

A note on q -analogue of Hermite-poly-Bernoulli numbers and polynomials

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ABSTRACT. In this paper, we introduce the Hermite-based poly-Bernoulli numbers and polynomials with q -parameter and give some of their basic properties including not only addition property, but also derivative properties and integral representations. We also define the Hermite-based λ -Stirling polynomials of the second kind and then provide some relations, identities of these polynomials related to the Stirling numbers of the second kind. We derive some symmetric identities for these families of special functions by applying the generating functions.

1. INTRODUCTION

Throughout this paper, we use the following notations, $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{1, 2, 3, \dots\}$ denotes the set of non negative integer, \mathbb{Z} denotes the set of integers and \mathbb{C} denotes the set of complex numbers respectively.

The 2-variable Kampé de Fériet generalization of the Hermite polynomials [3] and [8] are defined by

$$H_n(x, y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{y^r x^{n-2r}}{r!(n-2r)!}.$$

These polynomials are usually defined by the generating function:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}, \quad (1.1)$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [1]), when $y = -1$ and x is replaced by $2x$.

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As is well known, the Bernoulli polynomials are defined by the generating function as

$$\left(\frac{t}{e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see [1-27]}). \quad (1.2)$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

From (1.2), we have

$$B_n(x) = \sum_{m=0}^n \binom{n}{m} B_{n-m} x^m.$$

The classical polylogarithm function $\text{Li}_k(z)$ is

$$\text{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, \quad (k \in \mathbb{Z}), \quad (\text{see [10, 11, 12]}).$$

So, for $k \leq 1$,

$$\text{Li}_k(z) = -\ln(1-z), \quad \text{Li}_0(z) = \frac{z}{1-z}, \quad \text{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \dots$$

The poly-Bernoulli polynomials are given by

$$\frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad (\text{see [10, 15-19]}).$$

For $k = 1$, we have

$$\frac{\text{Li}_1(1 - e^{-t})}{e^t - 1} e^{xt} = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \quad (1.3)$$

From (1.2) and (1.3), we obtain

$$B_n^{(1)}(x) = B_n(x), \quad (n \geq 0).$$

Very recently, Khan et al. [14] introduced the Hermite poly-Bernoulli polynomials of two variables ${}_H B_n^{(k)}(x, y)$ defined by

$$\left(\frac{\text{Li}_k(1 - e^{-t})}{e^t - 1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H B_n^{(k)}(x, y) \frac{t^n}{n!},$$

which is essentially a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_H B_n(x, y)$ introduced by Dattoli et al. [8, p.386(1.6)] in the form:

$$\left(\frac{t}{e^t - 1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H B_n(x, y) \frac{t^n}{n!}.$$

The Stirling number of the first kind is given by

$$(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \geq 0), \quad (\text{see [4, 9, 26]})$$

and the Stirling number of the second kind is defined by generating function

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}.$$

This paper organized as follows. In Section 2, we introduce a new class of Hermite poly-Bernoulli numbers and polynomials with q -parameter. In Section 3, we establish some identities of these polynomials. In Section 4, we derive some properties of the Stirling numbers of the second kind. In Section 5, we derive symmetric identities for these generalized polynomials by using different analytical means on their respective generating functions.

2. A NOTE ON q -ANALOGUE OF HERMITE-POLY-BERNOULLI NUMBERS AND POLYNOMIALS

In this section, we define a q -analogue of Hermite-poly-Bernoulli numbers and polynomials and its properties.

Definition 2.1. For $n \geq 0$, $n, k \in \mathbb{Z}$, $0 \leq q < 1$, we introduce a q -analogue of Hermite-poly-Bernoulli polynomials by means of the following generating function:

$$\frac{\text{Li}_{k,q}(1 - e^{-t})}{e^t - 1} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x, y) \frac{t^n}{n!}, \tag{2.1}$$

where $\text{Li}_{k,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q^k!}$ is the k -th q -polylogarithm function (see [4, 6, 23]).

When $x = y = 0$, $B_{n,q}^{(k)} = {}_H B_{n,q}^{(k)}(0, 0)$ are called a q -analogue poly-Bernoulli numbers.

Remark 2.1. For $y = 0$ in (2.1), the result reduces to the known result of Hwang et al. [9] as follows:

$$\frac{\text{Li}_{k,q}(1 - e^{-t})}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x, y) \frac{t^n}{n!}. \tag{2.2}$$

From (2.1), we have

$$\lim_{q \rightarrow 1} \frac{\text{Li}_1(1 - e^{-t})}{e^t - 1} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H B_n(x, y) \frac{t^n}{n!}.$$

Thus by (2.1) and (2.2), we get

$${}_H B_n^{(k)}(x, y) = {}_H B_n(x, y) \quad (n \geq 0), \quad (\text{see [21, 22]}).$$

Theorem 2.1. (Addition Property), we have

$${}_H B_{n,q}^{(k)}(x_1 + x_2, y_1 + y_2) = \sum_{m=0}^n \text{binom}n{m} {}_H B_{n-m,q}^{(k)}(x_1, y_1) {}_H B_m(x_2, y_2). \tag{2.3}$$

Proof. Consider (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x_1 + x_2, y_1 + y_2) \frac{t^n}{n!} &= \frac{\text{Li}_{k,q}(1 - e^{-t})}{e^t - 1} e^{(x_1+x_2)t+(y_1+y_2)t^2} \\ &= \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x_1, y_1) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(x_2, y_2) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} {}_H B_{n-m,q}^{(k)}(x_1, y_1) H_m(x_2, y_2) \right) \frac{t^n}{n!}. \end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.3). \square

Theorem 2.2. (Derivative Properties) Each of the following formula holds true:

$$\frac{\delta {}_H B_{n,q}^{(k)}(x, y)}{\delta x} = n {}_H B_{n-1,q}^{(k)}(x, y)$$

and

$$\frac{\delta {}_H B_{n,q}^{(k)}(x, y)}{\delta y} = n(n-1) {}_H B_{n-2,q}^{(k)}(x, y).$$

Proof. The proof follows from (2.1). So we omit them. \square

Theorem 2.3. (Integral Representation) The following formula holds true:

$$\int_r^s {}_H B_{n,q}^{(k)}(x, y) dx = \frac{{}_H B_{n+1,q}^{(k)}(s, y) - {}_H B_{n+1,q}^{(k)}(r, y)}{n+1}$$

and

$$\int_{\lambda}^{\mu} {}_H B_{n,q}^{(k)}(x, y) dy = \frac{{}_H B_{n+2,q}^{(k)}(x, \mu) - {}_H B_{n+2,q}^{(k)}(x, \lambda)}{(n+1)^2}.$$

Proof. Using the derivative properties of ${}_H B_{n,q}^{(k)}(x, y)$ given in Theorem 2.2, we easily get the asserted results. So we omit them. \square

Theorem 2.4. The following formula holds true:

$${}_H B_{n,q}^{(k)}(x, y) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} B_{n-m,q}^{(k)}(x) y^m \frac{n!}{m!(n-2m)!}. \quad (2.4)$$

Proof. By (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x, y) \frac{t^n}{n!} &= \frac{\text{Li}_{k,q}(1 - e^{-t})}{e^t - 1} e^{xt+yt^2} \\ &= \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x) \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{y^m t^{2m}}{m!} \end{aligned}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} B_{n-m,q}^{(k)}(x)y^m \right) \frac{t^n}{m!(n-2m)!},$$

which implies the desired result (2.4). □

3. MAIN RESULTS

In this section, we establish some properties of q -analogue of Hermite-poly-Bernoulli polynomials by using generating function.

Theorem 3.1. For $n \geq 0, n, k \in \mathbb{Z}, 0 \leq q < 1$, we have

$${}_H B_{n,q}^{(k)}(x, y) = \sum_{m=0}^n \binom{n}{m} B_{n-m,q}^{(k)} H_m(x, y). \tag{3.1}$$

Proof. By using (2.1) and (1.1), we arrive at the desired result (3.1). □

Theorem 3.2. For $n \geq 0$, we have

$${}_H B_{n,1}^{(2)}(x, y) = \sum_{m=0}^n \binom{n}{m} \frac{B_m m!}{m+1} {}_H B_{n-m}(x, y). \tag{3.2}$$

Proof. Consider (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H B_{n,1}^{(k)}(x, y) \frac{t^n}{n!} &= \frac{\text{Li}_{k,1}(1 - e^{-t})}{e^t - 1} e^{xt+yt^2} \\ &= \frac{e^{xt+yt^2}}{e^t - 1} \underbrace{\int_0^t \frac{1}{e^z - 1} \int_0^t \frac{1}{e^z - 1} \cdots \int_0^t \frac{1}{e^z - 1}}_{(k-1)\text{-times}} \int_0^t \frac{z}{e^z - 1} dz \cdots dz. \end{aligned}$$

In particular for $k = 2$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H B_n^{(2)}(x, y) \frac{t^n}{n!} &= \frac{e^{xt+yt^2}}{e^t - 1} \int_0^t \frac{z}{e^z - 1} dz \\ &= \left(\sum_{m=0}^{\infty} \frac{t^m B_m}{m+1} \right) \frac{t}{e^t - 1} e^{xt+yt^2} \\ &= \left(\sum_{m=0}^{\infty} \frac{B_m m!}{m+1} \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} {}_H B_n(x, y) \frac{t^n}{n!} \right). \end{aligned}$$

Replacing n by $n - m$ in R.H.S. of above equation, we have

$$= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \frac{B_m m!}{m+1} {}_H B_{n-m}(x, y) \frac{t^n}{n!}.$$

On comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the result (3.2). □

Theorem 3.3. For $n \geq 0$, $n, k \in \mathbb{Z}$, $0 \leq q < 1$, we have

$${}_H B_{n,q}^{(k)}(x, y) = \sum_{l=0}^{\infty} \frac{1}{[l+1]_q^k} \sum_{a=0}^{l+1} \binom{l+1}{a} (-1)^a \frac{{}_H B_{n+1}(x-a, y)}{n+1}. \quad (3.3)$$

Proof. From equation (2.1), we have

$$\sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x, y) \frac{t^n}{n!} = \frac{\text{Li}_{k,q}(1-e^{-t})}{e^t-1} e^{xt+yt^2}.$$

Now

$$\begin{aligned} \frac{\text{Li}_{k,q}(1-e^{-t})}{e^t-1} e^{xt+yt^2} &= \sum_{l=1}^{\infty} \frac{(1-e^{-t})^l}{[l]_q^k} \frac{e^{xt+yt^2}}{e^t-1} \\ &= \sum_{n=0}^{\infty} \frac{1}{[l]_q^k} \sum_{a=0}^{l+1} \binom{l+1}{a} (-1)^a \frac{e^{(x-a)t+yt^2}}{e^t-1} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{1}{[l+1]_q^k} \sum_{a=0}^{l+1} \binom{l+1}{a} (-1)^a \frac{{}_H B_{n+1}(x-a, y)}{n+1} \right) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get the result (3.3). \square

Theorem 3.4. For $n \geq 0$, $n, k \in \mathbb{Z}$, $0 \leq q < 1$, we have

$${}_H B_{n,q}^{(k)}(x, y) = \sum_{l=0}^{\infty} \sum_{m=0}^l \sum_{a=0}^{m+1} \binom{m+1}{a} (-1)^a \frac{H_n(l-m-a+x, y)}{[m+1]_q^k}. \quad (3.4)$$

Proof. From equation (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x, y) \frac{t^n}{n!} &= \frac{\text{Li}_{k,q}(1-e^{-t})}{e^t-1} e^{xt+yt^2} \\ &= \left(\sum_{m=0}^{\infty} e^{mt} \right) \left(\sum_{l=0}^{\infty} \frac{(1-e^{-t})^{l+1}}{[l+1]_q^k} \right) e^{xt+yt^2} \\ &= \left(\sum_{l=0}^{\infty} \sum_{m=0}^l \frac{e^{(l-m)t}}{[m+1]_q^k} \right) \left(\sum_{a=0}^{m+1} \binom{m+1}{a} (-1)^a e^{(x-a)t+yt^2} \right) \\ &= \sum_{l=0}^{\infty} \sum_{m=0}^l \sum_{a=0}^{m+1} \binom{m+1}{a} (-1)^a \frac{e^{(l-m-1+x)t+yt^2}}{[m+1]_q^k} \\ &\quad \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x, y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^l \sum_{a=0}^{m+1} \binom{m+1}{a} (-1)^a \frac{H_n(l-m-a+x, y)}{[m+1]_q^k} \frac{t^n}{n!}. \end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!}$ in both sides, we get (3.4). \square

Theorem 3.5. The following implicit summation formulae for Hermite poly-Bernoulli polynomials ${}_H B_{n,q}^{(k)}(x, y)$ holds true:

$${}_H B_{l+p,q}^{(k)}(z, y) = \sum_{m,n=0}^{l,p} \binom{l}{m} \binom{p}{n} (z-x)^{m+n} {}_H B_{l+p-m-n,q}^{(k)}(x, y).$$

Proof. We replace t by $t + u$ and rewrite the generating function (2.1) as

$$\frac{\text{Li}_{k,q}(1 - (e)^{-(t+u)})}{e^{t+u} - 1} e^{y(t+u)^2} = e^{-x(t+u)} \sum_{l,p=0}^{\infty} {}_H B_{l+p,q}^{(k)}(x, y) \frac{t^l}{l!} \frac{u^p}{p!}.$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$e^{(z-x)(t+u)} \sum_{m,l=0}^{\infty} {}_H B_{l+p,q}^{(k)}(x, y) \frac{t^l}{l!} \frac{u^p}{p!} = \sum_{l,p=0}^{\infty} {}_H B_{l+p,q}^{(k)}(z, y) \frac{t^l}{l!} \frac{u^p}{p!}.$$

On expanding exponential function, (3.3) gives

$$\sum_{N=0}^{\infty} \frac{[(z-x)(t+u)]^N}{N!} \sum_{l,p=0}^{\infty} {}_H B_{l+p,q}^{(k)}(x, y) \frac{t^l}{l!} \frac{u^p}{p!} = \sum_{l,p=0}^{\infty} {}_H B_{l+p,q}^{(k)}(z, y) \frac{t^l}{l!} \frac{u^p}{p!},$$

which on using formula [27, p.52(2)]

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m) \frac{x^n y^m}{n! m!},$$

in the left hand side becomes

$$\begin{aligned} \sum_{m,n=0}^{\infty} \frac{(z-x)^{m+n} t^m u^n}{m! n!} \sum_{l,p=0}^{\infty} {}_H B_{l+p,q}^{(k)}(x, y) \frac{t^l}{l!} \frac{u^p}{p!} &= \\ &= \sum_{l,p=0}^{\infty} {}_H B_{l+p,q}^{(k)}(z, y) \frac{t^l}{l!} \frac{u^p}{p!}. \end{aligned} \tag{3.5}$$

Now replacing l by $l - m$, p by $p - n$ and using the lemma [27, p.100(1)] in the left hand side of (3.5), we get

$$\begin{aligned} \sum_{m,n=0}^{\infty} \sum_{l,p=0}^{\infty} \frac{(z-x)^{m+n}}{m! n!} {}_H B_{l+p-m-n,q}^{(k)}(x, y) \frac{t^l}{(l-m)!} \frac{u^p}{(p-n)!} &= \\ &= \sum_{l,p=0}^{\infty} {}_H B_{l+p,q}^{(k)}(z, y) \frac{t^l}{l!} \frac{u^p}{p!}. \end{aligned}$$

Finally on equating the coefficients of the like powers of t and u in the above equation, we get the required result. \square

Remark 3.1. By taking $l = 0$ in Equation (3.1), we immediately deduce the following result.

Corollary 3.1. The following implicit summation formula for Hermite poly-Bernoulli polynomials ${}_H B_{n,q}^{(k)}(z, y)$ holds true:

$${}_H B_{p,q}^{(k)}(z, y) = \sum_{n=0}^p \binom{p}{n} (z-x)^n {}_H B_{p-n,q}^{(k)}(x, y).$$

Remark 3.2. On replacing z by $z+x$ and setting $y = 0$ in Theorem 3.1, we get the following result involving poly-Bernoulli polynomials of one variable.

$${}_H B_{l+p,q}^{(k)}(z+x) = \sum_{m,n=0}^{l,p} \binom{l}{m} \binom{p}{n} z^{m+n} B_{l+p-m-n,q}^{(k)}(x),$$

where as, by setting $z = 0$ in Theorem 3.1, we get another result involving q -poly-Bernoulli polynomials of one and two variables.

$$B_{l+p,q}^{(k)}(y) = \sum_{m,n=0}^{l,p} \binom{l}{m} \binom{p}{n} (-x)^{m+n} {}_H B_{l+p-m-n,q}^{(k)}(x, y).$$

Remark 3.3. Along with the above results we will exploit extended forms of q -poly-Bernoulli polynomials $B_{l+p,q}^{(k)}(z)$ by setting $y = 0$ in Theorem 3.1 to get

$$B_{l+p,q}^{(k)}(z) = \sum_{m,n=0}^{l,p} \binom{l}{m} \binom{p}{n} (z-x)^{n+m} B_{l+p-m-n,q}^{(k)}(x).$$

4. THE q -ANALOGUE OF HERMITE-BASED STIRLING POLYNOMIALS OF THE SECOND KIND

In this section, we introduce q -analogue of Hermite-based Stirling polynomials of the second kind is defined by

$$\frac{(e^t - 1)^m}{m!} e^{xt+yt^2} = \sum_{n=0}^{\infty} S_2(n, m; x, y) \frac{t^n}{n!}. \quad (4.1)$$

For $x = y = 0$ in (4.1), $S_2(n, m) = S_2(n, m; 0, 0)$ are called the Stirling numbers of the second kind (see [4, 9, 26]).

We give some relations and properties belonging to the Hermite-based Stirling polynomials of the second kind by the following consecutive Theorems.

Theorem 4.1. We have

$$S_2(n, m; x, y) = \sum_{l=0}^n \binom{n}{l} S_2(l, m) H_{n-l}(x, y),$$

$$S_2(n, m; x, y) = \sum_{l=0}^n \binom{n}{l} S_2(l, m; 0, y) x^{n-l},$$

$$S_2(n, m; x, y) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{l} S_2(l, m; x, 0) \frac{n! y^l}{(n-2)!}.$$

Theorem 4.2. (Derivative Properties) Each of the following formula hold true:

$$\frac{\delta}{\delta x} S_2(n, m; x, y) = n S_2(n-1, m; x, y)$$

and

$$\frac{\delta}{\delta y} S_2(n, m; x, y) = n(n-1) S_2(n-2, m; x, y).$$

Theorem 4.3. (Integral Representations) The following equalities holds true:

$$\int_r^s S_2(n, m; x, y) dx = \frac{S_2(n+1, m; s, y) - S_2(n+1, m; r, y)}{n+1}$$

and

$$\int_v^\mu S_2(n, m; x, y) dy = \frac{S_2(n+1, m; x, \mu) - S_2(n+1, m; x, v)}{(n+1)^2}.$$

Theorem 4.4. For $n, k \in \mathbb{Z}$, $n \geq 0$, $0 \leq q < 1$, we have

$${}_H B_{n,q}^{(k)}(x, y) = \sum_{a=0}^n \sum_{l=1}^{a+1} (-1)^{l+a+1} \frac{l!}{[l]_q^k} \frac{S_2(a+1, l)}{(a+1)} {}_H B_{n-a}(x, y). \quad (4.2)$$

Proof. From (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x, y) \frac{t^n}{n!} &= \frac{\text{Li}_{k,q}(1 - e^{-t})}{e^t - 1} e^{xt+yt^2} \\ &= \sum_{n=0}^{\infty} \frac{(1 - e^{-t})^l}{[l]_q^k} \frac{e^{xt+yt^2}}{e^t - 1} = \sum_{n=1}^{\infty} \frac{(e^{-t}-1)^l}{[l]_q^k} \frac{e^{xt+yt^2}}{e^t - 1} \\ &= \sum_{n=1}^{\infty} \sum_{l=1}^n \frac{(-1)^{l+n}}{[l]_q^k} l! S_2(n, l) \frac{t^n}{n!} \frac{e^{xt+yt^2}}{e^t - 1} \\ &= \sum_{a=0}^{\infty} \left(\sum_{l=1}^{a+1} (-1)^{l+a+1} \frac{l!}{[l]_q^k} \frac{S_2(a+1, l)}{(a+1)} \right) \frac{t^a}{a!} \left(\sum_{n=0}^{\infty} {}_H B_n(x, y) \frac{t^n}{n!} \right). \end{aligned}$$

Replacing n by $n - a$ in above equation, we get

$$= \sum_{n=0}^{\infty} \left(\sum_{a=0}^n \sum_{l=1}^{a+1} (-1)^{l+a+1} \frac{l!}{[l]_q^k} \frac{S_2(a+1, l)}{(a+1)} {}_H B_{n-a}(x, y) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in above equation, we get (4.2). \square

Theorem 4.5. For $n, k \in \mathbb{Z}$, $n \geq 0$, $0 \leq q < 1$, we have

$${}_H B_{n,q}^{(k)}(x+u, y) = \sum_{l=0}^{\infty} \sum_{a=l}^n \binom{n}{a} (u)_l S_2(a, l) {}_H B_{n-a,q}^{(k)}(x, y). \quad (4.3)$$

Proof. Replacing x by $x+u$ in (2.1), we have

$$\begin{aligned} \frac{\text{Li}_{k,q}(1-e^{-t})}{e^t-1} e^{(x+u)t+yt^2} &= \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x+u, y) \frac{t^n}{n!} \\ &= \frac{\text{Li}_{k,q}(1-e^{-t})}{e^t-1} e^{xt+yt^2} \sum_{l=0}^{\infty} (u)_l \frac{(e^t-1)^l}{l!} \\ &= \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x, y) \frac{t^n}{n!} \sum_{l=0}^{\infty} (u)_l \sum_{a=0}^{\infty} S_2(a, l) \frac{t^a}{a!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty} \sum_{a=l}^n \binom{n}{a} (u)_l S_2(a, l) {}_H B_{n-a,q}^{(k)}(x, y) \right) \frac{t^n}{n!}. \end{aligned}$$

On comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get at the desired result (4.3). \square

Theorem 4.6. For $n \geq 1$, $n, k \in \mathbb{Z}$ and $0 \leq q < 1$, we have

$$\begin{aligned} &{}_H B_{n,q}^{(k)}(x+1, y) - {}_H B_{n,q}^{(k)}(x, y) = \\ &= \sum_{r=1}^n \binom{n}{r} \times \left(\sum_{l=0}^{r-1} \frac{(-1)^{r+l+1} (l+1)!}{[l+1]_q^k} S_2(r, l+1) H_{n-r}(x, y) \right). \quad (4.4) \end{aligned}$$

Proof. Using the definition (2.1), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x+1, y) \frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x, y) \frac{t^n}{n!} \\ &= \frac{\text{Li}_{k,q}(1-e^{-t})}{e^t-1} e^{(x+1)t+yt^2} - \frac{\text{Li}_{k,q}(1-e^{-t})}{e^t-1} e^{xt+yt^2}, \\ &= \text{Li}_{k,q}(1-e^{-t}) e^{xt+yt^2}, \\ &= \sum_{r=0}^{\infty} \left(\sum_{l=0}^{r-1} \frac{(-1)^{r+l+1} (l+1)!}{[l+1]_q^k} S_2(r, l+1) \right) \frac{t^r}{r!} e^{xt+yt^2}, \\ &= \sum_{n=0}^{\infty} \left(\sum_{r=1}^n \binom{n}{r} \left(\sum_{l=0}^{r-1} \frac{(-1)^{r+l+1} (l+1)!}{[l+1]_q^k} S_2(r, l+1) H_{n-r}(x, y) \right) \right) \frac{t^n}{n!}. \end{aligned}$$

On comparing the coefficients of $\frac{t^n}{n!}$ on either side, we get the result (4.4). \square

Theorem 4.7. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have

$$\begin{aligned} {}_H B_{n,q}^{(k)}(x, y) &= \sum_{p=0}^n \binom{n}{p} d^{n-p-1} \sum_{l=0}^{p+1} \sum_{a=0}^{d-1} \frac{(-1)^{l+p+1} l! S_2(p+1, l)}{[l]_q^k} (-1)^a \\ &\quad \times {}_H B_{n-p} \left(\frac{a+x}{d}, y \right). \end{aligned} \quad (4.5)$$

Proof. Equation (2.1) can be written as

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x, y) \frac{t^n}{n!} &= \frac{\text{Li}_{k,q}(1 - e^{-t})}{e^t - 1} e^{xt+yt^2} \\ &= \left(\frac{\text{Li}_{k,q}(1 - e^{-t})}{t} \right) \left(\frac{t}{e^{dt} - 1} \sum_{a=0}^{d-1} (-1)^a e^{(a+x)t+yt^2} \right), \\ &= \left(\sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1, l)}{[l]_q^k} \right) \frac{t^p}{p!} \right) \\ &\quad \times \left(\sum_{m=0}^{\infty} d^{m-1} \sum_{a=0}^{d-1} (-1)^a {}_H B_n \left(\frac{a+x}{d}, y \right) \frac{t^n}{n!} \right). \end{aligned}$$

Replacing n by $n - p$ in the R.H.S. of above equation and comparing the coefficient of $\frac{t^n}{n!}$ on either side, we get the result (4.5). \square

Theorem 4.8. For $n \geq 1$, $n, k \in \mathbb{Z}$ and $0 \leq q < 1$, we have

$${}_H B_{n,q}^{(k)}(x, y) = \sum_{m=0}^n \frac{(-1)^{m+n} m!}{[m+1]_q^k} S_2(n, m; x, y). \quad (4.6)$$

Proof. By using (2.1) and (4.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x, y) \frac{t^n}{n!} &= \frac{\text{Li}_{k,q}(1 - e^{-t})}{e^t - 1} e^{xt+yt^2} \\ &= \sum_{m=0}^{\infty} \frac{(1 - e^{-t})^m}{[m+1]_q^k} e^{xt+yt^2} = \sum_{n=0}^{\infty} \frac{(-1)^{m+n} m!}{[m+1]_q^k} \sum_{n=m}^{\infty} S_2(n, m; x, y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \frac{(-1)^{m+n} m!}{[m+1]_q^k} S_2(n, m; x, y) \right) \frac{t^n}{n!}. \end{aligned}$$

On comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we arrive at the desired result (4.6). \square

5. SYMMETRIC IDENTITIES

In this section, we give general symmetry identities for the q -poly-Bernoulli polynomials $B_{n,q}^{(k)}(x)$ and the Hermite poly-Bernoulli polynomials ${}_H B_{n,q}^{(k)}(x, y)$ with q parameter by applying the generating function (2.1) and (2.2). The results extend some known identities of Khan [11-14], Pathan and Khan [20-25].

Theorem 5.1. Let $a, b > 0$ and $a \neq b$, the following identity holds true:

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} b^m a^{n-m} {}_H B_{n-m,q}^{(k)}(bx, b^2y) {}_H B_{m,q}^{(k)}(ax, a^2y) \\ &= \sum_{m=0}^n \binom{n}{m} a^m b^{n-m} {}_H B_{n-m,q}^{(k)}(ax, a^2y) {}_H B_{m,q}^{(k)}(bx, b^2y). \end{aligned} \quad (5.1)$$

Proof. Start with

$$G(t) = \left(\frac{(\text{Li}_{k,q}(1 - e^{-at}))(\text{Li}_{k,q}(1 - e^{-bt}))}{(e^{at} - 1)(e^{bt} - 1)} \right) e^{abxt + a^2b^2yt^2}.$$

Then the expression for $G(t)$ is symmetric in a and b and we can expand $G(t)$ into series in two ways to obtain:

$$\begin{aligned} G(t) &= \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(bx, b^2y) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_H B_{m,q}^{(k)}(ax, a^2y) \frac{(bt)^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} \right) a^{n-m} b^m {}_H B_{n-m,q}^{(k)}(bx, b^2y) {}_H B_{m,q}^{(k)}(ax, a^2y) \frac{t^n}{n!}. \end{aligned}$$

On the similar lines, we can show that

$$\begin{aligned} G(t) &= \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(ax, a^2y) \frac{(bt)^n}{n!} \sum_{m=0}^{\infty} {}_H B_{m,q}^{(k)}(bx, b^2y) \frac{(at)^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} \right) a^m b^{n-m} {}_H B_{n-m,q}^{(k)}(ax, a^2y) {}_H B_{m,q}^{(k)}(bx, b^2y) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result (5.1). \square

Remark 5.1. On setting $b = 1$ in Theorem 5.1, we get the following corollary.

Corollary 5.1. For $n, k \in \mathbb{Z}$ and $n \geq 0$, the following identity holds true:

$$\sum_{m=0}^n \binom{n}{m} a^{n-m} {}_H B_{n-m,q}^{(k)}(x, y) {}_H B_{m,q}^{(k)}(ax, a^2y)$$

$$= \sum_{m=0}^n \binom{n}{m} a^m {}_H B_{n-m,q}^{(k)}(ax, a^2y) {}_H B_{m,q}^{(k)}(x, y).$$

Theorem 5.2. Let $a, b > 0$ and $a \neq b$, the following identity holds true:

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_H B_{n-m,q}^{(k)} \left(bx + \frac{b}{a}i + j, b^2z \right) B_{m,q}^{(k)}(ay) b^m a^{n-m} \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} {}_H B_{n-m,q}^{(k)} \left(ax + \frac{a}{b}i + j, a^2z \right) B_{m,q}^{(k)}(by) a^m b^{n-m}. \end{aligned} \quad (5.2)$$

Proof. Let

$$\begin{aligned} H(t) &= \left(\frac{(\text{Li}_{k,q}(1 - e^{-at}))(\text{Li}_{k,q}(1 - e^{-bt}))(e^{abt} - 1)^2}{(e^{at} - 1)^2(e^{bt} - 1)^2} \right) e^{ab(x+y)t + a^2b^2zt^2}. \\ H(t) &= \left(\frac{\text{Li}_{k,q}(1 - e^{-at})}{e^{at} - 1} \right) e^{abxt + a^2b^2zt^2} \left(\frac{e^{abt} - 1}{e^{bt} - 1} \right) \left(\frac{\text{Li}_{k,q}(1 - e^{-bt})}{e^{bt} - 1} \right) \\ &\quad \times e^{abyt} \left(\frac{e^{abt} - 1}{e^{at} - 1} \right) \\ &= \left(\frac{\text{Li}_{k,q}(1 - e^{-at})}{e^{at} - 1} \right) e^{abxt + a^2b^2zt^2} \sum_{i=0}^{a-1} e^{bti} \left(\frac{\text{Li}_{k,q}(1 - e^{-bt})}{e^{bt} - 1} \right) \\ &\quad \times e^{abyt} \sum_{j=0}^{b-1} e^{atj} \end{aligned} \quad (5.3)$$

$$\begin{aligned} &= \left(\frac{\text{Li}_{k,q}(1 - e^{-at})}{e^{at} - 1} \right) e^{a^2b^2zt^2} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} e^{(bx + \frac{b}{a}i + j)at} \sum_{m=0}^{\infty} B_{m,q}^{(k)}(ay) \frac{(bt)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_H B_{n,q}^{(k)} \left(bx + \frac{b}{a}i + j, b^2z \right) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} B_{m,q}^{(k)}(ay) \frac{(bt)^m}{m!} \\ H(t) &= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_H B_{n-m,q}^{(k)} \left(bx + \frac{b}{a}i + j, b^2z \right) \\ &\quad \times B_{m,q}^{(k)}(ay) b^m a^{n-m} \frac{t^n}{n!}. \end{aligned}$$

In similar method, we can be written as

$$\begin{aligned} H(t) &= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} {}_H B_{n-m,q}^{(k)} \left(ax + \frac{a}{b}i + j, a^2z \right) \\ &\quad \times B_{m,q}^{(k)}(by) a^m b^{n-m} \frac{t^n}{n!}. \end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result (5.2). \square

Theorem 5.3. For each pair of integers a and b and all integers $n \geq 0$, the following identity holds true:

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} a^{n-m} b^m \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_H B_{n-m,q}^{(k)} \left(bx + \frac{b}{a}i, b^2z \right) B_{m,q}^{(k)} \left(ay + \frac{a}{b}j \right) \\ &= \sum_{m=0}^n \binom{n}{m} b^{n-m} a^m \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} {}_H B_{n-m,q}^{(k)} \left(ax + \frac{a}{b}i, a^2z \right) B_{m,q}^{(k)} \left(by + \frac{b}{a}j \right). \end{aligned} \quad (5.4)$$

Proof. The proof is analogous to Theorem 5.2 but we need to write equation (5.3) in the form

$$\begin{aligned} H(t) &= \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_H B_{n,q}^{(k)} \left(bx + \frac{b}{a}i, b^2z \right) \frac{(at)^n}{n!} \\ &\quad \times \sum_{m=0}^{\infty} B_{m,q}^{(k)} \left(ay + \frac{a}{b}j \right) \frac{(bt)^m}{m!}. \end{aligned} \quad (5.5)$$

On the other hand equation (5.3) can be shown equal to

$$\begin{aligned} H(t) &= \sum_{n=0}^{\infty} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} {}_H B_{n,q}^{(k)} \left(ax + \frac{a}{b}i, a^2z \right) \frac{(bt)^n}{n!} \\ &\quad \times \sum_{m=0}^{\infty} B_{m,q}^{(k)} \left(by + \frac{b}{a}j \right) \frac{(at)^m}{m!}. \end{aligned} \quad (5.6)$$

Next making change of index and by equating the coefficients of t to zero in (5.5) and (5.6), we get the result (5.4). \square

Remark 5.2. By setting $y = 0$ in Theorem 5.3, we get the following corollary.

Corollary 5.2. Let $a, b > 0$ and $a \neq b$, the following identity holds true:

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} a^{n-m} b^m \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_H B_{n-m,q}^{(k)} \left(bx + \frac{b}{a}i, b^2z \right) B_{m,q}^{(k)} \left(\frac{a}{b}j \right) \\ &= \sum_{m=0}^n \binom{n}{m} b^{n-m} a^m \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} {}_H B_{n-m,q}^{(k)} \left(ax + \frac{a}{b}i, a^2z \right) B_{m,q}^{(k)} \left(\frac{b}{a}j \right). \end{aligned}$$

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