On exponentially (h_1, h_2) -convex functions and fractional integral inequalities related

MIGUEL VIVAS-CORTEZ,
JORGE ELIECER HERNÁNDEZ HERNÁNDEZ,
SERCAN TURHAN

ABSTRACT. In this work the concept of exponentially (h_1, h_2) -convex function is introduced and using it, the Hermite-Hadamard inequality and some bounds for the right side of this inequality, via Raina's fractional integral operator and generalized convex functions, are established.

1. INTRODUCTION

In many practical investigations it is necessary to bound one quantity by another. The classical inequalities are very useful for this purpose. An enormous amount of efforts has been devoted to the extension of the classical inequalities and to the applications of the same in diverse areas of science: estimation of integrals, special functions of mathematical physics, electrostatic field and capacitance, signal analysis, dynamical system stability and control and others.

One of the most discussed inequalities in recent work is the classic Hermite-Hadamard inequality. In [7], J. Hadamard stated his famous inequality in this way.

Theorem 1. *Let f be a convex function over $[a, b]$, $a < b$. If f is integrable over $[a, b]$, then*

$$(1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

2010 *Mathematics Subject Classification.* Primary: 26A33; Secondary: 26A51, 26D10, 33E12.

Key words and phrases. Exponentially (h_1, h_2) -convex function, Raina's fractional integral operator, fractional integral inequalities.

Full paper. Received 7 august 2019, revised 2 February 2020, accepted 24 February 2020, available online 29 February 2020.

*This paper was technically supported by Dirección de Investigación from Pontificia Universidad Católica del Ecuador and Consejo de Desarrollo Científico, Humanístico y Tecnológico from Universidad Centroccidental Lisandro Alvarado (Venezuela).

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during almost the past five decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables.

The inequalities involving more general fractional integral operators have also been considered in [2, 12, 16, 19]. Since work in this direction has received a lot of attention, as evidenced in the work of S. Turhan et. al. [13, 20] and J. E. Hernández Hernández and M. J. Vivas-Cortez [8, 9, 10, 21], in this work we establish a general expression of some Hermite-Hadamard type inequalities by the introduction of the concept of exponentially (h_1, h_2) -convex function and using the Raina's fractional integral operator.

2. PRELIMINARIES

2.1. About Fractional Integral Operator. In [16], R. K. Raina introduced a class of functions defined formally by

$$(2) \quad \mathcal{F}_{\rho, \lambda}^{\sigma}(x) = \mathcal{F}_{\rho, \lambda}^{\sigma(0), \sigma(1), \dots}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k$$

where $\rho, \lambda > 0, |x| < R$, (R is the set of real numbers), $\sigma = (\sigma(1), \dots, \sigma(k), \dots)$ is a bounded sequence of positive real numbers. Note that if we take in (2) $\rho = 1, \lambda = 0$ and $\sigma(k) = ((\alpha)_k(\beta)_k)/(\gamma)_k$ for $k = 0, 1, 2, \dots$, where α, β and γ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \dots$), and the symbol $(a)_k$ denote the quantity

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\dots(a+k-1), \quad k = 0, 1, \dots,$$

and restrict its domain to $|x| \leq 1$ (with $x \in \mathbb{C}$), then we have the classical Hypergeometric Function, that is

$$\mathcal{F}_{\rho, \lambda}^{\sigma}(x) = F(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} x^k,$$

also, if $\sigma(k) = (1, 1, 1, \dots)$ with $\rho = \alpha, (Re(\alpha) > 0)$, $\lambda = 1$ and restricting its domain to $z \in \mathbb{C}$ in (2) then we have the classical Mittag-Leffler function

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k.$$

When it is provided that the series converges uniformly then we can differentiate term wise, also integrate, to obtain

$$\left(\frac{d}{dx}\right)^n x^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma(wx^\rho) = x^{\lambda-n-1} \mathcal{F}_{\rho,\lambda-n}^\sigma(wx^\rho)$$

and

$$\int_0^x \dots \int_0^x t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma(wt^\rho)(dt)^n = x^{\lambda+n-1} \mathcal{F}_{\rho,\lambda+n}^\sigma(wx^\rho).$$

Using (2), in [2], R. P. Agarwal et. al., defined the following left-sided and right-sided fractional integral operators respectively, as follows

$$(3) \quad (\mathcal{J}_{\rho,\lambda,a+;w}^\sigma \varphi)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma[w(x-t)^\rho] \varphi(t) dt, \quad (x > a)$$

and

$$(4) \quad (\mathcal{J}_{\rho,\lambda,b-;w}^\sigma \varphi)(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma[w(t-x)^\rho] \varphi(t) dt, \quad (x < b),$$

where $\lambda, \rho > 0$, $w \in R$ and φ is such that the integral on the right side exists.

It is easy to verify that $\mathcal{J}_{\rho,\lambda,a+;w}^\sigma \varphi$ and $\mathcal{J}_{\rho,\lambda,b-;w}^\sigma \varphi$ are bounded integral operators on $L_p(a, b)$, ($1 \leq p \leq \infty$), if

$$\mathfrak{M} := \mathcal{F}_{\rho,\lambda+1}^\sigma[w(b-a)^\rho] < \infty.$$

Indeed, for $\varphi \in L_p((a, b))$ we have

$$\|\mathcal{J}_{\rho,\lambda,a+;w}^\sigma \varphi\|_p \leq \mathfrak{M} \|\varphi\|_p$$

and

$$\|\mathcal{J}_{\rho,\lambda,b-;w}^\sigma \varphi\|_p \leq \mathfrak{M} \|\varphi\|_p$$

where

$$\|\varphi\|_p = \left(\int_a^b |\varphi(x)|^p dx \right)^{1/p}.$$

Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. By example, the classical Riemann-Liouville fractional integrals I_{a+}^α and I_{b-}^α of order α

$$(I_{a+}^\alpha \varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \varphi(t) dt, \quad (x > a, \alpha > 0)$$

and

$$(I_{b-}^\alpha \varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \varphi(t) dt, \quad (x < b, \alpha > 0)$$

follow from (3) and (4) setting $\lambda = \alpha, \sigma(0) = 1$ and $w = 0$.

The Hermite-Hadamard integral inequality for the Raina's fractional integral operator is established in [22] as follows.

Theorem 2. Let $\lambda \in \mathbb{R}^+$, $a, b \in \mathbb{R}$, $a < b$ and $\phi : [a, b] \rightarrow \mathbb{R}$ be a convex function. Then

$$\begin{aligned} \phi\left(\frac{a+b}{2}\right) &\leq \frac{\left(\mathcal{J}_{\rho, \lambda, a+, w}^\sigma\right)(b) + \left(\mathcal{J}_{\rho, \lambda, b-, w}^\sigma\right)(a)}{(b-a)^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma[w(b-a)^\rho]} \\ &\leq \frac{\phi(a) + \phi(b)}{2}. \end{aligned}$$

2.2. About Generalized convexity. The well known concept of convex function is due to W. Jensen and it is established as follow.

Definition 1. A function $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is called convex on the interval I , if the following inequality holds

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

for all $a, b \in I$ and $t \in [0, 1]$.

From the work of S. S. Dragomir et. al. [5], we extract the following definition.

Definition 2. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative function where I is an interval. It is said that f belongs to the class $P(I)$ or f is a P -convex if for all $a, b \in I$ and $t \in [0, 1]$ the following inequality holds

$$f(ta + (1-t)b) \leq f(a) + f(b).$$

Also, H. Hudzik and L. Maligranda, in [11], disused about some properties of the following generalized concept of convexity.

Definition 3. Let $0 < s \leq 1$. A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$, is said to be s -convex in the first sense if

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t^s)f(b)$$

for all $a, b \in I$ and $t \in [0, 1]$. This is denoted by $f \in K_s^1$. A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, is said to be s -convex in the second sense if

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b)$$

for all $a, b \in I$ and $t \in [0, 1]$. This is denoted by $f \in K_s^2$.

The first class of functions in Definition 3 were introduced by Orlicz W. in [15], and the second class by W. W. Breckner in [3].

G. Cristescu et. al., in [4], in order to study bounds of the second degree cumulative frontier gaps of functions with generalized convexity functions, introduced the so-called (h_1, h_2) -convex functions.

Definition 4. Let $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}$ be two non-negative functions. A function $f : I \rightarrow \mathbb{R}_+$ is called an (h_1, h_2) -convex function if the inequality

$$f(ta + (1-t)b) \leq h_1(t)f(a) + h_2(t)f(b)$$

holds for all $a, b \in I$ and $t \in [0, 1]$. The functions that transform the inequality in an equality is called (h_1, h_2) -affine function.

Remark 1. If $h_1(t) = t$ and $h_2(t) = 1 - t$ for all $t \in [0, 1]$, then the (h_1, h_2) -convexity coincides with the classical convexity. If $h_1(t) = h_2(t) = 1$ for all $t \in [0, 1]$ the it is obtained the P -convexity. If $h_1(t) = t^s$ and $h_2(t) = 1 - t^s$ for all $t \in [0, 1]$, then we have the s -convexity in the first sense, and If $h_1(t) = t^s$ and $h_2(t) = (1 - t)^s$ for all $t \in [0, 1]$, we get the s -convexity in the second sense.

The exponentially convex functions are of interest for the development of this work. In the works of T. Antczac [1] and S. S. Dragomir [6] introduce this concept and find some results related to the Hermite-Hadamard inequality.

Definition 5. A positive function $f : I \rightarrow \mathbb{R}$ is said to be an exponentially convex function if the inequality

$$e^{f(ta+(1-t)b)} \leq te^{f(a)} + (1-t)e^{f(b)}$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

3. MAIN RESULTS

Definition 6. Let $h_1, h_2 : [0, 1] \rightarrow R$ be a two non negative functions. A positive function $f : I \rightarrow R$, where I is an interval include in $\mathbb{R}$, is called exponentially (h_1, h_2) -convex if the following inequality holds for all $x, y \in I$ and $t \in [0, 1]$

$$e^{f(tx+(1-t)y)} \leq h_1(t)e^{f(x)} + h_2(t)e^{f(y)}.$$

Remark 2. Note that:

- (1) If $h_1 = h_2 \equiv 1$ then we have an exponentially P -convex function.
- (2) If $h_1(t) = t$ and $h_2(t) = 1 - t$ for all $t \in [0, 1]$ we obtain an exponentially convex function.
- (3) If $h_1(t) = t^s$ and $h_2(t) = 1 - t^s$ for all $t \in [0, 1]$ and some $0 < s \leq 1$, then we have the exponentially s -convexity in the first sense.
- (4) If $h_1(t) = t^s$ and $h_2(t) = (1 - t)^s$ for all $t \in [0, 1]$ and some $0 < s \leq 1$, we get the exponentially s -convexity in the second sense.

First, we establish the Hermite-Hadamard inequality for exponentially convex function using Raina's fractional integral operator.

Theorem 3. Let $\lambda, \rho > 0, w \in R$, and $\sigma = \{\sigma(k)\}_{k=0}^{\infty}$ a sequence of non-negatives real numbers. Let $f : [a, b] \rightarrow R$ be an exponentially (h_1, h_2) -convex function, then the following inequalities holds

$$e^{f\left(\frac{a+b}{2}\right)} \leq \frac{\left(h_1(1/2) \left(\mathcal{J}_{\rho, \lambda, a+, w}^{\sigma} e^f\right)(b) + h_2(1/2) \left(\mathcal{J}_{\rho, \lambda, b-, w}^{\sigma} e^f\right)(a)\right)}{(b-a)^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma} [w(b-a)^{\rho}]}$$

and

$$\begin{aligned} & \frac{1}{(b-a)^\lambda} \left(\left(\mathcal{J}_{\rho, \lambda, a+, w}^\sigma e^f \right) (b) + \left(\mathcal{J}_{\rho, \lambda, b-, w}^\sigma e^f \right) (a) \right) \\ & \leq \left(e^{f(a)} + e^{f(b)} \right) \left(I(h_1) + I(h_2) \right), \end{aligned}$$

where

$$I(h_1) = \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w(b-a)^\rho t^\rho] h_1(t) dt$$

and

$$I(h_2) = \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w(b-a)^\rho t^\rho] h_2(t) dt.$$

Proof. Note that

$$\frac{a+b}{2} = \frac{ta + (1-t)b + tb + (1-t)a}{2}$$

for any $t \in [0, 1]$. Consequently, using the exponentially (h_1, h_2) -convexity of f we have

$$\begin{aligned} e^{f\left(\frac{a+b}{2}\right)} &= e^{f\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right)} \\ &\leq h_1(1/2) e^{f(ta+(1-t)b)} + h_2(1/2) e^{f(tb+(1-t)a)}. \end{aligned}$$

Multiplying by $t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w(b-a)^\rho t^\rho]$ in both sides of the above inequality

$$\begin{aligned} & e^{f\left(\frac{a+b}{2}\right)} t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w(b-a)^\rho t^\rho] \\ & \leq t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w(b-a)^\rho t^\rho] \left(h_1(1/2) e^{f(ta+(1-t)b)} + h_2(1/2) e^{f(tb+(1-t)a)} \right). \end{aligned}$$

Integrating over $t \in [0, 1]$ we have

$$\begin{aligned} (5) \quad & e^{f\left(\frac{a+b}{2}\right)} \mathcal{F}_{\rho, \lambda+1}^\sigma [w(b-a)^\rho] \\ & \leq h_1(1/2) \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w(b-a)^\rho t^\rho] e^{f(ta+(1-t)b)} dt \\ & \quad + h_2(1/2) \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w(b-a)^\rho t^\rho] e^{f(tb+(1-t)a)} dt. \end{aligned}$$

With a convenient change of variable we have

$$\begin{aligned} (6) \quad & \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w(b-a)^\rho t^\rho] e^{f(ta+(1-t)b)} dt \\ & \leq \frac{1}{b-a} \int_a^b \left(\frac{b-x}{b-a} \right)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma \left[w(b-a)^\rho \left(\frac{b-x}{b-a} \right)^\rho \right] e^{f(x)} dx \\ & = \frac{1}{(b-a)^\lambda} \int_a^b (b-x)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w(b-x)^\rho] e^{f(x)} dx \end{aligned}$$

$$= \frac{1}{(b-a)^\lambda} \left(\mathcal{J}_{\rho, \lambda, a+, w}^\sigma e^f \right) (b)$$

and

$$\begin{aligned}
 (7) \quad & \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] e^{f(tb+(1-t)a)} dt \\
 & \leq \frac{1}{b-a} \int_a^b \left(\frac{x-a}{b-a} \right)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma \left[w (b-a)^\rho \left(\frac{x-a}{b-a} \right)^\rho \right] e^{f(x)} dx \\
 & = \frac{1}{(b-a)^\lambda} \int_a^b (x-a)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-x)^\rho] e^{f(x)} dx \\
 & = \frac{1}{(b-a)^\lambda} \left(\mathcal{J}_{\rho, \lambda, b-, w}^\sigma e^f \right) (a).
 \end{aligned}$$

By replacement of (6) and (7) in (5) we have

$$\begin{aligned}
 & e^{f(\frac{a+b}{2})} \mathcal{F}_{\rho, \lambda+1}^\sigma [w (b-a)^\rho] \\
 & \leq \frac{1}{(b-a)^\lambda} \left(h_1(1/2) \left(\mathcal{J}_{\rho, \lambda, a+, w}^\sigma e^f \right) (b) + h_2(1/2) \left(\mathcal{J}_{\rho, \lambda, b-, w}^\sigma e^f \right) (a) \right).
 \end{aligned}$$

For the right side of the proposed inequality we have

$$\begin{aligned}
 e^{f(ta+(1-t)b)} & \leq h_1(t) e^{f(a)} + h_2(t) e^{f(b)} \\
 e^{f(tb+(1-t)a)} & \leq h_1(t) e^{f(b)} + h_2(t) e^{f(a)}.
 \end{aligned}$$

Multiplying by $t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho]$ both inequalities

$$\begin{aligned}
 & t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] e^{f(ta+(1-t)b)} \\
 & \leq t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] \left(h_1(t) e^{f(a)} + h_2(t) e^{f(b)} \right), \\
 & t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] e^{f(tb+(1-t)a)} \\
 & \leq t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] \left(h_1(t) e^{f(b)} + h_2(t) e^{f(a)} \right).
 \end{aligned}$$

Adding these inequalities and integrating over $t \in [0, 1]$ we obtain

$$\begin{aligned}
 & \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] \left(e^{f(ta+(1-t)b)} + e^{f(tb+(1-t)a)} \right) dt \\
 & \leq \left(e^{f(a)} + e^{f(b)} \right) \left(\int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] h_1(t) dt \right. \\
 & \quad \left. + \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] h_2(t) dt \right).
 \end{aligned}$$

With the change of variable $u = ta + (1-t)b$ and $v = tb + (1-t)a$ in the first integral of the above inequality it is obtained

$$\begin{aligned} & \frac{1}{(b-a)^\lambda} \left(\left(\mathcal{J}_{\rho, \lambda, a+, w}^\sigma e^f \right) (b) + \left(\mathcal{J}_{\rho, \lambda, b-, w}^\sigma e^f \right) (a) \right) \\ & \leq \left(e^{f(a)} + e^{f(b)} \right) \left(\int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] h_1(t) dt \right. \\ & \quad \left. + \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] h_2(t) dt \right), \end{aligned}$$

and letting

$$I(h_1) = \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] h_1(t) dt$$

and

$$I(h_2) = \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [w (b-a)^\rho t^\rho] h_2(t) dt$$

it is attained the desired result. $\square$

Remark 3. Doing $\lambda = \alpha$, $w = 1$ and $\sigma = (1, 0, \dots)$, then from Theorem 3, we obtain the Riemann-Liouville fractional integral version:

$$e^{f(\frac{a+b}{2})} \leq \frac{(h_1(1/2) (\mathcal{I}_{a+}^\alpha e^f) (b) + h_2(1/2) (\mathcal{I}_{b-}^\alpha e^f) (a))}{(b-a)^\alpha \Gamma(\alpha)^{-1}}$$

and

$$\frac{1}{(b-a)^\alpha} \left((\mathcal{I}_{a+}^\alpha e^f) (b) + (\mathcal{I}_{b-}^\alpha e^f) (a) \right) \leq (e^{f(a)} + e^{f(b)}) (I(h_1) + I(h_2)),$$

where $I(h_1)$ and $I(h_2)$ take the form

$$I(h_1) = \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} h_1(t) dt \text{ and } I(h_2) = \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} h_2(t) dt$$

additionally if $h_1(t) = t$ and $h_2(t) = 1 - t$ then

$$e^{f(\frac{a+b}{2})} \leq \frac{\Gamma(\alpha+1) ((\mathcal{I}_{a+}^\alpha e^f) (b) + (\mathcal{I}_{b-}^\alpha e^f) (a))}{2(b-a)^\alpha} \leq \frac{(e^{f(a)} + e^{f(b)})}{\Gamma(\alpha+1)},$$

and if we choose $\alpha = 1$ then it is obtained

$$2e^{f(\frac{a+b}{2})} \leq \frac{1}{b-a} \int_a^b e^{f(t)} dt \leq 2(e^{f(a)} + e^{f(b)})$$

making coincidence with Corollary 3.2 in [17]. If $h_1(t) = t^s$ and $h_2 = (1-t)^s$ for $t \in [0, 1]$ and some $s \in (0, 1]$ with $\lambda = \alpha = 1$, $w = 1$ and $\sigma = (1, 0, \dots)$ we find coincidence with Corollary 1 obtained by S. Rashid et.al. in [18].

The following Lemma will be useful to establish some others inequalities related with the right side of the Hermite-Hadamard inequality for exponentially convex functions using the Raina's fractional integral operator.

Lemma 3.1. *Let $\lambda, \rho > 0, w \in R$, and $\sigma = \{\sigma(k)\}_{k=0}^{\infty}$ a sequence of non-negatives real numbers. Let $f : [a, b] \rightarrow R$ be a differentiable mapping on (a, b) with $a < b$ and $\lambda > 0$. If $e^f \in L_1([a, b])$ then the following equality for the Raina's fractional integral operator holds*

$$\begin{aligned} & \left(\mathcal{J}_{\rho, \lambda, a+; w}^{\sigma} e^f \right) (b) + \left(\mathcal{J}_{\rho, \lambda, b-; w}^{\sigma} e^f \right) (a) \\ & - \left(\frac{\mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho})}{(b-a)^{1-\lambda}} \right) \left(e^{f(a)} + e^{f(b)} \right) \\ & = \int_0^1 t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) e^{f(ta+(1-t)b)} f'(ta+(1-t)b) dt \\ & - \int t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) e^{f((1-t)a+tb)} f'((1-t)a+tb) dt. \end{aligned}$$

Proof. Using integration by parts it follows that

$$\begin{aligned} I_1 &= \int_0^1 t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) e^{f(ta+(1-t)b)} f'(ta+(1-t)b) dt \\ &= \frac{t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) e^{f(ta+(1-t)b)}}{a-b} \Big|_0^1 \\ & - \frac{1}{a-b} \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^{\sigma}(w(b-a)^{\rho} t^{\rho}) e^{f(ta+(1-t)b)} dt \\ &= - \frac{\mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho}) e^{f(a)}}{b-a} + \frac{1}{(b-a)^{\lambda}} \left(\mathcal{J}_{\rho, \lambda, a+; w}^{\sigma} e^f \right) (b) \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) e^{f((1-t)a+tb)} f'((1-t)a+tb) dt \\ &= \frac{\mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho}) e^{f(b)}}{b-a} \\ & - \frac{1}{b-a} \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho, \lambda}^{\sigma}(w(b-a)^{\rho} t^{\rho}) e^{f((1-t)a+tb)} dt \\ &= \frac{\mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho}) e^{f(b)}}{b-a} - \frac{1}{(b-a)^{\lambda}} \left(\mathcal{J}_{\rho, \lambda, b-; w}^{\sigma} e^f \right) (a). \end{aligned}$$

Subtracting I_2 from I_1 it is attained the desired result. $\square$

Theorem 4. *Let $\lambda, \rho > 0, w \in R$, and σ a sequence of non-negatives real numbers. Let $f : [a, b] \rightarrow R$ be a differentiable mapping on (a, b) with $a < b$, and exponentially (h_1, h_2) -convex. If $e^f \in L_1([a, b])$ and $|f'|$ is*

(g_1, g_2) –convex then the following inequality for the Raina's fractional integral operator holds

$$\begin{aligned} & \left| \left(\mathcal{J}_{\rho, \lambda, a+; w}^{\sigma} e^f \right) (b) + \left(\mathcal{J}_{\rho, \lambda, b-; w}^{\sigma} e^f \right) (a) \right. \\ & \quad \left. - \left(\frac{\mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho})}{(b-a)^{1-\lambda}} \right) \left(e^{f(a)} + e^{f(b)} \right) \right| \\ & \leq (e^{f(a)} |f'(a)| + e^{f(b)} |f'(b)|) (I(h_1, g_1) + I(h_2, g_2)) \\ & \quad + (e^{f(a)} |f'(b)| + e^{f(b)} |f'(a)|) (I(h_1, g_2) + I(h_2, g_1)), \end{aligned}$$

where

$$\begin{aligned} I(h_1, g_1) &= \int_0^1 t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) h_1(t) g_1(t) dt, \\ I(h_2, g_2) &= \int_0^1 t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) h_2(t) g_2(t) dt, \\ I(h_1, g_2) &= \int_0^1 t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) h_1(t) g_2(t) dt, \\ I(h_2, g_1) &= \int_0^1 t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) h_2(t) g_1(t) dt. \end{aligned}$$

Proof. Using the Lemma 3.1 and the triangular inequality we have

$$\begin{aligned} (8) \quad & \left| \left(\mathcal{J}_{\rho, \lambda, a+; w}^{\sigma} e^f \right) (b) + \left(\mathcal{J}_{\rho, \lambda, b-; w}^{\sigma} e^f \right) (a) \right. \\ & \quad \left. - \left(\frac{\mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho})}{(b-a)^{1-\lambda}} \right) \left(e^{f(a)} + e^{f(b)} \right) \right| \\ & \leq \int_0^1 t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) |e^{f(ta+(1-t)b)} f'(ta+(1-t)b)| dt \\ & \quad + \int_0^1 t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) |e^{f((1-t)a+tb)} f'((1-t)a+tb)| dt. \end{aligned}$$

Now, we discuss the integrals involve in (8) using the exponentially (h_1, h_2) –convexity of f and the (g_1, g_2) –convexity of $|f'|$. First,

$$\begin{aligned} & \int_0^1 t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) |e^{f(ta+(1-t)b)} f'(ta+(1-t)b)| dt \\ & \leq \int_0^1 t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) \times \\ & \quad \left(h_1(t) e^{f(a)} + h_2(t) e^{f(b)} \right) (g_1(t) |f'(a)| + g_2(t) |f'(b)|) dt \\ & \leq \int_0^1 t^{\lambda} \mathcal{F}_{\rho, \lambda+1}^{\sigma}(w(b-a)^{\rho} t^{\rho}) \times \end{aligned}$$

$$\begin{aligned}
& \left(h_1(t)e^{f(a)}g_1(t)|f'(a)| + h_2(t)e^{f(b)}g_2(t)|f'(b)| \right. \\
& \quad \left. + h_1(t)e^{f(a)}g_2(t)|f'(b)| + h_2(t)e^{f(b)}g_1(t)|f'(a)| \right) dt \\
& = e^{f(a)}|f'(a)| \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) h_1(t) g_1(t) dt \\
& \quad + e^{f(b)}|f'(b)| \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) h_2(t) g_2(t) dt \\
& \quad + e^{f(a)}|f'(b)| \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) h_1(t) g_2(t) dt \\
& \quad + e^{f(b)}|f'(a)| \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) h_2(t) g_1(t) dt \\
(9) \quad & = e^{f(a)}|f'(a)|I(h_1, g_1) + e^{f(b)}|f'(b)|I(h_2, g_2) \\
& \quad + e^{f(a)}|f'(b)|I(h_1, g_2) + e^{f(b)}|f'(a)|I(h_2, g_1),
\end{aligned}$$

where

$$\begin{aligned}
I(h_1, g_1) &= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) h_1(t) g_1(t) dt, \\
I(h_2, g_2) &= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) h_2(t) g_2(t) dt, \\
I(h_1, g_2) &= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) h_1(t) g_2(t) dt, \\
I(h_2, g_1) &= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) h_2(t) g_1(t) dt.
\end{aligned}$$

Similarly, for the second integral we have

$$\begin{aligned}
& \int t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) |e^{f((1-t)a+tb)} f'((1-t)a+tb)| dt \\
(10) \quad & \leq e^{f(a)}|f'(a)|I(h_2, g_2) + e^{f(b)}|f'(b)|I(h_1, g_1) \\
& \quad + e^{f(a)}|f'(b)|I(h_2, g_1) + e^{f(b)}|f'(a)|I(h_1, g_2).
\end{aligned}$$

By replacement of (9) and (10) in (8) then it follows the result. $\square$

Using the previous Theorem some Corollary is deduced.

Corollary 1. *Let $\lambda, \rho > 0, w \in R$, and $\sigma = \{\sigma(k)\}_{k=0}^\infty$ a sequence of non-negatives real numbers. Let $f : [a, b] \rightarrow R$ be a differentiable mapping on (a, b) with $a < b$, and exponentially convex. If $e^f \in L_1([a, b])$ and $|f'|$ is a convex function then the following inequality for the Raina's fractional integral operator holds*

$$\left| \left(\mathcal{J}_{\rho, \lambda, a+; w}^\sigma e^f \right) (b) + \left(\mathcal{J}_{\rho, \lambda, b-; w}^\sigma e^f \right) (a) \right|$$

$$\begin{aligned}
& - \left(\frac{\mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho)}{(b-a)^{1-\lambda}} \right) \left(e^{f(a)} + e^{f(b)} \right) \Big| \\
& \leq (e^{f(a)}|f'(a)| + e^{f(b)}|f'(b)|) \mathcal{F}_{\rho, \lambda+1}^{2\sigma_1}(w(b-a)^\rho) \\
& \quad + (e^{f(a)}|f'(b)| + e^{f(b)}|f'(a)|) \mathcal{F}_{\rho, \lambda+1}^{2\sigma_2}(w(b-a)^\rho),
\end{aligned}$$

where

$$\begin{aligned}
\sigma_1(k) &= \frac{\sigma(k)}{k\rho + \lambda + 3}, \\
\sigma_2(k) &= \frac{\sigma(k)}{(k\rho + \lambda + 3)(k\rho + \lambda + 2)}, \quad k = 0, 1, 2, \dots
\end{aligned}$$

Proof. Letting $h_1(t) = g_1(t) = t$ and $h_2(t) = g_2(t) = 1 - t$ for all $t \in [0, 1]$ then

$$\begin{aligned}
I(h_1, g_1) = I(h_2, g_2) &= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) t^2 dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho + \lambda + 1)} w^k (b-a)^{k\rho} \int_0^1 t^{k\rho + \lambda + 2} dt \\
&= \mathcal{F}_{\rho, \lambda+1}^{\sigma_1}(w(b-a)^\rho),
\end{aligned}$$

where

$$\sigma_1(k) = \frac{\sigma(k)}{k\rho + \lambda + 3} \text{ for } k = 0, 1, 2, \dots$$

Similarly

$$\begin{aligned}
I(h_1, g_2) = I(h_2, g_1) &= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) t(1-t) dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho + \lambda + 1)} w^k (b-a)^{k\rho} \int_0^1 t^{k\rho + \lambda + 1} (1-t) dt \\
&= \mathcal{F}_{\rho, \lambda+1}^{\sigma_2}(w(b-a)^\rho)
\end{aligned}$$

where

$$\sigma_2(k) = \frac{\sigma(k)}{(k\rho + \lambda + 3)(k\rho + \lambda + 2)}, \quad k = 0, 1, 2, \dots$$

Making the corresponding substitutions in Theorem 4 it follows the desired result. $\square$

Corollary 2. Let $\lambda, \rho > 0, w \in \mathbb{R}$, and $\sigma = \{\sigma(k)\}_{k=0}^{\infty}$ a sequence of non-negatives real numbers. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$, and exponentially convex. If $e^f \in L_1([a, b])$ and $|f'|$ is convex and bounded by some $M > 0$, then the following inequality for the Raina's fractional integral operator holds

$$\left| \left(\mathcal{J}_{\rho, \lambda, a+; w}^\sigma e^f \right) (b) + \left(\mathcal{J}_{\rho, \lambda, b-; w}^\sigma e^f \right) (a) \right|$$

$$\begin{aligned}
& - \left(\frac{\mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho)}{(b-a)^{1-\lambda}} \right) \left(e^{f(a)} + e^{f(b)} \right) \Bigg| \\
& \leq M(e^{f(a)} + e^{f(b)}) \mathcal{F}_{\rho, \lambda+1}^{2\sigma'}(w(b-a)^\rho),
\end{aligned}$$

where

$$\sigma'(k) = \frac{\sigma(k)}{k\rho + \lambda + 2}, \quad k = 0, 1, 2, \dots$$

Proof. Noting that

$$\begin{aligned}
\sigma'(k) &= \sigma_1(k) + \sigma_2(k) \\
&= \frac{\sigma(k)}{k\rho + \lambda + 2}, \quad k = 0, 1, 2, \dots
\end{aligned}$$

then, using Corollary 1 easily it finds the result. $\square$

Corollary 3. Let $\lambda, \rho > 0, w \in R$, and $\sigma = \{\sigma(k)\}_{k=0}^\infty$ a sequence of non-negatives real numbers. Let $f : [a, b] \rightarrow R$ be a differentiable mapping on (a, b) with $a < b$, and exponentially convex. If $e^f \in L_1([a, b])$ and $|f'|$ is a P -convex function then the following inequality for the Raina's fractional integral operator holds

$$\begin{aligned}
& \left| \left(\mathcal{J}_{\rho, \lambda, a+; w}^\sigma e^f \right)(b) + \left(\mathcal{J}_{\rho, \lambda, b-; w}^\sigma e^f \right)(a) \right. \\
& \quad \left. - \left(\frac{\mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho)}{(b-a)^{1-\lambda}} \right) \left(e^{f(a)} + e^{f(b)} \right) \right| \\
& \leq \mathcal{F}_{\rho, \lambda+1}^{\sigma'}(w(b-a)^\rho) \left[(e^{f(a)} + e^{f(b)}) (|f'(a)| + |f'(b)|) \right],
\end{aligned}$$

where

$$\sigma'(k) = \frac{\sigma(k)}{k\rho + \lambda + 1}, \quad k = 0, 1, 2, \dots$$

Proof. Following the same demonstration scheme used in the Corollary 1 with Letting $h_1(t) = t$, $h_2(t) = 1 - t$, $g_1(t) = g_2(t) = 1$, for all $t \in [0, 1]$ then it is obtained that

$$\begin{aligned}
I(h_1, g_1) &= I(h_1, g_2) \\
&= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) t dt \\
&= \mathcal{F}_{\rho, \lambda+1}^{\sigma_1}(w(b-a)^\rho),
\end{aligned}$$

where

$$\sigma_1(k) = \frac{\sigma(k)}{k\rho + \lambda + 2}, \quad k = 0, 1, 2, \dots$$

and

$$I(h_2, g_1) = I(h_2, g_2) = \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) (1-t) dt = \mathcal{F}_{\rho, \lambda+1}^{\sigma_2}(w(b-a)^\rho),$$

where

$$\sigma_2(k) = \frac{\sigma(k)}{(k\rho + \lambda + 2)(k\rho + \lambda + 1)}, \quad k = 0, 1, 2, \dots$$

Note that

$$\begin{aligned} I(h_1, g_1) + I(h_2, g_2) &= \mathcal{F}_{\rho, \lambda+1}^{\sigma_1}(w(b-a)^\rho) + \mathcal{F}_{\rho, \lambda+1}^{\sigma_2}(w(b-a)^\rho) \\ &= \mathcal{F}_{\rho, \lambda+1}^{\sigma'}(w(b-a)^\rho), \end{aligned}$$

where

$$\sigma'(k) = \sigma_1(k) + \sigma_2(k) = \frac{\sigma(k)}{k\rho + \lambda + 1}, \quad k = 0, 1, 2, \dots$$

The proof is complete. $\square$

Corollary 4. Let $\lambda, \rho > 0, w \in R$, and $\sigma = \{\sigma(k)\}_{k=0}^\infty$ a sequence of non-negatives real numbers. Let $f : [a, b] \rightarrow R$ be a differentiable mapping on (a, b) with $a < b$, and exponentially convex. If $e^f \in L_1([a, b])$ and $|f'|$ is a s -convex function in the second sense then the following inequality for the Raina's fractional integral operator holds

$$\begin{aligned} &\left| \left(\mathcal{J}_{\rho, \lambda, a+; w}^\sigma e^f \right)(b) + \left(\mathcal{J}_{\rho, \lambda, b-; w}^\sigma e^f \right)(a) \right. \\ &\quad \left. - \left(\frac{\mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho)}{(b-a)^{1-\lambda}} \right) (e^{f(a)} + e^{f(b)}) \right| \\ &\leq (e^{f(a)}|f'(a)| + e^{f(b)}|f'(b)|) \times \\ &\quad (\mathcal{F}_{\rho, \lambda+1}^{\sigma_{s;1,1}}(w(b-a)^\rho) + \Gamma(s+2)\mathcal{F}_{\rho, \lambda+s+4}^{\sigma_1}(w(b-a)^\rho)) \\ &\quad + (e^{f(a)}|f'(b)| + e^{f(b)}|f'(a)|) \times \\ &\quad (\Gamma(s+1)\mathcal{F}_{\rho, \lambda+s+4}^{\sigma_2}(w(b-a)^\rho) + \mathcal{F}_{\rho, \lambda+1}^{\sigma_{s;2,1}}(w(b-a)^\rho)), \end{aligned}$$

where

$$\begin{aligned} \sigma_{s;1,1}(k) &= \frac{\sigma(k)}{k\rho + \lambda + s + 2}, \\ \sigma_1(k) &= \sigma(k)(k\rho + \lambda + 1), \\ \sigma_2(k) &= \sigma(k)(k\rho + \lambda + 2)(k\rho + \lambda + 1), \\ \sigma_{s;2,1}(k) &= \frac{\sigma(k)}{(k\rho + \lambda + s + 3)(k\rho + \lambda + s + 2)}, \end{aligned}$$

for $k = 0, 1, 2, \dots$

Proof. Letting $h_1(t) = t$, $h_2(t) = (1-t)$, $g_1(t) = t^s$ and $g_2(t) = (1-t)^s$ for all $t \in [0, 1]$ and some $s \in (0, 1]$ then

$$\begin{aligned} I(h_1, g_1) &= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) t^{s+1} dt \\ &= \sum_{k=0}^\infty \frac{\sigma(k)}{\Gamma(k\rho + \lambda + 1)} w^k (b-a)^{k\rho} \int_0^1 t^{k\rho + \lambda + s + 1} dt \end{aligned}$$

$$= \mathcal{F}_{\rho, \lambda+1}^{\sigma_{s;1,1}}(w(b-a)^\rho),$$

where

$$\sigma_{s;1,1}(k) = \frac{\sigma(k)}{k\rho + \lambda + s + 2}, \quad k = 0, 1, 2, \dots,$$

$$\begin{aligned} I(h_2, g_2) &= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho)(1-t)^{s+1} dt \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho + \lambda + 1)} w^k (b-a)^{k\rho} \int_0^1 t^{k\rho + \lambda + 1} (1-t)^{s+1} dt \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho + \lambda + 1)} w^k (b-a)^{k\rho} \frac{\Gamma(k\rho + \lambda + 2) \Gamma(s+2)}{\Gamma(k\rho + \lambda + s + 4)} \\ &= \Gamma(s+2) \mathcal{F}_{\rho, \lambda+s+4}^{\sigma_1}(w(b-a)^\rho) \end{aligned}$$

where

$$\sigma_1(k) = \sigma(k)(k\rho + \lambda + 1), \quad k = 0, 1, 2, \dots,$$

$$\begin{aligned} I(h_1, g_2) &= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho) t(1-t)^s dt \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho + \lambda + 1)} w^k (b-a)^{k\rho} \int_0^1 t^{k\rho + \lambda + 2} (1-t)^s dt \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho + \lambda + 1)} w^k (b-a)^{k\rho} \frac{\Gamma(k\rho + \lambda + 3) \Gamma(s+1)}{\Gamma(k\rho + \lambda + s + 4)} \\ &= \Gamma(s+1) \mathcal{F}_{\rho, \lambda+s+4}^{\sigma_2}(w(b-a)^\rho) \end{aligned}$$

where

$$\sigma_2(k) = \sigma(k)(k\rho + \lambda + 2)(k\rho + \lambda + 1), \quad k = 0, 1, 2, \dots,$$

and

$$\begin{aligned} I(h_2, g_1) &= \int_0^1 t^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma(w(b-a)^\rho t^\rho)(1-t)t^s dt \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho + \lambda + 1)} w^k (b-a)^{k\rho} \int_0^1 t^{k\rho + \lambda + s + 1} (1-t) dt \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho + \lambda + 1)} w^k (b-a)^{k\rho} \frac{\Gamma(k\rho + \lambda + s + 2) \Gamma(2)}{\Gamma(k\rho + \lambda + s + 4)} \\ &= \mathcal{F}_{\rho, \lambda+1}^{\sigma_{s;2,1}}(w(b-a)^\rho) \end{aligned}$$

where

$$\sigma_{s;2,1}(k) = \frac{\sigma(k)}{(k\rho + \lambda + s + 3)(k\rho + \lambda + s + 2)}, \quad k = 0, 1, 2, \dots$$

By replacement of these values in Theorem 4 it is attained the result. $\square$

Remark 4. Since in the preliminary section is mentioned the fact that from the Raina's fractional integral the fractional integrals of Riemann-Liouville and the classic integral of Riemman can be deduced then the results found in Theorem 4 and Corollaries 1,2, 3 and 4 are useful to express them in terms of these integrals.

4. CONCLUSION

In the present work we established the Hermite-Hadamard inequality for exponentially convex functions using the Raina's fractional integral and from this result we deduced some results found in [17, 18]. Also from Lemma 3.1 it was established a general theorem from which some fractional integral inequalities for exponentially convex functions, exponentially P -convex functions and exponentially s -convex functions in the second sense were found.

The usefulness of the theorems presented and the proposed technique can be applied to other types of generalized convex functions, for example, MT -convex functions [14].

Acknowledgement. Dr. Miguel J. Vivas-Cortez wants to thank to Dirección de Investigación from Pontificia Universidad Católica del Ecuador for the technical support given to the project entitled: Some inequalities using generalized convexity, and M.Sc. Jorge Eliecer Hernández Hernández thanks the Consejo de Desarrollo Científico, Humanístico y Tecnológico (CDCHT) from Universidad Centroccidental Lisandro Alvarado (Venezuela) for the technical support given in the development of this work.

The authors also thank the referees assigned for the evaluation of this article and in the same way to the entire editorial team of Mathematica Moravica.

Competing Interest and Author's Contribution. The authors declare that they have no competing interest, and all of them contributed equally to the development of this research paper.

REFERENCES

- [1] T. Antczak, (p, r) -*invex sets and functions*, Journal of Mathematical Analysis and Applications, 263 (2001), 355–379.
- [2] R. P. Agarwal, M-J. Luo and R. K. Raina, *On Ostrowski Type Inequalities*, Fasciculi Mathematici, 56 (2016), pp. 5–27
- [3] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen*, Publications de l'Institut Mathématique, 23 (1978), 13–20.
- [4] G. Gristescu, M. A. Noor and M. U. Awan, *Bounds of the second degree cumulative frontier gaps of functions with generalized convexity*, Carpathian Journal of Mathematics, 31 (2) (2015), 173–180.

-
- [5] S. S. Dragomir, J. Pečarić and L. E. Peerson, *Some Inequalities of Hadamard type*, Soochow Journal of Mathematics, 21 (3) (1995), 335–341.
 - [6] S. S. Dragomir and I. Gomm, *Some Hermite-Hadamard type inequalities for functions whose exponentials are convex*, Studia Universitatis Babeş-Bolyai Mathematica, 60 (2015), 527–534.
 - [7] J. Hadamard, *Etude sur les propriétés des fonctions entières et en particulier d’une fonction considérée par Riemann*, Journal de Mathématiques Pures et Appliquées, 4 (9) (1893), 171–216
 - [8] J.E. Hernández Hernández, *Some fractional integral inequalities of Hermite Hadamard and Minkowski type*, Selecciones Matemáticas (National University of Trujillo, Perú), 6 (1) (2019), 41–48.
 - [9] J. E. Hernández Hernández and M. J. Vivas-Cortez, *Hermite-Hadamard Inequalities type for Raina’s fractional integral operator using η -convex functions*, Revista Matemática: Teoría y Aplicaciones, 26 (1) (2019), 1–19
 - [10] J. E. Hernández Hernández and M. J. Vivas-Cortez, *On a Hardy’s inequality for a fractional integral operator*, Annals of the University of Craiova, Mathematics and Computer Science Series, 45 (2) (2018), 232–242
 - [11] H. Hudzik and L. Maligranda, *Some remarks on s -convex functions*, Aequationes Mathematicae, 48 (1994), 100–111.
 - [12] M. Iqbal, B. M. Iqbal and K. Nazeer, *Generalization of Inequalities Analogous to Hermite-Hadamard Inequality via Fractional Integrals*, Bulletin of the Korean Mathematical Society, 52 (3) (2015), 707–716
 - [13] M. Kunt, D. Karapinar, S. Turhan and I. Iscan, *The right Riemann-Liouville fractional Hermite-Hadamard type inequalities for convex functions*, Journal of Inequalities and Special Functions, 9 (1) (2018), 45–57
 - [14] W. Liu, W. Wen, J. Park, *Hermite-Hadamard type inequalities for MT -convex functions via classical integrals and fractional integrals*, Journal of Nonlinear Sciences and Applications, 9 (2016), 766–777.
 - [15] W. Orlicz, *A note on modular spaces I*, Bulletin L’Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques (BAPMAM), 9 (1961), 157–162.
 - [16] R. K. Raina, *On Generalized Wright’s Hypergeometric Functions and Fractional Calculus Operators*, East Asian Mathematics Journal, 21 (2) (2005), 191–203.
 - [17] S. Rashid, M. A. Noor and K. I. Noor, *Fractional Exponentially m -Convex Functions and Inequalities*, International Journal of Analysis and Applications, 17 (3) (2019), 464–478.
 - [18] S. Rashid, M. A. Noor, K. I. Noor and A.O. Akdemir, *Some New Generalizations for Exponentially s -Convex Functions and Inequalities via Fractional Operators*, Fractal and Fractional, MDPI, 3 (24) (2019), 1–16
 - [19] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, *Hermite-Hadamard’s inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, 57 (2017), 2403–2407

- [20] S. Turhan, I. Iscan and M. Kunt, *Hermite-Hadamard Type Inequalities for n -Times Differentiable Convex Functions via Riemann-Liouville Fractional Integrals*, Filomat, 32 (16) (2018), 5611–5622
- [21] M.J. Vivas-Cortez and J.E. Hernández Hernández, *On (m, h_1, h_2) -Convex Stochastic Processes using Fractional Integral Operator*, Applied Mathematics & Information Sciences, 12 (1) (2018), 45-53
- [22] H. Yaldiz and M. Z. Sarikaya, *On the Hermite - Hadamard type inequalities for fractional integral operator*: <https://www.researchgate.net/publication/309824275>

MIGUEL VIVAS-CORTEZ

PONTIFICIA UNIVERSIDAD CATÓLICA DEL ECUADOR

FACULTAD DE CIENCIAS EXACTAS Y NATURALES

ESCUELA DE FÍSICA Y MATEMÁTICA

TELF. 2991700 EXT. 2036

AV. 12 DE OCTUBRE 1076 APARTADO: 17-01-2184,

QUITO

ECUADOR

E-mail address: mjvivas@puce.edu.ec

JORGE ELIECER HERNÁNDEZ HERNÁNDEZ

UNIVERSIDAD CENTROCCIDENTAL LISANDRO ALVARADO

DECANATO DE CIENCIAS ECONÓMICAS Y EMPRESARIALES

DEPARTAMENTO DE TÉCNICAS CUANTITATIVAS,

AV. 20 ESQ. AV. MORAN, EDF. LOS MILITARES OFC 2.

3001, BARQUISIMETO

VENEZUELA

E-mail address: jorgehernandez@ucla.edu.ve

SERCAN TURHAN

GIRESUN UNIVERSITY

FACULTY OF ARTS AND SCIENCE

DEPARTMENT OF MATHEMATICS

28100, GIRESUN

TURKEY

E-mail address: sercan.turhan@giresun.edu.tr