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Inverse-C-class function on weak semi compatibility and fixed point theorems for expansive mappings in G-metric spaces

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Abstract. In this paper we introduce the concept of inverse C-class function in G-metric setting and established some fixed point theorems. We also put some examples in support of proved fixed point results.

1. Introduction and mathematical preliminaries

The study of metric fixed point have played very vital role with the applications in mathematics and applied science. Later it was found very essential to generalize the notion named metric space by many researchers. With this respect many generalization have come in the metric space like, D-metric space, 2-metric space, cone metric space, fuzzy metric space and manger space, etc. The 2-metric space was proposed by Gahler [12] and considered an independent theory rather than a generalization of metric space. One more generalization came in exist called D-metric space. It was introduced by Dhage [11]. He published lot of papers concerned to generalization of metric space, but many of his results found to be considered erroneous. This fact was set on by Mustafa and then, in collaboration with Sims [15] they demonstrated an improved form called G-metric space. Later Mustafa did more work in collaboration with Obiedat, Awahdeh, Shatanawi and Karpinar. For more information one can go through [13]–[21].

In the stream of generalization, in 2014 the concept of C-class function was introduced by Ansari [3]. Using this concept many fixed point results were generalized in different spaces. For more information one can go through [4]–[10]. This concept has enticed many fixed point theorists. Later the more generalized form called inverse C-class function was introduced by Saleem et al. [24].

The purpose of the present paper is to extend the concept of inverse C-class function in G-metric setting and proved some fixed point theorems by

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taking weak semi compatibility in G-metric space. These results include compatible mappings of type (E) also different compatible and commuting mappings.

Definition 1 ([13]). Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ iff $x = y = z$;
(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;
(G4) $G(x, y, z) = G(p\{x, y, z\})$, where p is any permutation of x, y, z (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

In 2014, the concept of C-class functions was introduced by A. H. Ansari [3]. By using this concept, we can generalize many fixed point theorems in the literature.

Definition 2 ([25]). Mappings f and g of a metric space (X, d) is said to be weak semi compatible if $\lim_{n\to\infty} fgx_n = gt$ or $\lim_{n\to\infty} gfx_n = ft$, whenever a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t in X.

Definition 3 ([23]). A self map pair (f, g) of a metric space (X, d) is said to be R-weak commuting of type (A_f) if there exists some positive real number R such that $d(fgx, ggx) \leq Rd(fx, gx)$, for all $x \in X$. Similarly, a self map pair (f, g) of a metric space (X, d) is said to be R-weak commuting of type (A_g) if there exists some positive real number R such that $d(gfx, ffx) \leq Rd(fx, gx)$, for all $x \in X$.

Definition 4 ([26]). Self mappings f and g are called f-compatible of type (E) if $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = gt$, whenever a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t in X. Similarly, self mappings f and g are called g-compatible of type (E) if $\lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = gt$, whenever a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t in X.

Definition 5 ([22]). Two self mappings f and g are called f-compatible if $\lim_{n\to\infty} d(fgx_n, ggx_n) = 0$, whenever a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t in X. Similarly, self mappings f and g are called f-compatible if $\lim_{n\to\infty} d(fgx_n, ggx_n) = 0$, whenever a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t in X.

Definition 6 ([1]). Mappings f and g satisfy E.A. property if there exists a sequence $\{x_n\} \in X$ which satisfy $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t in X.

Definition 7 ([3]). A mapping $f : [0,\infty)^2 \to \mathbb{R}$ is called C-*class* function if it is continuous and satisfies following axioms:

(1) $f(s,t) \leq s$;
(2) $f(s,t) = s$ implies that either $s = 0$ or $t = 0$, for all $s,t \in [0,\infty)$.

Note for some F we have that $F(0,0) = 0$.
We denote C-class functions as $\mathcal{C}$.

Definition 8 ([24]). A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called inverse-C-*class* function if it is continuous and satisfies following axioms:

(1) $F(s,t) \geq s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$, for all $s,t \in [0,\infty)$.

Note for some F we have that $F(0,0) = 0$.
We denote inverse C-class functions as $\mathcal{C}_{inv}$.
Let Φ denote the set of all functions $\varphi : [0,+\infty) \to [0,+\infty)$ that satisfy the following conditions:

(1) φ is lower semi-continuous on $[0,+\infty)$;
(2) $\varphi(0) = 0$;
(3) $\varphi(s) > 0$ for each $s > 0$.

Let Φ_1 denote the set of all functions $\varphi : [0,+\infty) \to [0,+\infty)$ that satisfy the following conditions:

(1) φ is lower semi-continuous on $[0,+\infty)$;
(2) $\varphi(0) \geq 0$;
(3) $\varphi(s) > 0$ for each $s > 0$.

Let Ψ denote all the functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy:

(i) $\psi(t) = 0$ if and only if $t = 0$;
(ii) ψ is continuous and monotinic increasing.

2. Main Results

Theorem 1. *Let f and g are weak semi compatible, R-weak commuting of type A_f self mappings of complete G-metric space (X,G) and suppose that $f : \cup_{i=1}^m A_i \to \cup_{i=1}^m A_i$ satisfies the following conditions: (where $A_{m+1} = A_1$):*

(a) $f(X) \subseteq g(X)$;
(b) *For all $x,y,z \in X$ holds*

$$\psi(G(gx,gy,gz)) \geq F(\psi(G(fx,fy,fz)),\varphi(G(fx,fy,fz))); \tag{1}$$

(c) *f and g are either f-compatible of type (E) or g-compatible of type (E).*

If functions $\varphi \in \Phi$, $\psi \in \Psi$ and $F \in \mathcal{C}_{inv}$ then f and g have a common fixed point in X.

Proof. Let x_0 be any point in X. Since $f(X) \subseteq g(X)$, there exist a point x_1 in X such that $fx_0 = gx_1$. Similarly, we can find a sequence fx_n in X such that $fx_n = gx_{n+1}$. Let $z_n = fx_n = gx_{n+1}$. Now we have to show that $\{z_n\}$ is a Cauchy sequence in X.

Step 1. We will prove that $\lim_{n\to\infty} G(z_n, z_{n+1}, z_{n+1}) = 0$.

For proving this, we take condition (b) with $x = x_n, y = x_{n+1}$ and $z = x_{n+1}$, so we have

$$\begin{aligned}\psi(G(z_{n-1}, z_n, z_n)) &= \psi(G(gx_n, gx_{n+1}, gx_{n+1})) \geq \\ &\geq F\big(\psi(G(fx_n, fx_{n+1}, fx_{n+1})), \varphi(G(fx_n, fx_{n+1}, fx_{n+1}))\big) \\ &= F(\psi(G(z_n, z_{n+1}, z_{n+1})), \varphi(G(z_n, z_{n+1}, z_{n+1}))) \\ &\geq \psi(G(z_n, z_{n+1}, z_{n+1})),\end{aligned} \tag{2}$$

and therefore

$$G(z_n, z_{n+1}, z_{n+1}) \leq G(z_{n-1}, z_n, z_n). \tag{3}$$

Hence, we conclude that $\{G(z_n, z_{n+1}, z_{n+1})\}$ is a nondecreasing sequence of non-negative real numbers. Thus, there exist a $r \geq 0$ such that

$$\lim_{n\to\infty} G(z_n, z_{n+1}, z_{n+1}) = r. \tag{4}$$

Letting $n \to \infty$ in (2) and by the continuity of ψ and φ, we have

$$\psi(r) \geq F(\psi(r), \varphi(r)) \geq \psi(r)$$

which implies either $\psi(r) = 0$ or $\varphi(r) = 0$. This gives

$$\lim_{n\to\infty} G(z_n, z_{n+1}, z_{n+1}) = 0. \tag{5}$$

Step 2. We will show that $\{z_n\}$ is a G-Cauchy sequence in X. Therefore, we will show that for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $m, n \geq k$, $G(z_m, z_n, z_n) < \varepsilon$.

Suppose that the above statement is false. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{z_{m(k)}\}$ and $\{z_{n(k)}\}$ of $\{z_n\}$ such that $n(k) > m(k) \geq k$:

(a) $m(k) = 3t$ and $n(k) = 3t' + 1$, where t and t' are nonnegative integers such that

$$G(z_{m(k)}, z_{n(k)}, z_{n(k)}) \geq \varepsilon. \tag{6}$$

(b) $n(k)$ is the smallest number such that the condition (b) holds:

$$G(z_{m(k)}, z_{n(k)-1}, z_{n(k)-1}) < \varepsilon. \tag{7}$$

From rectangle inequality and (7), we have

$$\begin{aligned} &G(z_{m(k)}, z_{n(k)}, z_{n(k)}) \leq \\ &\leq G(z_{m(k)}, z_{n(k)-1}, z_{n(k)-1}) + G(z_{n(k)-1}, z_{n(k)}, z_{n(k)}) < \\ &< \varepsilon + G(z_{n(k)-1}, z_{n(k)}, z_{n(k)+1}). \end{aligned} \tag{8}$$

If $k \to \infty$ in (8), from (5) and (7) we conclude that

$$\lim_{k\to\infty} G(z_{m(k)}, z_{n(k)}, z_{n(k)}) = \varepsilon. \tag{9}$$

Again, from rectangle inequality

$$\begin{aligned} &G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) \leq \\ &\leq G(z_{m(k)}, z_{n(k)}, z_{n(k)}) + G(z_{n(k)}, z_{n(k)}, z_{n(k)+1}) \leq \\ &\leq G(z_{m(k)}, z_{n(k)}, z_{n(k)}) + G(z_{n(k)}, z_{n(k)+1}, z_{n(k)+2}) \end{aligned} \tag{10}$$

and

$$G(z_{m(k)}, z_{n(k)}, z_{n(k)}) \leq G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}). \tag{11}$$

Hence in (10) and (11), if $k \to \infty$, using (5), (6) and (9), we have

$$\lim_{k\to\infty} G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) = \varepsilon. \tag{12}$$

On the other hand

$$\begin{aligned} &G(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}) \leq \\ &\leq G(z_{m(k)}, z_{n(k)}, z_{n(k)}) + G(z_{n(k)}, z_{n(k)+1}, z_{n(k)+1}) \end{aligned} \tag{13}$$

and

$$\begin{aligned} &G(z_{n(k)}, z_{n(k)+1}, z_{m(k)}) \leq \\ &\leq G(z_{n(k)}, z_{n(k)+1}, z_{n(k)+1}) + G(z_{n(k)+1}, z_{n(k)+1}, z_{m(k)}). \end{aligned} \tag{14}$$

Hence from (13) and (14), by applying limit $k \to \infty$, from (5), (9) and (12), we have

$$\lim_{k\to\infty} G(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}) = \varepsilon. \tag{15}$$

In a similar way, we have

$$\begin{aligned} &G(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}) \leq \\ &\leq G(z_{m(k)+1}, z_{m(k)}, z_{m(k)}) + G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) \leq \\ &\leq 2G(z_{m(k)}, z_{m(k)+1}, z_{m(k)+1}) + G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) \end{aligned} \tag{16}$$

and

$$\begin{aligned} &G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) \leq \\ &\leq G(z_{m(k)}, z_{m(k)+1}, z_{m(k)+1}) + G(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}), \end{aligned} \tag{17}$$

from (16) and (17) by taking limit as $k \to \infty$ and using (5) and (12), we have

$$\lim_{k\to\infty} G(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}) = \varepsilon. \tag{18}$$

Also

$$G(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}) \leq G(z_{m(k)}, z_{m(k)+1}, z_{n(k)+1}) \tag{19}$$

and

$$\begin{aligned} &G(z_{m(k)}, z_{m(k)+1}, z_{n(k)+1}) \leq \\ &\leq G(z_{m(k)}, z_{m(k)+1}, z_{m(k)+1}) + G(z_{m(k)+1}, z_{m(k)+1}, z_{n(k)+1}) \leq \\ &\leq G(z_{m(k)}, z_{m(k)+1}, z_{m(k)+1}) + G(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}). \end{aligned} \tag{20}$$

If $k \to \infty$ in (19) and (20), also using (5), (15) and (17), we have

$$\lim_{k\to\infty} G(z_{m(k)}, z_{m(k)+1}, z_{n(k)+1}) = \varepsilon. \tag{21}$$

Further

$$G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) \leq G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)}) \tag{22}$$

and

$$\begin{aligned} &G(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}) \leq \\ &\leq G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) + G(z_{n(k)+1}, z_{n(k)+1}, z_{n(k)}). \end{aligned} \tag{23}$$

So, from (5), (18), (21) and (22), we have

$$\lim_{k\to\infty} G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) = \varepsilon. \tag{24}$$

Finally

$$\begin{aligned} &G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+2}) \leq \\ &\leq G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) + G(z_{n(k)+1}, z_{n(k)+1}, z_{n(k)+2}) \leq \\ &\leq G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) + G(z_{n(k)}, z_{n(k)+1}, z_{n(k)+2}) \end{aligned} \tag{25}$$

and

$$G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) \leq G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+2}). \tag{26}$$

Hence, in (25) and (26), if $k \to \infty$ and using (5) and (24), we have

$$\lim_{k\to\infty} G(y_{m(k)+1}, z_{n(k)+1}, z_{n(k)+2}) = \varepsilon. \tag{27}$$

Now, suppose $x = x_{m(k)+1}$, $y = x_{n(k)+1}$ and $z = x_{n(k)+1}$ in (1), for all $k \geq 0$, we have

$$\begin{aligned} &\psi[G(z_{m(k)}, z_{n(k)}, z_{n(k)})] = \psi[G(gx_{m(k)+1}, gx_{n(k)+1}, gx_{n(k)+1})] \geq \\ &\geq F\big[\psi(G(fx_{m(k)+1}, fx_{n(k)+1}, fx_{n(k)+1})), \varphi G(fx_{m(k)+1}, fx_{n(k)+1}, fx_{n(k)+1}))\big] \\ &= F\big[\psi(G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1})), \varphi(G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}))\big]. \end{aligned} \tag{28}$$

Now, from (9) and (24), if $k \to \infty$ in (28), we have

$$\psi(\varepsilon) \geq F(\psi(\varepsilon), \varphi(\varepsilon)) \geq \psi(\varepsilon), \tag{29}$$

so $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$. Hence, $\varepsilon = 0$ is a contradiction. Consequently, $\{z_n\}$ is a G-Cauchy sequence.

Since (X, G) is complete, there exists a point $t \in X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = t$.

Case 1. If f and g are f-compatible of type (E).

Since f and g are weak semi compatible mappings, this yields $\lim_{n\to\infty} fgx_n = gt$ or $\lim_{n\to\infty} gfx_n = ft$. First, we take $\lim_{n\to\infty} gfx_n = ft$. Since f and g are f-compatible of type (E) then $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = gt$, Now, by (b),

$$\psi[G(gfx_n, gt, gt)] \geq F[\psi(G(ffx_n, ft, ft)), \varphi(G(ffx_n, ft, ft))].$$

On limiting $n \to \infty$, we have

$$\psi[G(ft, gt, gt)] \geq F[\psi(G(gt, ft, ft)), \varphi(G(gt, ft, ft))].$$

Since (X, G) is symmetric G-metric space, this yields

$$\psi[G(ft, gt, gt)] \geq F[\psi(G(ft, gt, gt)), \varphi(G(ft, gt, gt))].$$

Then by definition of function we have,

$$\psi[G(ft, gt, gt)] \geq F[\psi(G(ft, gt, gt)), \varphi(G(ft, gt, gt))] \geq \psi[G(ft, gt, gt)].$$

This implies further $\psi(G(ft, gt, gt)) = 0$ or $\varphi(G(ft, gt, gt)) = 0$. This yields $G(ft, gt, gt) = 0$ or $ft = gt$. R-weak commutativity of type A_f yields

$$d(fgt, ggt) \leq Rd(ft, gt),$$

which implies further $fgt = gft$ or $fgt = gft = fft = ggt$. Then by (b) we have

$$\begin{aligned} \psi[G(gft, gt, gt)] &\geq F[\psi(G(fft, ft, ft)), \varphi(G(fft, ft, ft))], \\ \psi[G(fft, ft, ft)] &\geq F[\psi(G(fft, ft, ft)), \varphi(G(fft, ft, ft))] \\ &\geq \psi[G(fft, ft, ft)]. \end{aligned}$$

This implies $\psi(G(fft, ft, ft)) = 0$ or $\varphi(G(fft, ft, ft)) = 0$ and gives finally $fft = ft$. Therefore, $fft = gft = ft$ and ft is common fixed point of f and g.

Now, we take $\lim fgx_n = gt$. Since f and g are f-compatible of type (E), we have $\lim ffx_n = \lim fgx_n = gt$. Moreover, since f and g are R-weak commuting of type A_f, it yields $d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n)$. Now, limiting $n \to \infty$ we have $d(gt, \lim ggx_n) \leq Rd(t, t)$, which implies $\lim ggx_n = gt$. Now, by (b)

$$\psi[G(ggx_n, gt, gt)] \geq F[\psi(G(fgx_n, ft, ft)), \varphi(G(fgx_n, ft, ft))].$$

Now limiting $n \to \infty$ yields

$$\psi[G(gt, gt, gt)] \geq F[\psi(G(gt, ft, ft)), \varphi(G(gt, ft, ft))]$$

or

$$\psi(0) \geq F[\psi(G(gt, ft, ft)), \varphi(G(gt, ft, ft))] \geq \psi(G(gt, ft, ft)).$$

Since ψ is monotonically increasing function, this yields

$$0 \geq G(gt, ft, ft) \text{ or } G(gt, ft, ft) = 0$$

and this implies $ft = gt$. By the preceding work it can be easily shown that ft is common fixed point of f and g.

Case 2. Let f and g be g-compatible of type (E). Since f and g are weak semi compatible mappings then $\lim fgx_n = gt$ or $\lim gfx_n = ft$.

First, we take $\lim gfx_n = ft$. Since f and g are g-compatible of type (E), this yields $\lim ggx_n = \lim gfx_n = ft$. Moreover, since f and g are R-weak commuting of type A_f, this yields

$$d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n).$$

On limiting $n \to \infty$, we get

$$d(\lim fgx_n, ft) \leq Rd(t, t)$$

or $\lim fgx_n = ft$. Since $f(X) \subseteq g(X)$, there exists a point $u \in X$ such that $ft = gu$. Now by (b) we have

$$\psi[G(ggx_n, gu, gu)] \geq F[\psi(G(fgx_n, fu, fu)), \varphi(G(fgx_n, fu, fu))].$$

Limiting $n \to \infty$ yields

$$\psi[G(ft, gu, gu)] \geq F[\psi(G(ft, fu, fu)), \varphi(G(ft, fu, fu))],$$

which implies $\psi(G(gu, gu, gu)) \geq \psi(G(gu, fu, fu))$ or $\psi(0) \geq \psi(G(gu, fu, fu))$. Since ψ is monotonically increasing, this yields $0 \geq G(gu, fu, fu)$ or $fu = gu$. Now, it can be easily shown that fu is common fixed point of f and g.

Now, we take $\lim fgx_n = gt$. By (b), we have

$$\psi[G(ggx_n, gt, gt)] \geq F[\psi(G(fgx_n, ft, ft)), \varphi(G(fgx_n, ft, ft))].$$

Limiting $n \to \infty$ yields

$$\psi[G(ft, gt, gt)] \geq F[\psi(G(gt, ft, ft)), \varphi(G(gt, ft, ft))].$$

Since (X, G) is symmetric G-metric space, this yields

$$\psi[G(gt, ft, ft)] \geq F[\psi(G(gt, ft, ft), \varphi(G(gt, ft, ft))] \geq \psi[G(gt, ft, ft)].$$

This implies further $\psi(G(gt, ft, ft)) = 0$ or $\varphi(G(gt, ft, ft)) = 0$, therefore $ft = gt$.

Now it can be easily shown that ft is common fixed point of f and g. □

Example 1. Let (X, G) be a G-metric space where G defines as $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$. Let $X = [1, 5]$ and we define mapping $f, g : X \to X$ by

$$fx = \begin{cases} 1, & x = 1 \text{ or } x \in [3, 5], \\ \frac{x+5}{2}, & x \in (1, 3), \end{cases} \qquad gx = \begin{cases} 1, & x = 1 \text{ or } x = 3, \\ x + 2, & x \in (1, 3), \\ x, & x \in (3, 5]. \end{cases}$$

Here functions f and g satisfy all the conditions of theorems with 1 is fixed point. To show weak semi compatibility and f-compatibility of type (E) of mappings f and g, a sequence $1+\epsilon_n$ is taken where $\epsilon_n \to \infty$ whenever $n \to \infty$. Here $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 3$ then $\lim_{n\to\infty} fgx_n = 1 = g(3)$ and $\lim_{n\to\infty} gfx_n = 3 \neq f(3)$ also $\lim_{n\to\infty} ffx_n = 1 = g(3)$. Moreover functions f and g satisfy R-weak commuting of type A_f.

Theorem 2. *Let f and g be two weak semi compatible, R-weak commuting of type A_f self mappings of G-metric space (X, G) satisfying following conditions:*

(a) $f(X) \subseteq g(X)$;

(b) *For all* $x, y, z \in X$;

$$\psi(G(gx, gy, gz)) \geq F(\psi(G(fx, fy, fz), \varphi(G(fx, fy, fz)));$$

(c) *f and g are either f-compatible or g-compatible;*

(d) *f and g satisfy E.A. property.*

If $\varphi \in \Phi$, $\psi \in \Psi$ and $F \in \mathcal{C}_{inv}$ then f and g have a common fixed point in X.

Proof. Since f and g satisfy E.A. property, then there exists a sequence $\{x_n\}$ in X such that $\lim fx_n = \lim gx_n = t$ for some t in X. Since $f(X) \subseteq g(X)$, then there exists a sequence $\{y_n\} \in X$ such that $\lim fx_n = \lim gy_n$ or $\lim fx_n = \lim gy_n = t$. Now by (b) we have

$$\psi(G(gx_n, gy_n, gy_n)) \geq F(\psi(G(fx_n, fy_n, fy_n), \varphi(G(fx_n, fy_n, fy_n))).$$

Limiting $n \to \infty$ and by the definition of functions F and ψ yields $\lim fy_n = t$. Thus

$$\lim fx_n = t, \ \lim gx_n = t, \ \lim gy_n = t \text{ and } \lim fy_n = t. \tag{30}$$

Case 1. If f and g are f-compatible.

Since f and g are weak semi compatible mappings, this yields either $\lim fgx_n = gt$ or $\lim gfx_n = ft$. First, we take $\lim fgx_n = gt$. Since f and g are f-compatible, this yields $\lim fgx_n = \lim ggx_n$. It gives $\lim ggx_n = gt$. Now by (b),

$$\psi(G(ggx_n, gt, gt)) \geq F(\psi(G(fgx_n, ft, ft), \varphi(G(fgx_n, ft, ft))).$$

Limiting $n \to \infty$ we get,

$$\psi(G(gt, gt, gt)) \geq F(\psi(G(gt, ft, ft), \varphi(G(gt, ft, ft))).$$

By the definition of F and ψ, it can be easily obtained $ft = gt$. Since f and g are R-weak commuting of type A_f, this gives

$$d(fgt, ggt) \leq Rd(ft, gt).$$

It further implies

$$fgt = gft \text{ or } fgt = gft = fft = ggt.$$

Now by (b), it is very easy to get $fft = gft = ft$ or ft is common fixed point of f and g.

Next we take $\lim gfx_n = ft$. This yields by 30, $\lim ggy_n = ft$. Since $f(X) \subseteq g(X)$. Then there exists a point u in X such that $ft = gu$. Now R-weak commuting of type A_f yields $d(fgy_n, ggy_n) \leq Rd(fy_n, gy_n)$. Limiting $n \to \infty$ yields $d(\lim fgy_n, gu) \leq Rd(t,t)$. It further gives $\lim fgy_n = gu$. Then by (b)

$$\psi(G(ggy_n, gu, gu)) \geq F(\psi(G(fgy_n, fu, fu), \varphi(G(fgy_n, fu, fu))).$$

Limiting $n \to \infty$ yields

$$\psi(G(gu, gu, gu)) \geq F(\psi(G(gu, fu, fu), \varphi(G(gu, fu, fu))).$$

By the definition of F and ψ, it can be easily obtained $fu = gu$. By (b) with R-weak commuting of type A_f, it can be easily shown that fu is common fixed point of f and g.

Case 2. if f and g are g-compatible.

Since f and g are weak semi compatible mappings, this yields either $\lim fgx_n = gt$ or $\lim gfx_n = ft$. First, we take $\lim fgx_n = gt$. By virtue of (30), $\lim ffx_n = gt$. Since f and g are g-compatible, this yields $\lim fgx_n = \lim ffx_n$ or $\lim gfx_n = gt$. Then by (b)

$$\psi(G(gfx_n, gt, gt)) \geq F(\psi(G(ffx_n, ft, ft), \varphi(G(ffx_n, ft, ft))).$$

Limiting $n \to \infty$ yields

$$\psi(G(gt, gt, gt)) \geq F(\psi(G(gt, ft, ft), \varphi(G(gt, ft, ft))).$$

By the definition of F and ψ , it can be easily shown that $ft = gt$. Then by (b) with R-weak commuting of type A_f, it can be easily shown that ft is common fixed point of f and g.

Finally we take $\lim gfx_n = ft$. Since $f(X) \subseteq g(X)$. Then there exists a point u in X such that $ft = gu$. By the previous working it is easy to show that fu is common fixed point of f and g. $\square$

Example 2. Let (X, G) be a G-metric space where $G(x, y, z)$ be defined as $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$. Let $X = [1, 4]$ and we define mapping $f, g : X \to X$ by

$$fx = \begin{cases} 1, & x = 1, \\ 2, & x \in (1, 2], \\ 3, & x \in (2, 4], \end{cases} \qquad gx = \begin{cases} 1, & x = 1, \\ x+1, & x \in (1, 2), \\ \frac{x+4}{2}, & x \in [2, 4]. \end{cases}$$

It is clear that functions f and g of the present example satisfy all the conditions of the theorem with 1 is a fixed point. This satisfy E.A. property for sequences $1 + \epsilon_n$ and $2 + \epsilon_n$ where $\epsilon_n \to \infty$ whenever $n \to \infty$. To show weak semi compatibility and f-compatibility, a sequence $1 + \epsilon_n$ is taken where $\epsilon_n \to \infty$ whenever $n \to \infty$. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 2$ with $\lim_{n\to\infty} fgx_n = 3 = g(2)$ and $\lim_{n\to\infty} gfx_n = 3 \neq f(2)$. Moreover, functions f and g satisfy R-weak commuting of type A_f.

The next theorem is none else but just the generalization of above theorems in which both compatibility conditions are removed and extend one more commutativity condition, with assuming that functions satisfy E.A. property.

Theorem 3. *Let f and g be two weak semi compatible self mappings of G-metric space (X,G)satisfying following conditions:*

(a) $f(X) \subseteq g(X)$;

(b) *For every $x,y,z \in X$, holds*

$$\psi(G(gx,gy,gz)) \geq F(\psi(G(fx,fy,fz),\varphi(G(fx,fy,fz)));$$

(c) *f and g are either R-weak commuting of type A_f or A_g;*

(d) *f and g satisfy E.A. property.*

If $\varphi \in \Phi$, $\psi \in \Psi$ and $F \in \mathcal{C}_{inv}$ then f and g have a common fixed point in X.

Proof. Since f and g satisfy E.A. property then there exists a sequence $\{x_n\}$ in X such that $\lim fx_n = \lim gx_n = t$ for some $t \in X$. Since $f(X) \subseteq g(X)$, then there exists a sequence $\{y_n\} \in X$ such that $\lim fx_n = \lim gy_n$ or $\lim fx_n = \lim gy_n = t$. Now by (b) we have

$$\psi(G(gx_n,gy_n,gy_n)) \geq F(\psi(G(fx_n,fy_n,fy_n),\varphi(G(fx_n,fy_n,fy_n))).$$

Limiting $n \to \infty$ yields

$$\psi(G(t,t,t)) \geq F(\psi(G(t,\lim fy_n,\lim fy_n),\varphi(G(t,\lim fy_n,\lim fy_n))).$$

This gives further

$$\psi(0) \geq \psi(G(t,\lim fy_n,\lim fy_n)).$$

Since ψ is a monotonic increasing function yields $\lim fy_n = t$. Thus

$$\lim fx_n = t, \quad \lim gx_n = t, \quad \lim gy_n = t, \quad \lim fy_n = t. \tag{31}$$

Case 1. If f and g are R-weak commuting of type A_g.

Since f and g are weak semi compatible mappings, this yields $\lim fgy_n = gt$ or $\lim gfy_n = ft$. Let us first assume that $\lim fgy_n = gt$. It is clear by 31 that $\lim fgy_n = \lim ffx_n = gt$. Since f and g are R-weak commuting of type A_g, this yields

$$d(gfx_n,ffx_n) \leq Rd(fx_n,gx_n).$$

Limiting $n \to \infty$ yields

$$d(\lim gfx_n,gt) \leq Rd(t,t).$$

Thus $\lim gfx_n = gt$ or $\lim ggy_n = gt$. Now by (b)

$$\psi(G(ggy_n,gt,gt)) \geq F(\psi(G(fgy_n,ft,ft),\varphi(G(fgy_n,ft,ft))).$$

Limiting $n \to \infty$ yields

$$\psi(G(gt,gt,gt)) \geq F(\psi(G(gt,ft,ft),\varphi(G(gt,ft,ft))).$$

It implies $\psi(0) \geq \psi(G(gt, ft, ft))$ or $gt = ft$. Since f and g are R-weak commuting of type A_g yields

$$d(gft, fft) \leq Rd(ft, gt).$$

Since $gt = ft$, it gives $gft = fft$ or $gft = fft = fgt = ggt$. Now by (b)

$$\psi(G(gft, gt, gt)) \geq F(\psi(G(fft, ft, ft), \varphi(G(fft, ft, ft))).$$

This implies

$$\begin{aligned}\psi(G(fft, gt, gt)) &\geq F(\psi(G(fft, ft, ft), \varphi(G(fft, ft, ft))) \\ &\geq \psi(G(fft, gt, gt).\end{aligned}$$

This gives $\psi(G(fft, ft, ft) = 0$ or $\varphi(G(fft, ft, ft)) = 0$. This yields $fft = ft$ or $fft = gft = ft$. Thus, ft is common fixed point of f and g.

Next we suppose that $\lim gfy_n = ft$. Since f and g are R-weak commuting of type A_g, this yields

$$d(gfy_n, ffy_n) \leq Rd(fy_n, gy_n).$$

Limiting $n \to \infty$ gives

$$d(ft, \lim ffy_n) \leq Rd(t, t) \text{ or } \lim ffy_n = ft.$$

Since $f(X) \subseteq g(X)$, then there exists some $u \in X$ such that $ft = gu$. Therefore, $\lim gfy_n = \lim ffy_n = ft = gu$. Then by (b), we have

$$\psi(G(gfy_n, gu, gu)) \geq F(\psi(G(ffy_n, fu, fu), \varphi(G(ffy_n, fu, fu))).$$

Limiting $n \to \infty$ yields

$$\begin{aligned}\psi(G(gu, gu, gu)) &\geq F(\psi(G(gu, fu, fu), \varphi(G(gu, fu, fu))) \\ &\geq \psi(G(gu, fu, fu))\end{aligned}$$

then

$$\psi(0) \geq \psi(G(gu, fu, fu)) \text{ or } gu = fu.$$

Now, it can be easily shown that fu is common fixed point of f and g.

Case 2. If f and g are R-weak commuting of type A_f.

Since f and g are weak semi compatible mappings, this yields $\lim fgx_n = gt$ or $\lim gfx_n = ft$. Let us first consider that $\lim fgx_n = gt$. Since f and g are R-weak commuting of type A_f, this gives

$$d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n).$$

Limiting $n \to \infty$ yields

$$d(gt, \lim ggx_n) \leq Rd(t, t) \text{ or } \lim ggx_n = gt.$$

Now, by (b), we have

$$\psi(G(ggx_n, gt, gt)) \geq F(\psi(G(fgx_n, ft, ft), \varphi(G(fgx_n, ft, ft))).$$

Limiting $n \to \infty$ yields

$$\psi(G(gt, gt, gt)) \geq F(\psi(G(gt, ft, ft), \varphi(G(gt, ft, ft))).$$

This implies

$$\psi(0) \geq F(\psi(G(gt, ft, ft), \varphi(G(gt, ft, ft))) \geq \psi(G(gt, ft, ft).$$

Since ψ is a monotonic increasing, this yields $ft = gt$. Now it can be easily shown that ft is common fixed point of f and g.

Next we suppose that $\lim gfx_n = ft$. Since $\lim fx_n = \lim gy_n$ therefore $\lim ggy_n = ft$. Since $f(X) \subseteq g(X)$, then there exists a point u in X such that $ft = gu$. Since f and g are R-weak commuting of type A_f, yields

$$d(fgy_n, ggy_n) \leq Rd(fy_n, gy_n).$$

Limiting $n \to \infty$ yields

$$d(\lim fgy_n, ft) \leq Rd(t, t).$$

It yields $\lim fgy_n = ft$. Then we have $\lim ggy_n = \lim fgy_n = ft = gu$. Now, by (b), we have

$$\psi(G(ggy_n, gu, gu)) \geq F(\psi(G(fgy_n, fu, fu), \varphi(G(fgy_n, fu, fu))).$$

Limiting $n \to \infty$ yields

$$\psi(G(gu, gu, gu)) \geq F(\psi(G(gu, fu, fu), \varphi(G(gu, fu, fu))).$$

Hence, holds

$$\psi(0) \geq F(\psi(G(gu, fu, fu), \varphi(G(gu, fu, fu))) \geq \psi(G(gu, fu, fu).$$

Since ψ is monotonic increasing function, this yield $0 \geq G(gu, fu, fu)$ or $fu = gu$.

Now, it is very easy to show that fu is common fixed point of f and g. $\square$

Example 3. Let (X, G) be a G-metric space where $G(x, y, z)$ be defined as $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$. Let $X = [2, 6]$ and we define mapping $f, g : X \to X$ by

$$fx = \begin{cases} 4, & x \in [2, 4], \\ 2, & x \in (4, 6], \end{cases} \qquad gx = \begin{cases} x + 2, & x \in (2, 4), \\ x, & x \in [4, 6] \text{ or } x = 2. \end{cases}$$

In the present example, the functions f and g satisfy all the given conditions of the theorem with 4 is fixed point. This satisfies E.A. property for sequence $2 + \epsilon_n$ where $\epsilon_n \to \infty$ whenever $n \to \infty$. To show weak semi compatibility a sequence $2 + \epsilon_n$ is taken where $\epsilon_n \to \infty$ whenever $n \to \infty$. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 4$ with $\lim_{n\to\infty} fgx_n = 2 \neq g(4)$ and $\lim_{n\to\infty} gfx_n = 4 = f(4)$. Moreover, functions f and g satisfy R-weak commuting of type A_f.

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