in L_{\tau_{\mathcal{I}}}$, where (x) is constant sequence.

- (ii) Suppose $((x_n), x) \in L_{\tau_{\mathcal{I}}}$ then there exists a subsequence (x_{n_k}) of (x_n) such that $((x_{n_k}), x) \in L_{\tau_{\mathcal{I}}}$.
- (iii) Let $x \in X$ and $A \subset X$. Suppose that $((x_n), x) \notin L_{\tau_{\mathcal{I}}}$ for each $(x_n) \in S[A]$, then $((y_n), y) \notin L_{\tau_{\mathcal{I}}}$ for each $(y_n) \in S[\{y \in X | ((x_n), y) \in L_{\tau_{\mathcal{I}}} \text{ for some } (x_n) \in S[A]\}]$.

Then, it is clear that $L_{\tau_{\mathcal{I}}} \in STC[X]$ and two topological spaces $(X, \tau_{\mathcal{I}})$ and $(X, L_{\tau_{\mathcal{I}}})$ are precisely same, since $(X, \tau_{\mathcal{I}})$ is a \mathcal{I} -Fréchet-Urysohn space. Hence every \mathcal{I} -Fréchet-Urysohn space is a sequential \mathcal{I} -convergence space. And, for each $L_{\mathcal{I}} \in STC[X]$, define a mapping $c_{L_{\mathcal{I}}}$ of the power set $\mathcal{P}(X)$ of X into itself as follows:

$$c_{L_{\mathcal{I}}}(A) = \{x \in X | ((x_n), x) \in L_{\mathcal{I}} \text{ for some } (x_n) \in S[A]\}$$

The following Lemma 2.1 gives the properties of the operator $c_{L_{\mathcal{I}}}$.

Lemma 2.1. *Let (X, τ) be a topological space and $A, B \subset X$. Then the following hold.*

- (a) $c_{L_{\mathcal{I}}}(\emptyset) = \emptyset$.
 (b) $A \subset c_{L_{\mathcal{I}}}(A)$.
 (c) $c_{L_{\mathcal{I}}}(c_{L_{\mathcal{I}}}(A)) \subset c_{L_{\mathcal{I}}}(A)$.
 (d) $A \subset B \Rightarrow c_{L_{\mathcal{I}}}(A) \subset c_{L_{\mathcal{I}}}(B)$.
 (e) $c_{L_{\mathcal{I}}}(A \cup B) = c_{L_{\mathcal{I}}}(A) \cup c_{L_{\mathcal{I}}}(B)$.

Proof. (a) $c_{L_{\mathcal{I}}}(\emptyset) = \emptyset$ is clear.

(b) Suppose $x \in A$, and consider $(x_n) = (x, x, \dots, x)$.

Then $(x_n) \in S[A]$ and $((x_n), x) \in L_{\mathcal{I}}$. Therefore, $x \in c_{L_{\mathcal{I}}}(A)$.

(c) Suppose $x \in c_{L_{\mathcal{I}}}(c_{L_{\mathcal{I}}}(A))$, then there exists a sequence $(x_n) \in S[c_{L_{\mathcal{I}}}(A)]$ such that $((x_n), x) \in L_{\mathcal{I}}$. Suppose that $x \notin c_{L_{\mathcal{I}}}(A)$. Then for each $(y_n) \in S[A]$, $((y_n), x) \notin L_{\mathcal{I}}$. By (STC3), $((x_n), x) \notin L_{\mathcal{I}}$, which is a contradiction. Therefore, $x \in c_{L_{\mathcal{I}}}(A)$. Hence $c_{L_{\mathcal{I}}}(c_{L_{\mathcal{I}}}(A)) \subset c_{L_{\mathcal{I}}}(A)$.

(d) Suppose $x \in c_{L_{\mathcal{I}}}(A)$, then $((x_n), x) \in L_{\mathcal{I}}$ for some $(x_n) \in S[A]$. Since $A \subset B$, $(x_n) \in B$ and so $x \in c_{L_{\mathcal{I}}}(B)$. Therefore, $c_{L_{\mathcal{I}}}(A) \subset c_{L_{\mathcal{I}}}(B)$.

(e) We have $c_{L_{\mathcal{I}}}(A) \cup c_{L_{\mathcal{I}}}(B) \subset c_{L_{\mathcal{I}}}(A \cup B)$, by (d). Let $x \in c_{L_{\mathcal{I}}}(A \cup B)$. Then $((x_n), x) \in L_{\mathcal{I}}$ for some $(x_n) \in A \cup B$. Note that either A or B contains infinitely many terms of (x_n) . If A contains infinitely many terms of (x_n) , then there exists a subsequence (x_{nm}) of (x_n) in A with $((x_{nm}), x) \in L_{\mathcal{I}}$, by (STC2). Therefore, $x \in c_{L_{\mathcal{I}}}(A)$. Similarly, $x \in c_{L_{\mathcal{I}}}(B)$. Hence $x \in c_{L_{\mathcal{I}}}(A) \cup c_{L_{\mathcal{I}}}(B)$. Thus, $c_{L_{\mathcal{I}}}(A \cup B) = c_{L_{\mathcal{I}}}(A) \cup c_{L_{\mathcal{I}}}(B)$. \square

Thus, $c_{L_{\mathcal{I}}}$ is a Kuratowski closure operator on X and $(X, c_{L_{\mathcal{I}}})$ is a \mathcal{I} -Fréchet-Urysohn space as it satisfies the above properties. Let $\mathcal{L}(c_{L_{\mathcal{I}}})$ denote the set of all pairs $((x_n), x) \in S[X] \times X$ such that $(x_n) \xrightarrow{\mathcal{I}} x$ in $(X, c_{L_{\mathcal{I}}})$. By the following Example 2.1, $L_{\mathcal{I}} \subsetneq \mathcal{L}(c_{L_{\mathcal{I}}})$, in general. Hence not every sequential \mathcal{I} -convergence space $(X, L_{\mathcal{I}})$ need be a \mathcal{I} -Fréchet-Urysohn space even if $(X, L_{\mathcal{I}})$ determines a \mathcal{I} -Fréchet-Urysohn space $(X, c_{L_{\mathcal{I}}})$ as above.

Example 2.1. Let \mathbb{Q} be the set of all rational numbers and $L_{\mathcal{I}} = \{((x), x) \mid x \in \mathbb{Q}\} \cup \{((x_n), x) \in S[\mathbb{Q}] \times \mathbb{Q} \mid (x_n) \xrightarrow{\mathcal{I}} x \in \mathbb{Q}\}$ with the usual topology and (x_n) is either increasing or decreasing }.

Let $(x_n) = (\frac{1}{n})$ and $(x) = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \dots)$. Then there is an open set U of 0 such that $\{n \mid (x_n) = \frac{1}{n} \notin (U \cap \mathbb{Q})\} \in \mathcal{I}$ where $\mathcal{I} = \{J \subset \mathbb{N} \mid d(J) = 0\}$. Therefore, $((x), \frac{1}{2}) \in L_{\mathcal{I}}$ satisfy (SIC1). Choose $(x_{n_i}) = (\frac{1}{2}, \frac{1}{4}, \dots)$ of (x_n) . Then $((x_{n_i}), 0) \in L_{\mathcal{I}}$ satisfy (SIC2). Suppose $(x_n) = (-1, 1, -1, \dots)$ does not \mathcal{I} -converge in $L_{\mathcal{I}}$ but $(y_n) = \{1 \mid (1, 1, \dots)\} \xrightarrow{\mathcal{I}} 1$ for some $(x_n) \in S[\mathbb{Q}]$. Therefore, (SIC3) is satisfied. Thus, $L_{\mathcal{I}} \in SIC[\mathbb{Q}]$ but $L_{\mathcal{I}} \not\subseteq \mathcal{L}(c_{L_{\mathcal{I}}}) = \{((x_n), 0) \in [S\mathbb{Q}] \times \mathbb{Q} \mid (x_n) \xrightarrow{\mathcal{I}} 0 \text{ in } \mathbb{Q} \text{ with the usual topology}\}$.

Lemma 2.2. Let $L_{\mathcal{I}} \in SIC[X]$ and $x \in A \subset X$. If A is a neighbourhood of x in $(X, c_{L_{\mathcal{I}}})$, then for each $((x_n), x) \in L_{\mathcal{I}}$, there is an open set U in $(X, c_{L_{\mathcal{I}}})$ containing x such that $\{n \mid x_n \notin U\} \in \mathcal{I}$.

Proof. Let A be a neighbourhood of x in $(X, c_{L_{\mathcal{I}}})$ and $((x_n), x) \in L_{\mathcal{I}}$. Then there exists an open set U in $(X, c_{L_{\mathcal{I}}})$ such that $x \in U \subset A$. It follows that $c_{L_{\mathcal{I}}}(X \setminus U) = X \setminus U$, and there does not exist (y_n) in $X \setminus U$ such that $((y_n), x) \in L_{\mathcal{I}}$, by the definition of $c_{L_{\mathcal{I}}}$. Now we show that $\{k \in \mathbb{N} \mid x_k \in X \setminus U\} \in \mathcal{I}$. If $\{k \in \mathbb{N} \mid x_k \in X \setminus U\} \notin \mathcal{I}$, then there exists a subsequence (x_{n_i}) of (x_n) in $X \setminus U$. Since $((x_n), x) \in L_{\mathcal{I}}$, $((x_{n_i}), x) \in L_{\mathcal{I}}$, which is a contradiction. Therefore, $\{k \in \mathbb{N} \mid x_k \in X \setminus U\} \in \mathcal{I}$. □

Theorem 2.1. If $L_{\mathcal{I}} \in SIC[X]$, then the following hold:

- (a) $L_{\mathcal{I}} \subset \mathcal{L}(c_{L_{\mathcal{I}}})$.
- (b) $c_{L_{\mathcal{I}}} = c_{\mathcal{L}(c_{L_{\mathcal{I}}})}$.

Proof. (a) Let $((x_n), x) \in L_{\mathcal{I}}$. Then by Lemma 2.2, for each neighborhood A of x in $(X, c_{L_{\mathcal{I}}})$, there is an open set U in $(X, c_{L_{\mathcal{I}}})$ such that $\{n \mid x_n \notin U\} \in \mathcal{I}$. Therefore, $((x_n), x) \in \mathcal{L}(c_{L_{\mathcal{I}}})$.

(b) Let A be a non-empty subset of X . Then by Lemma 2.1(b) and (d), $c_{L_{\mathcal{I}}}(A) \subset c_{\mathcal{L}(c_{L_{\mathcal{I}}})}(A)$. Conversely, let $x \in c_{\mathcal{L}(c_{L_{\mathcal{I}}})}(A)$. Then $((x_n), x) \in \mathcal{L}(c_{L_{\mathcal{I}}})$ for some $(x_n) \in S[A]$. By the definition of $\mathcal{L}(c_{L_{\mathcal{I}}})$, $(x_n) \xrightarrow{\mathcal{I}} x$ in $(X, c_{L_{\mathcal{I}}})$. Therefore, $x \in c_{L_{\mathcal{I}}}(A)$. □

Theorem 2.2. For each \mathcal{I} -Fréchet-Urysohn topology $\tau_{\mathcal{I}}$ on X , $L_{\tau_{\mathcal{I}}} = \mathcal{L}(c_{L_{\tau_{\mathcal{I}}}})$ $\in SIC[X]$, where $L_{\tau_{\mathcal{I}}} = \{((x_n), x) \in S[X] \times X \mid (x_n) \xrightarrow{\mathcal{I}} x \text{ in } (X, \tau_{\mathcal{I}})\}$.

Proof. Note that $c_{L_{\tau_{\mathcal{I}}}}$ is the closure operator for $(X, \tau_{\mathcal{I}})$. Since $\tau_{\mathcal{I}}$ is an \mathcal{I} -Fréchet-Urysohn topology and $L_{\tau_{\mathcal{I}}} \in SIC[X]$, $L_{\tau_{\mathcal{I}}} \subset \mathcal{L}(c_{L_{\tau_{\mathcal{I}}}})$, by Theorem 2.1. □

Corollary 2.1. Let $F_{\mathcal{I}}[X]$ denote the set of all \mathcal{I} -Fréchet-Urysohn topologies on X and $TSL[X] = \{\mathcal{L}(c_{L_{\mathcal{I}}}) \mid L_{\mathcal{I}} \in SIC[X]\}$. Then partially ordered sets

$F_{\mathcal{I}}[X]$ and $TSIC[X]$ endowed with the set inclusion order are dual isomorphic under the correspondence $\tau_{\mathcal{I}} \rightarrow L_{\tau_{\mathcal{I}}}$.

Proof. Since $c_{L_{\tau_{\mathcal{I}}}}$ is the closure operator for $(X, \tau_{\mathcal{I}})$, $L_{\tau_{\mathcal{I}_1}} = L_{\tau_{\mathcal{I}_2}}$ implies $\tau_{\mathcal{I}_1} = \tau_{\mathcal{I}_2}$. Hence this correspondence is one-to-one. Take any $L_{\mathcal{I}}$ in $SIC[X]$ and let $\tau_{c_{L_{\mathcal{I}}}}$ be the \mathcal{I} -Fréchet topology on X with the closure operator $c_{L_{\mathcal{I}}}$. Then $L_{\tau_{c_{L_{\mathcal{I}}}}} = \mathcal{L}(c_{L_{\mathcal{I}}})$. Hence this correspondence is onto. \square

Theorem 2.3. *There exists an one-to-one correspondence between the set of all \mathcal{I} -Fréchet-Urysohn topologies on a set X and $\{c_{L_{\mathcal{I}}} \mid L_{\mathcal{I}} \in SIC[X]\}$.*

Proof. Follows from Corollary 2.1. \square

3. PRODUCT OF SEQUENTIAL \mathcal{I} -CONVERGENCE SPACES

In general, the product of two sequential \mathcal{I} -convergence spaces need not be a sequential \mathcal{I} -convergence space, but we give a sufficient condition for the product of two sequential \mathcal{I} -convergence spaces to be a sequential \mathcal{I} -convergence space. The following Example 3.1 shows that the product of two sequential \mathcal{I} -convergence spaces is not a sequential \mathcal{I} -convergence space.

Example 3.1. Let $X = \mathbb{R}/\mathbb{Z}$, \mathbb{R} is the real line (equipped with the usual topology) with the integers identified and let $I = [0, 1]$ be the closed unit interval in the real line. Since $\mathcal{I} = \{J \subset \mathbb{N} \mid d(J) = 0\}$ and every point in the closure of I is an \mathcal{I} -limit of a sequence of points of I , I is \mathcal{I} -Fréchet-Urysohn space. A quotient map $\phi : \mathbb{R} \rightarrow X$ is pseudo-open. Hence by Lemma 1.1, X is an \mathcal{I} -Fréchet-Urysohn space. For each $n \in \mathbb{N}$, let $A_n = \{(n - \frac{1}{k}, \frac{1}{n}) \mid k \in \mathbb{N}\}$ and let $A = \bigcup\{A_n \mid n \in \mathbb{N}\}$. Then $(0, 0) \in \bar{A}$, but no sequence in A \mathcal{I} -converging to $(0, 0)$. Hence $X \times I$ is not \mathcal{I} -Fréchet-Urysohn. Next we show that $X \times I$ is not a sequential \mathcal{I} -convergence space. For each $n, k \in \mathbb{N}$, let $z_{n_k} = (n - \frac{1}{k+1}, \frac{1}{n})$ and let $A = \{z_{n_k} \mid n, k \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$, $(z_{n_k}) \xrightarrow{\mathcal{I}} (0, \frac{1}{n})$ in $X \times I$ and the sequence $((0, \frac{1}{n})) \xrightarrow{\mathcal{I}} (0, 0)$ in $X \times I$. But there does not exist a (z_{n_k}) in A such that $(z_{n_k}) \xrightarrow{\mathcal{I}} (0, 0)$. Therefore, $X \times I$ is not a sequential \mathcal{I} -convergence space.

The following condition (*) is sufficient for the product of two sequential \mathcal{I} -convergence spaces to be a sequential \mathcal{I} -convergence space.

(*) Let $((x_n), x) \in L_{\mathcal{I}}$ and let $((x_{n_m}), x_n) \in L_{\mathcal{I}}$ for each $n \in \mathbb{N}$. It is possible to choose a cross-sequence $(x_{n_m(n)})$ in the double sequence (x_{n_m}) such that (i) $((x_{n_m(n)}), x) \in L_{\mathcal{I}}$, (ii) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (iii) $((x_{n_k(n)}), x) \in L_{\mathcal{I}}$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$.

(*) implies (SIC3) Let $((x_n), x) \notin L_{\mathcal{I}}$ for all $(x_n) \in A$ and $(y_n) \in S[B]$ where $B = \{y \in X \mid ((X_n), y) \in L_{\mathcal{I}} \text{ for some } (x_n) \text{ in } A\}$. Then $y_i \in B$ for all $i \in \mathbb{N}$ and for each i , there exists $(x_{n_i}) \in S[A]$ such that $((x_{n_i}), y_i) \in L_{\mathcal{I}}$. By hypothesis, $((x_{n_i}), x) \notin L_{\mathcal{I}}$ for all i . Suppose $((y_n), y) \in L_{\mathcal{I}}$, then (i) $((x_{n_i(n)}), x) \in L_{\mathcal{I}}$ (ii) $i(n) \geq n$ for all n and (iii) $((x_{n_k(n)}), x) \in L_{\mathcal{I}}$ if

$k(n) \geq m(n)$ for all $n \in \mathbb{N}$, which is a contradiction to $((x_n), x) \notin L_{\mathcal{I}}$ for all i . Therefore, $((y_n), y) \notin L_{\mathcal{I}}$.

Example 3.2. Let $L_{\mathcal{I}_{\mathbb{Q}}} = \{((x_n), x) \in S[\mathbb{Q}] \times \mathbb{R} \mid (x_n) \text{ } \mathcal{I}\text{-converges to } x \text{ in the real line } \mathbb{R}\}$. By Example 2.1, $(\mathbb{R}, L_{\mathcal{I}_{\mathbb{Q}}})$ is a sequential \mathcal{I} -convergence space satisfying $(*)$, but not an \mathcal{I} -Fréchet-Urysohn space

Theorem 3.1. Let $(X, L_{\mathcal{I}_X})$ and $(Y, L_{\mathcal{I}_Y})$ be any two sequential \mathcal{I} -convergence spaces satisfying $(*)$ and let

$$L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y} = \{((x_n, y_n), (x, y)) \mid ((x_n), x) \in L_{\mathcal{I}_X} \text{ and } ((y_n), y) \in L_{\mathcal{I}_Y}\}$$

Then $(X \times Y, L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y})$ is a sequential \mathcal{I} -convergence space satisfying $(*)$.

Proof. Suppose that $(x_n) = (x)$, $(y_n) = (y)$. Then $(X \times Y, L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y})$ satisfies (SIC1). Choose the subsequences (x_{n_i}) and (y_{n_i}) of (x_n) and (y_n) , $L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y} = \{((x_{n_i}, y_{n_i}), (x, y)) \mid ((x_{n_i}), x) \in L_{\mathcal{I}_X} \text{ and } ((y_{n_i}), y) \in L_{\mathcal{I}_Y}\}$. Thus, $L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$ satisfies (SIC2). Since $(*)$ implies (SIC3), it is enough to show that $L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$ satisfies $(*)$.

Let $((x_n, y_n), (x, y)) \in L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$ and let $((x_{n_m}, y_{n_l}), (x_n, y_n)) \in L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$ for each $n \in \mathbb{N}$. Then, by the definition of $L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$, $((x_n), x) \in L_{\mathcal{I}_X}$, $((y_n), y) \in L_{\mathcal{I}_Y}$, $((x_{n_m}), x_n) \in L_{\mathcal{I}_X}$ for each $n \in \mathbb{N}$ and $((y_{n_l}), y_n) \in L_{\mathcal{I}_Y}$ for each $n \in \mathbb{N}$. Since $(X, L_{\mathcal{I}_X})$ and $(Y, L_{\mathcal{I}_Y})$ satisfy $(*)$, there are two cross-sequences $x_{n_m(n)}$ and $y_{n_l(n)}$ in the double sequences (x_{n_m}) and (y_{n_l}) , respectively, such that (i) $((x_{n_m(n)}), x) \in L_{\mathcal{I}_X}$ and $((y_{n_l(n)}), y) \in L_{\mathcal{I}_Y}$ and those cross-sequences also satisfy the properties (ii) and (iii), respectively.

Let $p(n) = \max \{m(n), l(n)\}$ for each $n \in \mathbb{N}$. Then $((x_{n_p(n)}), x) \in L_{\mathcal{I}_X}$ and $((y_{n_p(n)}), y) \in L_{\mathcal{I}_Y}$, and we obtain a cross-sequence $(x_{n_p(n)}, y_{n_p(n)})$ in the double sequence (x_{n_m}, y_{n_l}) such that (i) $((x_{n_p(n)}, y_{n_p(n)}), (x, y)) \in L_{\mathcal{I}_X} \times L_{\mathcal{I}_Y}$; (ii) $p(n) \geq n$ for all $n \in \mathbb{N}$; (iii) $((x_{n_q(n)}, y_{n_q(n)}), (x, y))$, for all $n \in \mathbb{N}$. \square

4. SEQUENTIAL \mathcal{I} -CONVERGENCE GROUPS

Definition 4.1. A sequential \mathcal{I} -convergence space $(X, L_{\mathcal{I}})$ is called Hausdorff if $L_{\mathcal{I}}$ satisfies the following property:

If $((x_n), x) \in L_{\mathcal{I}}$ and $((x_n), y) \in L_{\mathcal{I}}$, then $x = y$.

Definition 4.2. Let $(X, L_{\mathcal{I}})$ be a Hausdorff sequential \mathcal{I} -convergence space satisfying $(*)$ and let \cdot be a commutative group operator on X . The triple $(X, \cdot, L_{\mathcal{I}})$ is called a sequential \mathcal{I} -convergence group if it satisfies the following property:

(SIG) For each $((x_n), x) \in L_{\mathcal{I}}$ and $((y_n), y) \in L_{\mathcal{I}}$, $((x_n y_n^{-1}), xy^{-1}) \in L_{\mathcal{I}}$.

Remark 4.1. Define a mapping ϕ of $X \times X$ onto $(X, \cdot, L_{\mathcal{I}})$ by taking $\phi(x, y) = xy^{-1}$ for each $(x, y) \in X \times X$. Then (SIG) is equivalent to the following:

(SIG') For each $((x_n, y_n), (x, y)) \in L_{\mathcal{I}} \times L_{\mathcal{I}}$, $(\phi(x_n, y_n), \phi(x, y)) \in L_{\mathcal{I}}$.

Let $(X, \cdot, L_{\mathcal{I}})$ be a sequential \mathcal{I} -convergence group. A sequence $(x_n) \in S[X]$ is called \mathcal{I} -Cauchy if for each subsequences (x_{n_i}) and (x_{n_j}) of (x_n) , $((x_{n_i}x_{n_j}^{-1}), e) \in L_{\mathcal{I}}$ where e is the identity element of the group (X, \cdot) . Let $C_{\mathcal{I}}[X]$ denote the set of all \mathcal{I} -Cauchy sequences in $(X, \cdot, L_{\mathcal{I}})$. A sequential \mathcal{I} -convergence group $(X, \cdot, L_{\mathcal{I}})$ is called \mathcal{I} -complete if for each $(x_n) \in C_{\mathcal{I}}[X]$, $((x_n), x) \in L_{\mathcal{I}}$ for some $x \in X$. Let $(X, \cdot, L_{\mathcal{I}})$ be a sequential \mathcal{I} -convergence group and let \sim be an equivalence relation on $C_{\mathcal{I}}[X]$ defined by $(x_n) \sim (y_n)$ if and only if $((x_{n_i}y_{n_j}^{-1}), e) \in L_{\mathcal{I}}$ for each subsequence (x_{n_i}) of (x_n) and each subsequence (y_{n_j}) of (y_n) . We denote the class of all \mathcal{I} -Cauchy sequences which are equivalent to an $(x_n) \in C_{\mathcal{I}}[X]$ by $[(x_n)]$.

In particular, for each constant sequence (x) , x will be used for the equivalence class $[(x)]$. Let $X^* = \{[(x_n)] \mid (x_n) \in C_{\mathcal{I}}[X]\}$ and $\phi : X \rightarrow X^*$ defined by $\phi(x) = x$ for all $x \in X$. Then ϕ is injective and if $(X, \cdot, L_{\mathcal{I}})$ is \mathcal{I} -complete, then $\phi(x) = x$ for each $(x_n) \in C_{\mathcal{I}}[X]$. Thus, ϕ is bijective. Define an (group) operator $*$ on X^* by $[(x_n)] * [(y_n)] = [(x_n y_n)] = [(y_n x_n)] = [(y_n)] * [(x_n)]$. It follows that $(X^*, *)$ is a commutative group.

Lemma 4.1. *Let $(X, \cdot, L_{\mathcal{I}})$ be a sequential \mathcal{I} -convergence group. Then the following hold:*

- (a) $((x_n), x) \in L_{\mathcal{I}}$ if and only if $(x_n) \in C_{\mathcal{I}}[X]$ and $(x_n) \in x$.
- (b) If $(x_n) \in C_{\mathcal{I}}[X]$ and $(y_n) \in C_{\mathcal{I}}[X]$, then $(x_n y_n) \in C_{\mathcal{I}}[X]$
- (c) If $(x_n) \in C_{\mathcal{I}}[X]$ and $(y_n) \in S[X]$ with $((x_n y_n^{-1}), e) \in L_{\mathcal{I}}$, then $(y_n) \in C_{\mathcal{I}}[X]$ and $(y_n) \in [(x_n)]$.

Proof. (a) Let $((x_n), x) \in L_{\mathcal{I}}$ and $(x_{n_i}), (x_{n_j})$ be subsequences of (x_n) . Then by (SI2), $((x_{n_i}), x) \in L_{\mathcal{I}}$ and $((x_{n_j}), x) \in L_{\mathcal{I}}$. Hence $((x_{n_i}x_{n_j}^{-1}), xx^{-1}) \in L_{\mathcal{I}}$, by (SIG) and so $((x_{n_i}x_{n_j}^{-1}), e) \in L_{\mathcal{I}}$. Therefore, $(x_n) \in C_{\mathcal{I}}[X]$. Since $((x_n), x) \in L_{\mathcal{I}}$, $(x, x) \in L_{\mathcal{I}}$, $((x_n x^{-1}), xx^{-1}) \in L_{\mathcal{I}}$. Therefore, $((x_n x^{-1}), e) \in L_{\mathcal{I}}$. Hence $(x_n) \in x$.

Conversely, let $(x_n) \in C_{\mathcal{I}}[X]$ with $(x_n) \in x$. Then for each subsequence (x_{n_i}) of (x_n) , $((x_{n_i}^{-1}), x) \in L_{\mathcal{I}}$, since $(x_n) \in x$. Hence $((x_{n_i}^{-1}), e) \in L_{\mathcal{I}}$. It follows that $((x_n), x) \in L_{\mathcal{I}}$, by (SIC1).

(b) Let $(x_n) \in C_{\mathcal{I}}[X]$ and $(y_n) \in C_{\mathcal{I}}[X]$. Then for each subsequences (x_{n_i}) and (x_{n_j}) of (x_n) and subsequences (y_{n_i}) and (y_{n_j}) of (y_n) , $((x_{n_i}x_{n_j}^{-1}), e) \in L_{\mathcal{I}}$ and $((y_{n_i}y_{n_j}^{-1}), e) \in L_{\mathcal{I}}$. Hence $((x_{n_i}x_{n_j}^{-1})(y_{n_i}y_{n_j}^{-1}), e) = (((x_{n_i}y_{n_i})(x_{n_j}y_{n_j})^{-1}), e) \in L_{\mathcal{I}}$. Therefore, $(x_n y_n) \in C_{\mathcal{I}}[X]$.

(c) Suppose that $(x_n) \in C_{\mathcal{I}}[X]$ and $(y_n) \in S[X]$ with $((x_n y_n^{-1}), e) \in L_{\mathcal{I}}$. Then by (SIC2), for each subsequences of $(x_{n_i}y_{n_i}^{-1})$ and $(x_{n_j}y_{n_j}^{-1})$ of $(x_n y_n^{-1})$ such that $((x_{n_i}y_{n_i}^{-1}), e) \in L_{\mathcal{I}}$ and $((x_{n_j}y_{n_j}^{-1}), e) \in L_{\mathcal{I}}$. Since $y_{n_i}y_{n_j}^{-1} = (x_{n_i}^{-1}y_{n_i})(x_{n_j}y_{n_j}^{-1})(x_{n_i}x_{n_j}^{-1})$ and $x_{n_i}y_{n_j}^{-1} = (x_{n_i}x_{n_j}^{-1})(x_{n_j}y_{n_j}^{-1})$ and $(x_n) \in C_{\mathcal{I}}[X]$. Hence $(x_{n_j}y_{n_j}^{-1}), e) \in L_{\mathcal{I}}$ and $((y_{n_i}y_{n_j}^{-1}), e) \in L_{\mathcal{I}}$. Therefore, $(y_n) \in C_{\mathcal{I}}[X]$ and $(y_n) \in [(x_n)]$. □

Theorem 4.1. *Let $(X, \cdot, L_{\mathcal{I}})$ be a sequential \mathcal{I} -convergence group. Then $(X^*, *)$ is a commutative group containing (X, \cdot) as a subgroup.*

Proof. First we shall show that $(X^*, *)$ is a commutative group.

(1) Since X^* is non-empty, there exists $[(x_n)] \in X^*$. If we take $a = [(x_n)]$ and $b = [(x_n)]$, then $a * b^{-1} = [(x_n)] * [(x_n)]^{-1} = e$ where e is the identity element. Therefore, $e \in X^*$.

(2) Let $[(x_n)] \in X^*$. If $a = e$ and $b = [(x_n)]$, it follows that $a * b^{-1} = e * [(x_n)]^{-1} = [(x_n)]^{-1} \in X^*$.

(3) Let $[(x_n)], [(y_n)] \in X^*$. Then $[(y_n)]^{-1} \in X^*$. Hence $[(x_n)] * [(y_n)]^{-1} = [(x_n)] * [(y_n)]^{-1} \in X^*$. So X^* is closed under the operation $*$.

(4) Let $[(x_n)], [(y_n)]$ and $[(z_n)]$ in X^* . Then $[(x_n)] * ([[(y_n)] * [(z_n)]] = [(x_n)] * ([[(y_n * z_n)]] = [(x_n(y_n z_n))] = [(x_n y_n)(z_n)] = [(x_n y_n)] * [(z_n)]$. Therefore, X^* satisfies associative property.

(5) If $[(x_n)], [(y_n)] \in X^*$, then $[(x_n)] * [(y_n)] = [(x_n y_n)] = [(y_n x_n)] = [(y_n)] * [(x_n)]$. Therefore, $(X^*, *)$ is a commutative group. Hence $(X^*, *)$ is a commutative group containing (X, \cdot) , since (X, \cdot) is a subgroup of $(X^*, *)$.

Now we will construct a sequential \mathcal{I} -convergence structure $L_{\mathcal{I}}^*$ on X^* . Let $L_{\mathcal{I}}^*$ be the set of all pairs $((\alpha_n), \alpha) \in S[X^*] \times X^*$ satisfying the condition that there exists $(x_n) \in C_{\mathcal{I}}[X]$ such that $\alpha_m x_m^{-1} = \alpha [(x_n)]^{-1}$ for each $m \in \mathbb{N}$. \square

Lemma 4.2. *If $L_{\mathcal{I}}^*$ is a sequential \mathcal{I} -convergence structure on X^* , then the following hold:*

- (a) For each $((x_n), x) \in L_{\mathcal{I}}$, $((x_n), x) \in L_{\mathcal{I}}^*$
- (b) For each $(x_n) \in C_{\mathcal{I}}[X]$, $((x_n), [(x_n)]) \in L_{\mathcal{I}}^*$
- (c) $L_{\mathcal{I}}^*$ satisfies (SIC1) and (SIC2).
- (d) If $((\alpha_n), \alpha) \in L_{\mathcal{I}}^*$, then $\alpha_n \alpha_m^{-1} \in X$ for each $n, m \in \mathbb{N}$.
- (e) If $((\alpha_n), \alpha) \in L_{\mathcal{I}}^*$ and $((\beta_n), \beta) \in L_{\mathcal{I}}^*$, then $((\alpha_n \beta_n^{-1}), \alpha \beta^{-1})$, that is, L^* satisfied (SIG).

Proof. (a) Suppose $((x_n), x) \in L_{\mathcal{I}}$. By Lemma 4.1(a), $(x_n) \in C_{\mathcal{I}}[X]$ and $(x_n) \in x$. Therefore, for each subsequence (x_{n_i}) of (x_n) , $((x_{n_i}^{-1}), e) \in L_{\mathcal{I}}$. Therefore, $x \in [(x_n)]$ implies $x [(x_n)]^{-1} = e$. Thus, $((x_n), x) \in L_{\mathcal{I}}^*$

(b) For each $(x_n) \in C_{\mathcal{I}}[X]$, $[(x_n)] \in X^*$. Therefore, $((x_n), [(x_n)]) \in L_{\mathcal{I}}^*$, since $x_m x_m^{-1} = [(x_n)] [(x_n)]^{-1}$

(c) Since $(x) \in C_{\mathcal{I}}[X]$, $((x), x) \in L_{\mathcal{I}}^*$ for each $x \in X$. If $((x_n), x) \in L_{\mathcal{I}}^*$, then there exists $(y_n) \in C_{\mathcal{I}}[X]$ such that $x_m y_m^{-1} = x [(y_n)]^{-1}$ for each $m \in \mathbb{N}$. Let (x_{n_i}) be a subsequence of (x_n) . Then $x_{n_i} y_{n_i}^{-1} = x [(y_{n_i})]^{-1}$ for each $n_i \in \mathbb{N}$. Therefore, $((x_{n_i}), x) \in L_{\mathcal{I}}^*$.

(d) Since $((\alpha_n), \alpha) \in L_{\mathcal{I}}^*$, there exists $(x_n) \in C_{\mathcal{I}}[X]$ such that $\alpha_m x_m^{-1} = \alpha [(x_n)]^{-1}$ for each $m \in \mathbb{N}$. Hence $(\alpha_n x_n^{-1})(\alpha_m x_m^{-1})^{-1} = e$ for each $n, m \in \mathbb{N}$. Thus, $\alpha_n \alpha_m^{-1} = x_n x_m^{-1} \in X$ for each $n, m \in \mathbb{N}$.

(e) Since $((\alpha_n), \alpha) \in L_{\mathcal{I}}^*$ and $((\beta_n), \beta) \in L_{\mathcal{I}}^*$, there are $(x_n) \in C_{\mathcal{I}}[X]$ and $(y_n) \in C_{\mathcal{I}}[X]$ such that $\alpha_m x_m^{-1} = \alpha [(x_n)]^{-1}$ for each $m \in \mathbb{N}$ and $\beta_k y_k^{-1} = \beta [(y_n)]^{-1}$ for each $k \in \mathbb{N}$. Hence $(\alpha_m \beta_k^{-1})(x_m y_k^{-1})^{-1} = (\alpha \beta)^{-1} [(x_n y_n^{-1})]^{-1}$

for each $m, k \in \mathbb{N}$. Thus, $((\alpha_n \beta_n^{-1}), \alpha \beta^{-1}) \in L_{\mathcal{I}}^*$, by (SIG) and Lemma 4.1(b). \square

Lemma 4.3. *Let $((\alpha_n), \alpha) \in L_{\mathcal{I}}^*$ and let (x_{n_m}) be a double sequence in X with $[(x_{n_m})] = \alpha_n$ for each $n \in \mathbb{N}$. Then there exists a cross-sequence $(x_{n_m(n)})$ in (x_{n_m}) such that (i) $[(x_{n_m(n)})] = \alpha$, (ii) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (iii) $[(x_{n_k(n)})] = \alpha$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$.*

Proof. We show that in particular case $\alpha_n = \alpha$ for each $n \in \mathbb{N}$. By Lemma 4.2(e), we have $(x_{n_m} x_{1_m}^{-1})$ with $((x_{n_m} x_{1_m}^{-1}), e) \in L_{\mathcal{I}}^*$ for each $n \in \mathbb{N}$. First we prove that $L_{\mathcal{I}}$ satisfy (*). (SIC1) obviously. Let $((x_n), x) \in L_{\mathcal{I}}$, $((x_{n_i}), x) \in L_{\mathcal{I}}$ and $((x_{n_m}), x_n) \in L_{\mathcal{I}}$ for each $n \in \mathbb{N}$. Choose a cross-sequence $(x_{n_m(ni)})$ in the double sequence (x_{n_m}) such that (i) $((x_{n_m(ni)}), x) \in L_{\mathcal{I}}$, (ii) $m_{ni} \geq ni$ for all $ni \in \mathbb{N}$ and (iii) $((x_{n_k(ni)}), x) \in L_{\mathcal{I}}$ if $k(ni) \geq m(ni)$ for all $n \in \mathbb{N}$, (SIC3) already satisfied (*). Thus, $L_{\mathcal{I}}$ satisfy (*). Therefore, there exists a cross-sequence $(x_{n_m(n)} x_{l_m(n)}^{-1}) \in C_{\mathcal{I}}[X]$ in $(x_{n_m} x_{l_m}^{-1})$ such that (1) $((x_{n_m(n)} x_{l_m(n)}^{-1}), e) \in L_{\mathcal{I}}$, (2) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (3) $((x_{n_k(n)} x_{l_k(n)}^{-1}), e) \in L_{\mathcal{I}}$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$. Since $[(x_{l_m})] = \alpha$ and $L_{\mathcal{I}}^*$ satisfies (SZ2), $[(x_{l_m(n)})] = \alpha$ so $[(x_{n_m(n)})] = \alpha$, by Lemma 4.2(d). Thus, we have a cross-sequence $(x_{n_m(n)})$ in x_{n_m} such that (i) $[(x_{n_m(n)})] = \alpha$, (ii) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (iii) $[(x_{n_k(n)})] = \alpha$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$.

Now we prove this lemma in general case. Since $((\alpha_n), \alpha) \in L_{\mathcal{I}}^*$ and by definition of $L_{\mathcal{I}}^*$, there exists $(y_n) \in C_{\mathcal{I}}[X]$ such that $\alpha_m y_m^{-1} = \alpha[(y_n)]^{-1}$ for each $m \in \mathbb{N}$. Hence a double sequence $(x_{n_m} y_n^{-1})$ in X with $[(x_{p_m} y_p^{-1})] = \alpha[(y_n)]^{-1}$ for each $p \in \mathbb{N}$. By the above particular case, there exists a cross-sequence $(x_{n_m(n)} y_n^{-1}) \in C_{\mathcal{I}}[X]$ such that (i) $[(x_{n_m(n)})] = \alpha[(y_n)]^{-1}$, (ii) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (iii) $[(x_{n_k(n)})] = \alpha[(y_n)]^{-1}$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$. Hence by Lemma 4.2(d), a cross-sequence $(x_{n_m(n)})$ in (x_{n_m}) is such that (i) $\alpha = \alpha[(y_n)]^{-1}[(y_n)]$, where $\alpha[(y_n)]^{-1} = [(x_{n_m(n)} y_n^{-1})]$. Therefore, $\alpha = [(x_{n_m(n)} y_n^{-1})][(y_n)] = [(x_{n_m(n)} y_n^{-1}) y_n] = [(x_{n_m(n)})]$, (ii) $m(n) \geq n$ for all $n \in \mathbb{N}$ and (iii) $[(x_{n_k(n)})] = \alpha$ if $k(n) \geq m(n)$ for all $n \in \mathbb{N}$. \square

Theorem 4.2. *Assume that $L_{\mathcal{I}}^*$ satisfies the following condition:*

(**) *Let $\alpha \in X^*$ and (α_{nm}) be a double sequence in X^* with $((\alpha_{nm}), \alpha) \in L_{\mathcal{I}}^*$ for each $n \in \mathbb{N}$. It is possible to choose $(x_n) \in C_{\mathcal{I}}[X]$ satisfying the property that for each $p \in \mathbb{N}$, there exists a sequence $(x_{n_p(m)})$ of (x_n) such that $\alpha_{pm} x_{n_p(m)}^{-1} = \alpha[(x_n)]^{-1}$ for all $m \in \mathbb{N}$.*

Then $(X^, *, L_{\mathcal{I}}^*)$ is an \mathcal{I} -complete sequential \mathcal{I} -convergence group containing $(X, \cdot, L_{\mathcal{I}})$.*

Proof. First, we prove that $L_{\mathcal{I}}^*$ satisfies (*). Let $((\alpha_n), \alpha) \in L_{\mathcal{I}}^*$ and let (α_{nm}) be a double sequence in X^* with $((\alpha_{nm}), \alpha_n) \in L_{\mathcal{I}}^*$ for each $n \in \mathbb{N}$. Since $((\alpha_n), \alpha) \in L_{\mathcal{I}}^*$, there exists $(x_n) \in C_{\mathcal{I}}[X]$ such that $\alpha_m x_m^{-1} = \alpha[(x_n)]^{-1}$ for each $m \in \mathbb{N}$.

Let $\alpha[(x_n)]^{-1} = \beta$. Then by (**), there exists $(y_n) \in C_{\mathcal{I}}[X]$ satisfying the property that for each $p \in \mathbb{N}$, there exists a subsequence $(y_{n_p(m)})$ of (y_n) such that $\alpha_{p_m} x_p^{-1} y_{n_p(m)}^{-1} = \beta[(y_n)]^{-1}$ for all $m \in \mathbb{N}$. Since $(y_n) \in C_{\mathcal{I}}[X]$ and by Lemma 4.2(b) and (c), $((y_n), [(y_n)]) \in L_{\mathcal{I}}^*$ and so $[(y_{n_p(m)})] = [(y_n)]$ for each $p \in \mathbb{N}$ and hence by Lemma 4.3, there exists a cross-sequence $(y_{n_m(im)}) \in C_{\mathcal{I}}[X]$ in the double sequence $(y_{n_p(m)})$ such that (1) $[(y_{n_m(i(m))})] = [(y_n)]$, (2) $i(m) \geq m$ for all $m \in \mathbb{N}$ and (3) $[(y_{n_m(j(m))})] = [(y_n)]$ if $j(m) \geq i(m)$ for all $m \in \mathbb{N}$.

Thus, $((\alpha_{m_i(m)} x_m^{-1} y_{n_m(i(m))}^{-1}), \beta[(y_n)]^{-1}) \in L_{\mathcal{I}}^*$.

That is, $\alpha_{1_i(1)} x_1^{-1} y_{n_1(i(1))}^{-1}, \alpha_{2_i(2)} x_2^{-1} y_{n_2(i(2))}^{-1}, \dots, \xrightarrow{\mathcal{I}} \beta[(y_n)]^{-1}$. Hence by Lemma 4.2(e), $((\alpha_{n_i(n)}, \alpha) \in L_{\mathcal{I}}^*$. Since the cross-sequence $y_{n_m(i(m))}$ satisfies the above three properties, $i(n) \geq n$ for all $n \in \mathbb{N}$ and $((\alpha_{n_j(n)}, \alpha) \in L_{\mathcal{I}}^*$ if $j(n) \geq i(n)$ for all $n \in \mathbb{N}$. Therefore, $L_{\mathcal{I}}^*$ satisfies (*).

Therefore, by Theorem 4.1 and Lemma 4.2, $(X^*, *, L_{\mathcal{I}}^*)$ is a sequential \mathcal{I} -convergence group containing $(X, \cdot, L_{\mathcal{I}})$.

Next we show that $(X^*, *, L_{\mathcal{I}}^*)$ is \mathcal{I} -complete. Let (α_n) be an \mathcal{I} -Cauchy sequence in X^* . Then, by the definition of \mathcal{I} -Cauchy, $((\alpha_{i_n} \alpha_{j_n}), e) \in L_{\mathcal{I}}^*$ for each subsequences (α_{i_n}) and (α_{j_n}) of (α_n) . The proof will be divided into two cases.

Case 1. There exists a subsequence (α_{i_n}) of (α_n) such that $\alpha_{i_n} \alpha_{i_m}^{-1}$ for each $n, m \in \mathbb{N}$. Then by Lemma 4.2(b), $(\alpha_{i_n} \alpha_{i_1})^{-1} \in C_{\mathcal{I}}[X]$ and so

$$((\alpha_{i_n} \alpha_{i_1}^{-1}), [(\alpha_{i_n} \alpha_{i_1}^{-1})]) \in L_{\mathcal{I}}^*.$$

Therefore, $((\alpha_n), [(\alpha_{i_n})]) \in L_{\mathcal{I}}^*$, by Lemma 4.2(e).

Case 2. There does not exist a subsequence (α_{n_j}) of (α_n) such that $\alpha_{i_n} \alpha_{i_m}^{-1}$ for each $n, m \in \mathbb{N}$. Without loss of generality, we assume that $\alpha_n \alpha_m^{-1} \notin X$ for each $n \neq m \in \mathbb{N}$. Now, we determine a subsequence (α_{i_n}) of (α_n) . Let $\alpha_{i_1} = \alpha_2$, then we can choose α_{i_2} in $\{\alpha_3, \alpha_4\}$. Satisfying $(\alpha_1 \alpha_{i_1}^{-1})(\alpha_2 \alpha_{i_2}^{-1})^{-1} \notin X$.

Suppose we choose $k-1$ natural numbers i_m such that $2^{m-1} \leq i_m \leq 2^m$ and $(\alpha_n \alpha_{i_n}^{-1})(\alpha_m \alpha_{i_m}^{-1})^{-1} \notin X$ for each $n \neq m \in \{1, 2, \dots, k-1\}$.

Now, we show that there exists $\alpha_{i_k} \in \{\alpha_{2^{k-1}+1}, \alpha_{2^k-1+2}, \dots, \alpha_{2^k}\}$ such that $(\alpha_n \alpha_{i_n}^{-1})(\alpha_k \alpha_{i_k}^{-1})^{-1} \notin X$ for each $n \in \{1, 2, \dots, k-1\}$. Assume that there does not exist such α_{i_k} . Then $(\alpha_n \alpha_{i_n}^{-1})(\alpha_k \alpha_p^{-1})^{-1} \in X$ for all $p \in \{2^{k-1}+1, 2^{k-1}+2, \dots, 2^k\}$ and for all $n \in \{1, 2, \dots, k-1\}$. Therefore, $\{(\alpha_n \alpha_{i_n}^{-1})(\alpha_k \alpha_p^{-1})^{-1}\} \{(\alpha_n \alpha_{i_n}^{-1})(\alpha_k \alpha_q^{-1})^{-1}\}^{-1} = \alpha_p \alpha_q^{-1} \in X$ which is a contradiction to $\alpha_n \alpha_m^{-1} \notin X$. Thus, by induction, we have a subsequence (α_{i_n}) of (α_n) such that $(\alpha_n \alpha_{i_n}^{-1})(\alpha_m \alpha_{i_m}^{-1})^{-1} \notin X$ for all $n \neq m \in \mathbb{N}$.

Since (α_n) is \mathcal{I} -Cauchy sequence in X^* , $((\alpha_n \alpha_{i_n})^{-1}, e) \in L_{\mathcal{I}}^*$. Therefore, by Lemma 4.2(d), $(\alpha_n \alpha_{i_n}^{-1})(\alpha_m \alpha_{i_m}^{-1})^{-1} \in X$ for each $n, m \in \mathbb{N}$, which is a contradiction to $(\alpha_n \alpha_{i_n}^{-1})(\alpha_m \alpha_{i_m}^{-1})^{-1} \notin X$. It follows that the case 2 can not occur. Hence $(X^*, *, L_{\mathcal{I}}^*)$ is an \mathcal{I} -complete. \square

Example 4.1. Let $L_{\mathcal{I}} = \{((x_n), x) \in S[\mathbb{R}] \times \mathbb{R} \mid (x_n) \text{ } \mathcal{I}\text{-converges to } x \in \mathbb{R} \text{ with the usual topology}\}$, where $\mathcal{I} = \{J \subset \mathbb{N} \mid d(J) = 0\}$. Let \cdot be the usual multiplication on \mathbb{R} . Suppose $(x_n) = (\frac{1}{n})$ and $(y_n) = (\frac{1}{n+1})$. Then $(\mathbb{R}, \cdot, L_{\mathcal{I}})$ is a sequential \mathcal{I} -convergence group and $L_{\mathcal{I}}^*$ satisfies (**). Hence $(\mathbb{R}, \cdot, L_{\mathcal{I}})$ is \mathcal{I} -complete.

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