On $G$–transitive version of perfectly meager sets

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Abstract. We study the $G$– invariant version of perfectly meager sets (a generalization of the notion of AFC’ sets). We find the necessary and sufficient conditions for the inclusion $AFC_G’ \subseteq I$. In particular, we partially characterize for which groups $G$ of automorphisms of the Cantor space every $AFC_G’$ set is Lebesgue null.

1. Definitions and notation

We consider the Cantor space $2^\infty$ as a topological group (where $(x + y)(k) = x(k) + y(k) \mod 2$). By $2^{<\infty}$ let us denote the collection of all finite binary sequences: $2^{<\infty} = \{f : n \rightarrow 2 \text{ where } n \in \omega\}$.

For any $s \in 2^{<\infty}$ by $[s]$ denote the base open set determined by $s$: $[s] = \{x \in 2^\infty : s \subseteq x\}$. Let Perf stand for the family of all perfect subsets of the space $2^\infty$. Recall that a proper collection of subsets of $2^\infty$: $I \subseteq P(2^\infty)$ is called a $\sigma$-ideal iff it is closed under taking subsets and countable sums. Throughout the paper we assume that every $\sigma$-ideal $I$ contains all singletons: $\forall x \in X \{x\} \in I$.

Let $I \subseteq P(2^\infty)$ be a $\sigma$-ideal. Define the following cardinal numbers:

**Definition 1.** $cov(I) = \min\{|A| : A \subseteq I \wedge \bigcup A = 2^\infty\}$

and

$cof(I) = \min\{|A| : A \subseteq I \wedge \forall Z \in I \exists A \in A Z \subseteq A\}$.

Notice that we always have $cov(I) \leq cof(I)$.

We assume that the reader is familiar with basic concept of arithmetic of cardinal numbers. In particular, we need the notion of cofinality; recall that an uncountable cardinal number $\kappa$ is called regular iff $cf(\kappa) = \kappa$.

By $Hom(X)$ we denote the group of all homeomorphisms of the topological space $X$. We always assume that $G$ is a fixed subgroup of $Hom(2^\infty)$.

The following additional terminology will be useful in our proof.

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For an arbitrary \( g \in G \) and \( Q \in Perf \) we often abbreviate the image \( g(Q) = \{ gx : x \in Q \} \) and write simply \( gQ \). Also for any \( t \in 2^\omega \) and \( A \subseteq 2^\omega \) we write \( A + t = \{ x + t : x \in A \} \).

We denote by \( \mathcal{M}(P) \) the collection of all first category sets on \( P \), where \( P \in Perf(X) \).

We use a letter \( N \) to denote the sigma ideal of Lebesgue measure zero sets of \( 2^\omega \).

We denote by \( Trans(2^\omega) \) the subgroup of all translations of \( 2^\omega \).

2. Introduction

Let us start with the following, classical definition:

**Definition 2.** A subset \( S \subseteq 2^\omega \) is a Sierpiński set if, and only if, it is uncountable and has countable intersection with any set of measure zero.

Notice that under the assumption of Continuum Hypothesis there exists a Sierpiński set (see [9]) and, on the other hand, it is consistent that there is no Sierpiński set.

A special variation of the notion of a Sierpiński set is a \( \kappa \)-Sierpiński set with respect to the \( \sigma \)-ideal \( I \), namely:

**Definition 3.** Suppose that \( \kappa \) is a cardinal number and \( \mathcal{I} \subseteq P(2^\omega) \) a \( \sigma \)-ideal. A set \( X \subseteq 2^\omega \) is called a \( \kappa \)-Sierpiński set \( X \) with respect to \( \mathcal{I} \) iff \( |X| = \kappa \) and \( \forall A \in \mathcal{I} |A \cap X| < \kappa \).

Notice that if \( \mathcal{T} \) is a \( \sigma \)-ideal (which contains singletons) and \( \kappa = \text{cof}(\mathcal{T}) = \text{cov}(\mathcal{T}) \) then there exists a \( \kappa \)-Sierpiński set \( X \) with respect to \( \mathcal{T} \).

Recall the classical definition of perfectly meager sets (called also always of the first category sets):

**Definition 4.** A set \( X \) of \( 2^\omega \) is a perfectly meager (AFC) set iff for every \( P \in Perf \), \( X \cap P \) is a first category set in the topology of \( P \).

The following notion of sets was first defined in [5] and then it has been studied most extensively in papers [6] and [7].

**Definition 5.** A set \( X \subseteq 2^\omega \) is an AFC’-set if for each perfect set \( P \) there exists an \( F_\sigma \)-set \( F \) such that \( X \subseteq F \) and for each \( t \in 2^\omega \), \( (F + t) \cap P \) is a first category set in the topology of \( P \).

Notice that the notion AFC’ is a strengthening of the classical perfectly meager sets.

The following notion was first defined by Karel Prikry: (see [3], introduction):

**Definition 6.** A set \( X \subseteq 2^\omega \) is called strongly meager (SFC) iff for every measure zero set \( A \subseteq 2^\omega \), there exists \( t \in 2^\omega \), such that \( (X + t) \cap A = \emptyset \).
Notice that K. Prikry conjectured that the collection of strongly meager sets form a \( \sigma \)-ideal but it turned out that it is consistent that strongly meager sets are exactly the countable sets (see [3]) and that it is consistent that even the sum of two strongly meager sets need not be strongly meager set (see [2]).

It is known (see for example [5] and [7]), that \( AFC' \subseteq AFC \) and every strongly meager set is an \( AFC' \) set.

It is also known (see [8]) that every Sierpiński set is strongly meager.

We can summarize all these inclusions in Fig. 1

\[
\text{Sierpiński set} \rightarrow \text{SFC} \rightarrow \text{AFC}' \rightarrow \text{AFC}
\]

**Figure 1.** Basic relations.

Let us define the main notion of this article.

**The AFC\(_G'\) - sets**

Suppose that \( G \) is a subgroup of \( \text{Hom}(2^\omega) \) and let \( X \) be an arbitrary subset of \( 2^\omega \).

**Definition 7.** Suppose that \( X \subseteq 2^\omega \). We write \( X \in AFC'\_G \) iff for every \( Q \in Perf \) there exists \( F \supseteq X, F \in F_\sigma \) such that \( \forall g \in G gQ \cap F \in \mathcal{M}(gQ) \).

This notion is a natural generalization of the notion of AFC\(_G'\) sets.

**Remarks:**

It is obvious that

\[
\text{AFC}'_{\text{Trans}(2^\omega)} = \text{AFC}' \quad \text{AFC}'_{\{\text{id}\}} = \text{AFC} \quad \text{AFC}'_{\text{Hom}(2^\omega)} = [2^\omega]^{\leq \omega}.
\]

It is also evident that if \( G_1 \subseteq G_2 \), then \( AFC'_{G_1} \supseteq AFC'_{G_2} \).

All inclusions are summarized in Fig. 2 (where arrows denote inclusions).

\[
\frac{\text{AFC}'_{\text{Hom}(2^\omega)}}{[2^\omega]^{\leq \omega}} \rightarrow \frac{\text{AFC}'_{\text{Trans}(2^\omega)}}{\text{AFC}'} \rightarrow \frac{\text{AFC}'_{\{\text{id}\}}}{\text{AFC}}
\]

**Figure 2.** Relations between various versions of perfectly meager sets.

Let us define:
Definition 8. Suppose that $I$ is a $\sigma$-ideal of subsets of the space $2^{\omega}$.

We say that a group $G \leq \text{Hom}(2^{\omega})$ has the $(Em)_I$ property iff there exists a perfect set $Q \in \text{Perf}$ such that for each $P \in \text{Perf} \setminus I$ there exists $g \in G$ such that $P \cap gQ \not\in \mathcal{M}(gQ)$.

Remarks:
One can prove that $\text{Trans}(2^{\omega})$ does not have the $(Em)_N$ property.
Without loss of generality we may assume that in Definition 8 $P$ is only closed set such that $P \notin I$.
We will start with the following theorem.

Theorem 1. Let $I$ be an arbitrary $\sigma$-ideal of subsets of $2^{\omega}$ such that $\forall x \in 2^{\omega}\{x\} \in I$.

Moreover, let $G \leq \text{Hom}(2^{\omega})$ be a subgroup of $\text{Hom}(2^{\omega})$ with the property $(Em)_I$.

Then we have: $\text{AFC}_G' \subseteq I$.

Proof. Let $X \subseteq 2^{\omega}$ be a set such that $X \notin I$. By the definition of the notion $(Em)_I$ there is a perfect set $Q$ such that for each closed $E \notin I$ we have $\exists g \in G : E \cap gQ \not\in \mathcal{M}(gQ)$.

Let $F \subseteq 2^{\omega}$ be an $F_\sigma$ set such that $X \subseteq F$. We have

$$F = \bigcup_{n<\omega} F_n,$$

where $cl(F_n) = F_n$, so there exists $n_0 < \omega$ such that $F_{n_0} \notin I$. Now there exists $g \in G$ such that $F_{n_0} \cap gQ$ is not meager in $gQ$. So we conclude, that $X$ is not an $\text{AFC}_G'$ set. \qed

The implication given in Theorem 1 is reversible under some additional set theoretical assumptions. Indeed, we have the following theorem.

Theorem 2. Let us assume like in Theorem 1 that $I$ is an arbitrary $\sigma$-ideal of subsets of $2^{\omega}$ such that $\forall x \in 2^{\omega}\{x\} \in I$ and $G \leq \text{Hom}(2^{\omega})$ is a subgroup of $\text{Hom}(2^{\omega})$. Moreover, assume that

1. $\text{cof}(I) = \text{cov}(I) \leq \text{non}(\text{AFC}_G')$,
2. $\forall P \in \text{Perf} \setminus I \exists |C| \leq \omega 2^{\omega} \setminus (P + C) \in I$,
3. $\text{Trans}(2^{\omega}) \subseteq G$.

Then the following conditions are equivalent:

1. $\text{AFC}_G' \subseteq I$
2. $G$ fulfills $(Em)_I$.

Proof. Theorem 1 gives us immediately the implication (2) $\Rightarrow$ (1).

Now suppose that $G$ fulfills $-\text{(Em)}_I$. Since $\kappa = \text{cof}(I) = \text{cov}(I)$ and $I$ contains singletons we conclude that there exists a $\kappa$-Sierpiński set $X$ with respect to $I$ (see Def. 3). Let $Q \in \text{Perf}$ be arbitrary. From the assumption
\( \neg(Em)_I \) there exists a perfect set \( P \) such that \( P \notin I \) and \( \forall g \in G gQ \cap P \in \mathcal{M}(gQ) \). Pick a countable set \( C \subseteq 2^\omega \) such that \( 2^\omega \setminus (C + P) \in I \).

We have
\[
X = \left[ [2^\omega \setminus (P + C)] \cap X \right] \cup \left[ (P + C) \cap X \right].
\]

Since \( 2^\omega \setminus (P + C) \in I \) we obtain \( [2^\omega \setminus (P + C)] \cap X | \kappa. \) Moreover, if \( c \in C \) and \( g \in G \), then \( hQ \cap P \in \mathcal{M}(hQ) \), where \( h \in G \) is defined by \( h(x) = g(x) - c. \) Hence \( gQ \cap (P + c) \in \mathcal{M}(gQ) \), thus \( gQ \cap (P + C) \in \mathcal{M}(gQ) \) for each \( g \in G \).

Since \( \kappa \leq \text{non}(\text{AFC}_G') \) we obtain \( [2^\omega \setminus (P + C)] \cap X \in \text{AFC}_G', \) so there exists \( E \in F_\sigma, E \supseteq X \setminus (P + C) \) such that \( \forall g \in G gQ \cap E \in \mathcal{M}(gQ) \). Finally, define \( E^* = E \cup (P + C) \). It is easy to see that \( X \subseteq E^* \) and \( \forall g \in G gQ \cap E^* \in \mathcal{M}(gQ) \). Hence \( X \in \text{AFC}_G' \) and the proof is completed, since \( X \) does not belong to \( I \). \( \square \)

Unfortunately, we don’t know whether this theorem is true under weaker assumptions. Thus we think that the following question may be of some interest.

**Question 3.** Can we prove the equivalence from Theorem 2 under weaker assumptions?

For any \( F \subseteq \text{Perf} \) let us define the following cardinal coefficient:

**Definition 9.** \( Em(F, G) = \min\{ |G| : g \subseteq \text{Perf} \land \forall P \in F \exists g \in G \exists Q \in gQ \subseteq P \} \)

Let us formulate a characterization of the property \((Em)\) in terms of the coefficient \( Em(F, G)\).

Assume that \( G \) has the property that for each \( x \in 2^\omega \) the orbit \( Gx \) is dense in \( 2^\omega \). Then the following conditions are equivalent:

1. \( G \) fulfills \((Em)_I\);
2. \( |Em(\text{Perf} \setminus I, G)| \leq \aleph_0 \).

We will need the following technical lemma (folklore for the group \( G = \text{Trans}(2^\omega) \)):

**Lemma 1.** If \( G \leq \text{Hom}(2^\omega) \) is a group such that for each \( x \in 2^\omega \), \( Gx \) is dense in \( 2^\omega \), then for every sequence \( \langle Q_n \rangle \) of perfect subsets of \( 2^\omega \) there exists a perfect \( P \in \text{Perf} \) such that \( \forall n \in \omega \exists x \in gQ_n \cap P \notin \mathcal{M}(P) \).

**Proof.** Let \( v_k = [(0, \ldots, 0, 1)] \) (0 \( k \) times). For each \( k \) choose \( x_k \in Q_k \) and \( g_k \in G \) such that \( g_k x_k \in V_k \). Define \( P = \bigcup_{k \in \omega} g_k Q_k \cap V_k \), then \( P \) is a perfect set and if \( k \in \omega \) then \( g_k Q_k \cap P \supseteq g_k Q_k \cap V_k \notin \mathcal{M}(P) \). \( \square \)

**Proof.** (1) \( \Rightarrow \) (2)

Assume that \( G \) has the \((Em)_I\) property, i.e. there exists \( Q \in \text{Perf} \) such that \( \forall P \in \text{Perf} \setminus I \exists g \in G P \cap gQ \notin \mathcal{M}(gQ) \). Let us define perfect sets: \( G = \{ Q \cap [s] : Q \cap [s] \neq \emptyset \land s \in 2^{<\omega} \} \). Then \( |G| \leq \aleph_0 \) and if \( P \in \text{Perf} \setminus I \) then there exists \( g \in G \) such that \( P \cap gQ \notin \mathcal{M}(gQ) \), so \( P \cap gQ \supseteq W \cap gQ \neq \emptyset \) for
some open set $W$. Then $g^{-1}[W] \cap Q \neq \emptyset$ so there exists $Q_1 \in \mathcal{G}$ such that $Q_1 \subseteq g^{-1}[W] \cap Q$. Hence $g[Q_1] \subseteq W \cap g[Q] \subseteq P \cap g[Q]$. This proves (2).

(2) $\Rightarrow$ (1). \hfill $\square$

Next we give an useful characterization of the property $(Em)_\mathcal{N}$.

**Theorem 4.** Let $G$ be a subgroup of $\text{Hom}(2^\omega)$ which contains the subgroup $\text{Trans}(2^\omega)$. The following two conditions are equivalent:

1. $\neg(Em)_\mathcal{N}$,
2. For every $Q \in \text{Perf}$ and for every $\epsilon > 0$ there exists an open set $U$, such that $\mu(U) < \epsilon$ and $\forall g \in G gQ \cap U \neq \emptyset$

**Proof.** (1) $\Rightarrow$ (2)

Assume that $\forall Q \in \text{Perf} \exists P \in \text{Perf} \forall g \in G gQ \cap P \in \mathcal{M}(gQ)$

Let $Q \in \text{Perf}$ be any perfect set and let $\epsilon > 0$. Pick a perfect set $P$, $\mu(P) > 0$ such that $\forall g \in G gQ \cap P \in \mathcal{M}(gQ)$. We can find finite $C \subseteq 2^\omega$ such that $\mu(2^\omega \setminus (C + P)) < \epsilon$. Now put $U = 2^\omega \setminus (C + P)$.

By way of contradiction suppose that there exists $g \in G$ such that $gQ \cap U = \emptyset$. Then $gQ \subseteq C + P$, hence there exists $c_0 \in C$ and an open set $I$ such that $\emptyset \neq I \cap gQ \subseteq P + c_0$. Define $h(x) = g(x) - c_0$, obviously $h \in G$. Next, $hQ = gQ - c_0$ thus $\emptyset \neq hQ \cap (I - c_0) \subseteq P$, which is a contradiction with $hQ \cap P \in \mathcal{M}(hQ)$.

(2) $\Rightarrow$ (1)

Assume (2). Let $R$ be any perfect set. Let $\{I_m\}_{m<\omega}$ be an enumeration of all basic clopen sets of $2^\omega$. Let

$$\epsilon_m = \frac{1}{2^{m+2}}.$$  

For any $m < \omega$ we choose, using the assumption (2), an open set $U_m$ such that

$$\forall g \in G R \cap I_m \neq \emptyset \Rightarrow U_m \cap g(R \cap I_m) \neq \emptyset$$

and $\mu(U_m) < \epsilon_m$. This can be done, since $I_m \cap R$ is a perfect or an empty set.

Now put

$$U = \bigcup_{m<\omega} U_m.$$  

We see that

$$\mu(U) \leq \sum_{m<\omega} \frac{1}{2^{m+2}} \leq 2 \cdot \frac{1}{4} < 1.$$  

Define $F = 2^\omega \setminus U$, then we have $\mu(F) > 0$ so choose a perfect $P \subseteq F$ of positive measure.

Let $g \in G$ and $I_{m_0}$ be given such that $R \cap I_{m_0} \neq \emptyset$. 


Now $U_{m_0} \cap g(R \cap I_{m_0}) \neq \emptyset$. Moreover, since $U_{m_0} \cap P = \emptyset$ we obtain that $g(R \cap I_{m_0}) \not\subseteq P$. This means that (1) holds.

Notice that in the proof of implication (2)$\Rightarrow$(1) we did not use the assumption that $\text{Trans}(2^\omega) \leq G$.

In the next part we will prove theorems about relations between AFC$'_G$ and different classes of peculiar small sets of the real line.

**Theorem 5.** Assume that $G$ is a subgroup of $\text{Hom}(2^\omega)$ which contains $\text{Trans}(2^\omega)$. If $G$ fulfills $\neg (\text{Em})_\mathcal{N}$, then every strongly meager set is an AFC$'_G$ set.

**Proof.** Let $X$ be a strongly meager set and let $Q$ be an arbitrary perfect set. Since $\neg (\text{Em})_\mathcal{N}$ we obtain that there exists a perfect set $P$ such that $\mu(P) > 0$ and $\forall g \in G g(Q) \cap P \in \mathcal{M}(g(Q))$. Let $C \subseteq 2^\omega$ be a countable set such that $2^\omega \setminus (P + C) \in \mathcal{N}$. Then there exists $x_0$ such that $(x_0 + X) \cap [2^\omega \setminus (P + C)] = \emptyset$, so $x_0 + X \subseteq P + C$, hence $X \subseteq P + C - x_0$. Let $g \in G$ be an arbitrary and let $c \in C$. Define $h \in G$ by $h(x) = g(x) - c + x_0$. Then $h(Q) \cap P \in \mathcal{M}(h(Q))$, hence $(g(Q) - c + x_0) \cap P \in \mathcal{M}(g(Q) - c + x_0)$, thus $g(Q) \cap (P + c - x_0) \in \mathcal{M}(g(Q))$. Since $c \in C$ was taken arbitrary, we conclude that $g(Q) \cap (P + C - x_0) \in \mathcal{M}(g(Q))$. This implies that $X \in \text{AFC}'_G$, since $P + C - x_0 \in F_\sigma$. 

**Remark:**

This implication is reversible under CH. Namely:

**Theorem 6.** Suppose that $G \leq \text{Hom}(2^\omega)$ and assume that $G$ has the $(\text{Em})_\mathcal{N}$ property. Moreover, assume CH. Then there exists a strongly meager set $X \subseteq 2^\omega$ such that $X \notin \text{AFC}'_G$.

**Proof.** Let $X \subseteq 2^\omega$ be arbitrary Sierpiński set. Then $X$ is strongly meager ([8]). From the $(\text{Em})_\mathcal{N}$ property we obtain that there exists $Q \in \text{Perf}$ such that

$$\forall P \in \text{Perf} \setminus \mathcal{N} \exists g \in G P \cap g(Q) \notin \mathcal{M}(g(Q)).$$

Suppose that $E$ is an $F_\sigma$–set such that $X \subseteq E$. Since $X \notin \mathcal{N}$ it follows that $E \notin \mathcal{N}$. Hence there exists $P \in \text{Perf} \setminus \mathcal{N}$ such that $P \subseteq E$.

Therefore $\exists g \in G P \cap g(Q) \notin \mathcal{M}(g(Q))$, hence $E \cap g(Q) \notin \mathcal{M}(g(Q))$. This yields $X \notin \text{AFC}'_G$, which finishes the proof. 

**Corollary 1.** Assume that $\text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N})$ and $\text{cov}(\mathcal{N})$ is a regular cardinal. Let $G \leq \text{Hom}(2^\omega)$ and suppose that $\text{Trans}(2^\omega) \leq G$. Then the following conditions are equivalent:

1. $G$ has the $(\text{Em})_\mathcal{N}$ property.
2. $\text{AFC}'_G \subseteq \mathcal{N}$. 
Proof. The implication $(1) \Rightarrow (2)$ follows immediately from Theorem 1. Assume $\neg (Em)^N_{\mathcal{N}}$. Since $cov(\mathcal{N}) = cof(\mathcal{N})$, there exists a $cof(\mathcal{N})$ – Sierpiński set. By Lemma 8.5.4 from [1] if there exists a $\kappa$ – Sierpiński set and $cf(\kappa) = \kappa > \omega$, then every set of size $< \kappa$ is strongly meager. Hence by Theorem 5 we conclude that $non(AFC'_G) \geq cof(\mathcal{N})$ thus all assumptions of Theorem 2 are satisfied. \[\square\]

References


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