

## Initial Coefficient estimates for a classes of $m$ -fold symmetric bi-univalent functions involving Mittag-Leffler function

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ABSTRACT. The main object of the present paper is to use Mittag-Leffler function to introduce and study two new classes

$$\mathcal{R}_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \alpha) \quad \text{and} \quad \mathcal{R}_{\Sigma_m}^*(\gamma, \lambda, \eta, \delta, \tau; \beta)$$

of  $\Sigma_m$  consisting of analytic and  $m$ -fold symmetric bi-univalent functions defined in the open unit disk  $U$ . Also, we determine the estimates on the initial coefficients  $|a_{m+1}|$  and  $|a_{2m+1}|$  for functions in each of these new classes. Furthermore, we indicate certain special cases for our results.

### 1. INTRODUCTION

Let  $\mathcal{A}$  stands for the class of functions  $f$  that are analytic in the open unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$ , are normalized by the conditions  $f(0) = f'(0) - 1 = 0$ , and have the form:

$$(1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let  $S$  be the subclass of  $\mathcal{A}$  consisting of functions of the form (1) which are also univalent in  $U$ . The Koebe one-quarter theorem (see [5]) states that the image of  $U$  under every function  $f \in S$  contains a disk of radius  $\frac{1}{4}$ . Therefore, every function  $f \in S$  has an inverse  $f^{-1}$  which satisfies  $f^{-1}(f(z)) = z$ , ( $z \in U$ ) and  $f(f^{-1}(w)) = w$ , ( $|w| < r_0(f)$ ,  $r_0(f) \geq \frac{1}{4}$ ), where

$$(2) \quad g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots.$$

A function  $f \in \mathcal{A}$  is said to be bi-univalent in  $U$  if both  $f$  and  $f^{-1}$  are univalent in  $U$ . We denote by  $\Sigma$  the class of bi-univalent functions in  $U$

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satisfying (1). In fact, Srivastava et al. [18] have actually revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Murugusundaramoorthy et al. [11], Caglar et al. [4], Adegani et al. [1] and others (see, for example [8, 13, 14, 17, 25]).

For each function  $f \in S$ , the function  $h(z) = (f(z^m))^{\frac{1}{m}}$ , ( $z \in U, m \in \mathbb{N}$ ) is univalent and maps the unit disk  $U$  into a region with  $m$ -fold symmetry. A function is said to be  $m$ -fold symmetric (see [7]) if it has the following normalized form:

$$(3) \quad f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in U, m \in \mathbb{N}).$$

We denote by  $S_m$  the class of  $m$ -fold symmetric univalent functions in  $U$ , which are normalized by the series expansion (3). In fact, the functions in the class  $S$  are one-fold symmetric.

In [20] Srivastava et al. defined  $m$ -fold symmetric bi-univalent functions analogues to the concept of  $m$ -fold symmetric univalent functions. They gave some important results, such as each function  $f \in \Sigma$  generates an  $m$ -fold symmetric bi-univalent function for each  $m \in \mathbb{N}$ . Furthermore, for the normalized form of  $f$  given by (3), they obtained the series expansion for  $f^{-1}$  as follows:

$$(4) \quad g(w) = w - a_{m+1} w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}] w^{2m+1} \\ - \left[ \frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \right] w^{3m+1} + \dots,$$

where  $f^{-1} = g$ . We denote by  $\Sigma_m$  the class of  $m$ -fold symmetric bi-univalent functions in  $U$ . It is easily seen that for  $m = 1$ , the formula (4) coincides with the formula (2) of the class  $\Sigma$ . Some examples of  $m$ -fold symmetric bi-univalent functions are given as follows:

$$\left( \frac{z^m}{1-z^m} \right)^{\frac{1}{m}}, \quad \left[ \frac{1}{2} \log \left( \frac{1+z^m}{1-z^m} \right) \right]^{\frac{1}{m}}, \quad [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left( \frac{w^m}{1+w^m} \right)^{\frac{1}{m}}, \quad \left( \frac{e^{2w^m} - 1}{e^{2w^m} + 1} \right)^{\frac{1}{m}}, \quad \left( \frac{e^{w^m} - 1}{e^{w^m}} \right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subclasses of  $m$ -fold bi-univalent functions (see [2, 6, 12, 15, 16, 19, 22, 23, 24, 26, 27]).

The Mittag-Leffler function  $E_{\lambda}(z)$ , ( $z \in \mathbb{C}$ ) (see [9, 10]) is defined by

$$E_{\lambda}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\lambda k + 1)}, \quad (\lambda \in \mathbb{C}, \operatorname{Re}(\lambda) > 0).$$

Recently, Srivastava and Tomovski [21] introduced the function  $E_{\lambda,\eta}^{\delta,\tau}(z)$ , ( $z \in \mathbb{C}$ ) in the form:

$$E_{\lambda,\eta}^{\delta,\tau}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_{k\tau} z^k}{\Gamma(\lambda k + \eta) k!}, \quad (\lambda \in \mathbb{C}, \operatorname{Re}(\lambda) > 0).$$

where  $\lambda, \eta, \delta \in \mathbb{C}$ ,  $\operatorname{Re}(\lambda) > \max\{0, \operatorname{Re}(\tau) - 1\}$ ,  $\operatorname{Re}(\tau) > 0$  and  $(x)_k$  is the Pochhammer symbol defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1, & \text{for } k = 0; \\ x(x+1)\cdots(x+n-1), & \text{for } k \in \mathbb{N}. \end{cases}$$

In 2016, Attiya [3] introduced and investigated a linear operator  $\mathcal{H}_{\lambda,\eta}^{\delta,\tau} : \mathcal{A} \rightarrow \mathcal{A}$  by using the Hadamard product (or convolution) and defined as follows

$$\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z) = Q_{\lambda,\eta}^{\delta,\tau}(z) * f(z), \quad (z \in U),$$

where "\*" indicate the Hadamard product (or convolution) of two series and

$$Q_{\lambda,\eta}^{\delta,\tau}(z) = \frac{\Gamma(\lambda + \eta)}{(\delta)_\tau} \left( E_{\lambda,\eta}^{\delta,\tau}(z) - \frac{1}{\Gamma(\eta)} \right),$$

$\lambda, \eta, \delta \in \mathbb{C}$ ,  $\operatorname{Re}(\lambda) > \max\{0, \operatorname{Re}(\tau) - 1\}$ ,  $\operatorname{Re}(\tau) > 0$ .

By some easy computations, we conclude that

$$\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\delta + k\tau)\Gamma(\lambda + \eta)}{\Gamma(\delta + \tau)\Gamma(\lambda k + \eta)\Gamma(k + 1)} a_k z^k.$$

It is easily verified that if  $f \in S_m$ , then we have

$$\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z) = z + \sum_{k=1}^{\infty} \frac{\Gamma(\delta + (mk + 1)\tau)\Gamma(\lambda + \eta)}{\Gamma(\delta + \tau)\Gamma(\lambda(mk + 1) + \eta)\Gamma(mk + 2)} a_{mk+1} z^{mk+1}.$$

We require the following lemma to prove our main results.

**Lemma 1.** [5] *If  $h \in \mathcal{P}$ , then  $|c_k| \leq 2$  for each  $k \in \mathbb{N}$ , where  $\mathcal{P}$  is the family of all functions  $h$  analytic in  $U$  for which*

$$\operatorname{Re}(h(z)) > 0, \quad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad (z \in U).$$

## 2. COEFFICIENT ESTIMATES FOR THE FUNCTION CLASS $\mathcal{R}_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \alpha)$

**Definition 1.** For  $0 < \alpha \leq 1$ ,  $0 \leq \gamma \leq 1$  and  $m \in \mathbb{N}$ , a function  $f \in \Sigma_m$  given by (3) is said to be in the class  $\mathcal{R}_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \alpha)$ , if it satisfies the following conditions:

$$(5) \quad \left| \arg \left( \gamma z \left( \mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z) \right)'' + (2\gamma + 1) \left( \mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z) \right)' - 2\gamma \right) \right| < \frac{\alpha\pi}{2},$$

and

$$(6) \quad \left| \arg \left( \gamma w \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} g(w) \right)'' + (2\gamma + 1) \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} g(w) \right)' - 2\gamma \right) \right| < \frac{\alpha\pi}{2},$$

where the function  $g = f^{-1}$  is given by (4).

In particular, for one-fold symmetric bi-univalent functions, we denote the class

$$\mathcal{R}_{\Sigma_1}(\gamma, \lambda, \eta, \delta, \tau; \alpha) \equiv \mathcal{R}_{\Sigma}(\gamma, \lambda, \eta, \delta, \tau; \alpha).$$

**Remark 1.** If we put  $\gamma = \lambda = 0$  and  $\eta = \delta = \tau = 1$ :

- (1) The class  $\mathcal{R}_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \alpha)$  reduces to the class  $\mathcal{H}_{\Sigma, m}^{\alpha}$  which was given by Srivastava et al. [20];
- (2) The class  $\mathcal{R}_{\Sigma}(\gamma, \lambda, \eta, \delta, \tau; \alpha)$  reduces to the class  $\mathcal{H}_{\Sigma}^{\alpha}$  which was given by Srivastava et al. [18].

**Theorem 1.** Let  $f \in \mathcal{R}_{\Sigma_m}(\gamma, \lambda, \eta, \delta, \tau; \alpha)$  ( $0 < \alpha \leq 1$ ,  $0 \leq \gamma \leq 1$ ,  $m \in \mathbb{N}$ ) be given by (3). Then

$$(7) \quad \frac{|a_{m+1}| \leq 2\alpha\Gamma(\delta + \tau)\Gamma(\lambda(m + 1) + \eta)\Gamma(m + 2)\sqrt{\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))}}{\sqrt{|\alpha\Omega_1(\gamma, \lambda, \eta, \delta, \tau, m) + (1 - \alpha)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))\Omega_2^2(\gamma, \lambda, \eta, \delta, \tau, m)|}}$$

and

$$(8) \quad |a_{2m+1}| \leq \frac{2\alpha^2(m + 1)\Gamma^2(\delta + \tau)\Gamma^2(\lambda(m + 1) + \eta)\Gamma^2(m + 2)}{\Omega_2^2(\gamma, \lambda, \eta, \delta, \tau, m)} + \frac{2\alpha\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))}{\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma(4m(m + 1) + 2) + 2m(\gamma + 1) + 1]},$$

where

$$(9) \quad \begin{aligned} \Omega_1(\gamma, \lambda, \eta, \delta, \tau, m) &= (m + 1)\Gamma(\delta + \tau)\Gamma^2(\lambda(m + 1) + \eta) \times \\ &\times \Gamma^2(m + 2)\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) \times \\ &\times [\gamma(4m(m + 1) + 2) + 2m(\gamma + 1) + 1], \end{aligned}$$

$$(10) \quad \begin{aligned} \Omega_2(\gamma, \lambda, \eta, \delta, \tau, m) &= \Gamma(\delta + (m + 1)\tau)\Gamma(\lambda + \eta) \times \\ &\times \left[ \gamma \left( (m + 1)^2 + 1 \right) + m(\gamma + 1) + 1 \right]. \end{aligned}$$

*Proof.* It follows from conditions (5) and (6) that

$$(11) \quad \gamma z \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} f(z) \right)'' + (2\gamma + 1) \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} f(z) \right)' - 2\gamma = [p(z)]^{\alpha}$$

and

$$(12) \quad \gamma w \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} g(w) \right)'' + (2\gamma + 1) \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} g(w) \right)' - 2\gamma = [q(w)]^{\alpha},$$

where  $g = f^{-1}$  and  $p, q$  in  $\mathcal{P}$  have the following series representations:

$$(13) \quad p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \dots$$

and

$$(14) \quad q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \dots$$

Comparing the corresponding coefficients of (11) and (12) yields

$$(15) \quad \frac{\Gamma(\delta + (m+1)\tau)\Gamma(\lambda + \eta) \left[ \gamma \left( (m+1)^2 + 1 \right) + m(\gamma + 1) + 1 \right]}{\Gamma(\delta + \tau)\Gamma(\lambda(m+1) + \eta)\Gamma(m+2)} a_{m+1} = \alpha p_m,$$

$$(16) \quad \frac{\Gamma(\delta + (2m+1)\tau)\Gamma(\lambda + \eta) \left[ \gamma (4m(m+1) + 2) + 2m(\gamma + 1) + 1 \right]}{\Gamma(\delta + \tau)\Gamma(\lambda(2m+1) + \eta)\Gamma(2(m+1))} a_{2m+1} \\ = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2,$$

$$(17) \quad -\frac{\Gamma(\delta + (m+1)\tau)\Gamma(\lambda + \eta) \left[ \gamma \left( (m+1)^2 + 1 \right) + m(\gamma + 1) + 1 \right]}{\Gamma(\delta + \tau)\Gamma(\lambda(m+1) + \eta)\Gamma(m+2)} a_{m+1} = \alpha q_m$$

and

$$(18) \quad \frac{\Gamma(\delta + (2m+1)\tau)\Gamma(\lambda + \eta) \left[ \gamma (4m(m+1) + 2) + 2m(\gamma + 1) + 1 \right]}{\Gamma(\delta + \tau)\Gamma(\lambda(2m+1) + \eta)\Gamma(2(m+1))} \times \\ \times \left( (m+1)a_{m+1}^2 - a_{2m+1} \right) = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2.$$

In view of (15) and (17), we find that

$$(19) \quad p_m = -q_m$$

and

$$(20) \quad \frac{2\Gamma^2(\delta + (m+1)\tau)\Gamma^2(\lambda + \eta) \left[ \gamma \left( (m+1)^2 + 1 \right) + m(\gamma + 1) + 1 \right]^2}{\Gamma^2(\delta + \tau)\Gamma^2(\lambda(m+1) + \eta)\Gamma^2(m+2)} a_{m+1}^2 \\ = \alpha^2(p_m^2 + q_m^2).$$

Also, from (16), (18) and (20), we obtain

$$\frac{(m+1)\Gamma(\delta + (2m+1)\tau)\Gamma(\lambda + \eta) \left[ \gamma (4m(m+1) + 2) + 2m(\gamma + 1) + 1 \right]}{\Gamma(\delta + \tau)\Gamma(\lambda(2m+1) + \eta)\Gamma(2(m+1))} a_{m+1}^2 \\ = \alpha(p_{2m} + q_{2m}) + \frac{\alpha(\alpha-1)}{2} (p_m^2 + q_m^2) = \alpha(p_{2m} + q_{2m}) + \\ + \frac{(\alpha-1)\Gamma^2(\delta + (m+1)\tau)\Gamma^2(\lambda + \eta) \left[ \gamma \left( (m+1)^2 + 1 \right) + m(\gamma + 1) + 1 \right]^2}{\alpha\Gamma^2(\delta + \tau)\Gamma^2(\lambda(m+1) + \eta)\Gamma^2(m+2)} a_{m+1}^2.$$

Therefore, we have

$$(21) \quad a_{m+1}^2 = \frac{\alpha^2\Gamma^2(\delta + \tau)\Gamma^2(\lambda(m+1) + \eta)\Gamma^2(m+2)\Gamma(\lambda(2m+1) + \eta)\Gamma(2(m+1))(p_{2m} + q_{2m})}{\alpha\Omega_1(\gamma, \lambda, \eta, \delta, \tau, m) + (1-\alpha)\Gamma(\lambda(2m+1) + \eta)\Gamma(2(m+1))\Omega_2^2(\gamma, \lambda, \eta, \delta, \tau, m)},$$

where  $\Omega_1(\gamma, \lambda, \eta, \delta, \tau, m)$  and  $\Omega_2(\gamma, \lambda, \eta, \delta, \tau, m)$  are given by (9) and (10), respectively.

Now, taking the absolute value of (21) and applying Lemma 1 for the coefficients  $p_{2m}$  and  $q_{2m}$ , we deduce that

$$|a_{m+1}| \leq \frac{2\alpha\Gamma(\delta + \tau)\Gamma(\lambda(m + 1) + \eta)\Gamma(m + 2)\sqrt{\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))}}{\sqrt{|\alpha\Omega_1(\gamma, \lambda, \eta, \delta, \tau, m) + (1 - \alpha)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))\Omega_2^2(\gamma, \lambda, \eta, \delta, \tau, m)|}}$$

This gives the desired estimate for  $|a_{m+1}|$  as asserted in (7).

In order to find the bound on  $|a_{2m+1}|$ , by subtracting (18) from (16), we get

$$(22) \quad \frac{\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma(4m(m + 1) + 2) + 2m(\gamma + 1) + 1]}{\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))} \times \\ \times (2a_{2m+1} - (m + 1)a_{m+1}^2) = \alpha(p_{2m} - q_{2m}) + \frac{\alpha(\alpha - 1)}{2} (p_m^2 - q_m^2).$$

It follows from (19), (20) and (22) that

$$(23) \quad a_{2m+1} = \frac{\alpha^2(m + 1)\Gamma^2(\delta + \tau)\Gamma^2(\lambda(m + 1) + \eta)\Gamma^2(m + 2) (p_m^2 + q_m^2)}{4\Omega_2^2(\gamma, \lambda, \eta, \delta, \tau, m)} \\ + \frac{\alpha\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1)) (p_{2m} - q_{2m})}{2\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma(4m(m + 1) + 2) + 2m(\gamma + 1) + 1]}.$$

Taking the absolute value of (23) and applying Lemma 1 once again for the coefficients  $p_m, p_{2m}, q_m$  and  $q_{2m}$ , we obtain

$$|a_{2m+1}| \leq \frac{2\alpha^2(m + 1)\Gamma^2(\delta + \tau)\Gamma^2(\lambda(m + 1) + \eta)\Gamma^2(m + 2)}{\Omega_2^2(\gamma, \lambda, \eta, \delta, \tau, m)} \\ + \frac{2\alpha\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))}{\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma(4m(m + 1) + 2) + 2m(\gamma + 1) + 1]},$$

which completes the proof of Theorem 1. □

**Remark 2.** In Theorem 1, if we choose  $\gamma = \lambda = 0$  and  $\eta = \delta = \tau = 1$ , then we obtain the results which was given by Srivastava et al. [20, Theorem 2].

For one-fold symmetric bi-univalent functions, Theorem 1 reduces to the following corollary:

**Corollary 1.** *Let  $f \in \mathcal{R}_\Sigma(\gamma, \lambda, \eta, \delta, \tau; \alpha)$  ( $0 < \alpha \leq 1, 0 \leq \gamma \leq 1$ ) be given by (1). Then*

$$|a_2| \leq \frac{4\alpha\Gamma(\delta + \tau)\Gamma(2\lambda + \eta)\sqrt{6\Gamma(3\lambda + \eta)}}{\sqrt{|\alpha\Omega_3(\gamma, \lambda, \eta, \delta, \tau) + 24(1 - \alpha)\Gamma(3\lambda + \eta)\Gamma^2(\delta + 2\tau)\Gamma^2(\lambda + \eta)(3\gamma + 1)^2|}}$$

and

$$|a_3| \leq \frac{4\alpha^2\Gamma^2(\delta + \tau)\Gamma^2(2\lambda + \eta)}{\Gamma^2(\delta + 2\tau)\Gamma^2(\lambda + \eta)(3\gamma + 1)^2} + \frac{4\alpha\Gamma(\delta + \tau)\Gamma(3\lambda + \eta)}{\Gamma(\delta + 3\tau)\Gamma(\lambda + \eta)(4\gamma + 1)},$$

where

$$\Omega_3(\gamma, \lambda, \eta, \delta, \tau) = 24\Gamma(\delta + \tau)\Gamma^2(2\lambda + \eta)\Gamma(\delta + 3\tau)\Gamma(\lambda + \eta)(4\gamma + 1).$$

**Remark 3.** In Corollary 1, if we choose  $\gamma = \lambda = 0$  and  $\eta = \delta = \tau = 1$ , then we obtain the results which was given by Srivastava et al. [18, Theorem 1].

3. COEFFICIENT ESTIMATES FOR THE FUNCTIONS CLASS

$$\mathcal{R}_{\Sigma_m}^*(\gamma, \lambda, \eta, \delta, \tau; \beta)$$

**Definition 2.** For  $0 \leq \beta < 1$ ,  $0 \leq \gamma \leq 1$  and  $m \in \mathbb{N}$ , a function  $f \in \Sigma_m$  given by (3) is said to be in the class  $\mathcal{R}_{\Sigma_m}^*(\gamma, \lambda, \eta, \delta, \tau; \beta)$  if it satisfies the following conditions:

$$(24) \quad \operatorname{Re} \left\{ \gamma z \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} f(z) \right)'' + (2\gamma + 1) \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} f(z) \right)' - 2\gamma \right\} > \beta$$

and

$$(25) \quad \operatorname{Re} \left\{ \gamma w \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} g(w) \right)'' + (2\gamma + 1) \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} g(w) \right)' - 2\gamma \right\} > \beta,$$

where the function  $g = f^{-1}$  is given by (4).

In particular, for one-fold symmetric bi-univalent functions, we denote the class

$$\mathcal{R}_{\Sigma_1}^*(\gamma, \lambda, \eta, \delta, \tau; \beta) \equiv \mathcal{R}_{\Sigma}^*(\gamma, \lambda, \eta, \delta, \tau; \beta).$$

**Remark 4.** If we put  $\gamma = \lambda = 0$  and  $\eta = \delta = \tau = 1$ :

- (1) The class  $\mathcal{R}_{\Sigma_m}^*(\gamma, \lambda, \eta, \delta, \tau; \beta)$  reduces to the class  $\mathcal{H}_{\Sigma, m}(\beta)$  which was given by Srivastava et al. [20];
- (2) The class  $\mathcal{R}_{\Sigma}^*(\gamma, \lambda, \eta, \delta, \tau; \beta)$  reduces to the class  $\mathcal{H}_{\Sigma}(\beta)$  which was given by Srivastava et al. [18].

**Theorem 2.** Let  $f \in \mathcal{R}_{\Sigma_m}^*(\gamma, \lambda, \eta, \delta, \tau; \beta)$  ( $0 \leq \beta < 1$ ,  $0 \leq \gamma \leq 1$ ,  $m \in \mathbb{N}$ ) be given by (3). Then

$$(26) \quad \begin{aligned} & |a_{m+1}| \\ & \leq 2 \sqrt{\frac{\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))(1 - \beta)}{(m + 1)\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta)[\gamma(4m(m + 1) + 2) + 2m(\gamma + 1) + 1]}} \end{aligned}$$

and

$$(27) \quad \begin{aligned} |a_{2m+1}| & \leq \frac{2\Gamma^2(\delta + \tau)\Gamma^2(\lambda(m + 1) + \eta)\Gamma^2(m + 2)(1 - \beta)^2(m + 1)}{\Omega_2^2(\gamma, \lambda, \eta, \delta, \tau, m)} \\ & + \frac{2\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))(1 - \beta)}{\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta)[\gamma(4m(m + 1) + 2) + 2m(\gamma + 1) + 1]}, \end{aligned}$$

where  $\Omega_2(\gamma, \lambda, \eta, \delta, \tau, m)$  is given by (10).

*Proof.* It follows from conditions (24) and (25) that there exist  $p, q \in \mathcal{P}$  such that

$$(28) \quad \gamma z \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} f(z) \right)'' + (2\gamma + 1) \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} f(z) \right)' - 2\gamma = \beta + (1 - \beta)p(z)$$

and

$$(29) \quad \gamma w \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} g(w) \right)'' + (2\gamma + 1) \left( \mathcal{H}_{\lambda, \eta}^{\delta, \tau} g(w) \right)' - 2\gamma = \beta + (1 - \beta)q(w),$$

where  $p(z)$  and  $q(w)$  have the forms (13) and (14), respectively. Equating coefficients (28) and (29) yields

$$(30) \quad \frac{\Gamma(\delta + (m + 1)\tau)\Gamma(\lambda + \eta) \left[ \gamma \left( (m + 1)^2 + 1 \right) + m(\gamma + 1) + 1 \right]}{\Gamma(\delta + \tau)\Gamma(\lambda(m + 1) + \eta)\Gamma(m + 2)} a_{m+1} \\ = (1 - \beta)p_m,$$

$$(31) \quad \frac{\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma (4m(m + 1) + 2) + 2m(\gamma + 1) + 1]}{\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))} a_{2m+1} \\ = (1 - \beta)p_{2m},$$

$$(32) \quad - \frac{\Gamma(\delta + (m + 1)\tau)\Gamma(\lambda + \eta) \left[ \gamma \left( (m + 1)^2 + 1 \right) + m(\gamma + 1) + 1 \right]}{\Gamma(\delta + \tau)\Gamma(\lambda(m + 1) + \eta)\Gamma(m + 2)} a_{m+1} \\ = (1 - \beta)q_m$$

and

$$(33) \quad \frac{\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma (4m(m + 1) + 2) + 2m(\gamma + 1) + 1]}{\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))} \times \\ \times \left( (m + 1)a_{m+1}^2 - a_{2m+1} \right) = (1 - \beta)q_{2m}.$$

From (30) and (32), we get

$$(34) \quad p_m = -q_m$$

and

$$(35) \quad \frac{2\Gamma^2(\delta + (m + 1)\tau)\Gamma^2(\lambda + \eta) \left[ \gamma \left( (m + 1)^2 + 1 \right) + m(\gamma + 1) + 1 \right]^2}{\Gamma^2(\delta + \tau)\Gamma^2(\lambda(m + 1) + \eta)\Gamma^2(m + 2)} a_{m+1}^2 \\ = (1 - \beta)^2 (p_m^2 + q_m^2).$$

Adding (31) and (33), we obtain

$$\frac{(m + 1)\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma (4m(m + 1) + 2) + 2m(\gamma + 1) + 1]}{\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))} \times \\ \times a_{m+1}^2 = (1 - \beta)(p_{2m} + q_{2m}).$$

Therefore, we have

$$a_{m+1}^2 = \frac{\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))(1 - \beta)(p_{2m} + q_{2m})}{(m + 1)\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma (4m(m + 1) + 2) + 2m(\gamma + 1) + 1]}.$$

Applying Lemma 1 for the coefficients  $p_{2m}$  and  $q_{2m}$ , we obtain

$$|a_{m+1}| \leq 2\sqrt{\frac{\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))(1 - \beta)}{(m + 1)\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma (4m(m + 1) + 2) + 2m(\gamma + 1) + 1]}}.$$



This gives the desired estimate for  $|a_{m+1}|$  as asserted in (26).

In order to find the bound on  $|a_{2m+1}|$ , by subtracting (33) from (31), we get

$$\frac{\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma(4m(m + 1) + 2) + 2m(\gamma + 1) + 1]}{\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))} \times \\ \times (2a_{2m+1} - (m + 1)a_{m+1}^2) = (1 - \beta)(p_{2m} - q_{2m}),$$

or equivalently

$$a_{2m+1} = \frac{m + 1}{2} a_{m+1}^2 \\ + \frac{\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))(1 - \beta)(p_{2m} - q_{2m})}{2\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma(4m(m + 1) + 2) + 2m(\gamma + 1) + 1]}.$$

Upon substituting the value of  $a_{m+1}^2$  from (35), it follows that

$$a_{2m+1} = \frac{\Gamma^2(\delta + \tau)\Gamma^2(\lambda(m + 1) + \eta)\Gamma^2(m + 2)(1 - \beta)^2(m + 1)(p_m^2 + q_m^2)}{4\Omega_2^2(\gamma, \lambda, \eta, \delta, \tau, m)} \\ + \frac{\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))(1 - \beta)(p_{2m} - q_{2m})}{2\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma(4m(m + 1) + 2) + 2m(\gamma + 1) + 1]},$$

where  $\Omega_2(\gamma, \lambda, \eta, \delta, \tau, m)$  is given by (10).

Applying Lemma 1 once again for the coefficients  $p_m, p_{2m}, q_m$  and  $q_{2m}$ , we obtain

$$|a_{2m+1}| \leq \frac{2\Gamma^2(\delta + \tau)\Gamma^2(\lambda(m + 1) + \eta)\Gamma^2(m + 2)(1 - \beta)^2(m + 1)}{\Omega_2^2(\gamma, \lambda, \eta, \delta, \tau, m)} \\ + \frac{2\Gamma(\delta + \tau)\Gamma(\lambda(2m + 1) + \eta)\Gamma(2(m + 1))(1 - \beta)}{\Gamma(\delta + (2m + 1)\tau)\Gamma(\lambda + \eta) [\gamma(4m(m + 1) + 2) + 2m(\gamma + 1) + 1]},$$

which completes the proof of Theorem 2.  $\square$

**Remark 5.** In Theorem 2, if we choose  $\gamma = \lambda = 0$  and  $\eta = \delta = \tau = 1$ , then we obtain the results which was given by Srivastava et al. [20, Theorem 3].

For one-fold symmetric bi-univalent functions, Theorem 2 reduces to the following corollary:

**Corollary 2.** Let  $f \in \mathcal{R}_{\Sigma}^*(\gamma, \lambda, \eta, \delta, \tau; \beta)$  ( $0 \leq \beta < 1, 0 \leq \gamma \leq 1$ ) be given by (1). Then

$$|a_2| \leq 2\sqrt{\frac{\Gamma(\delta + \tau)\Gamma(3\lambda + \eta)(1 - \beta)}{\Gamma(\delta + 3\tau)\Gamma(\lambda + \eta)(4\gamma + 1)}}$$

and

$$|a_3| \leq \frac{8\Gamma^2(\delta + \tau)\Gamma^2(2\lambda + \eta)(1 - \beta)^2}{\Gamma^2(\delta + 2\tau)\Gamma^2(\lambda + \eta)(3\gamma + 1)^2} + \frac{4\Gamma(\delta + \tau)\Gamma(3\lambda + \eta)(1 - \beta)}{\Gamma(\delta + 3\tau)\Gamma(\lambda + \eta)(4\gamma + 1)}.$$

**Remark 6.** In Corollary 2, if we choose  $\gamma = \lambda = 0$  and  $\eta = \delta = \tau = 1$ , then we obtain the results which was given by Srivastava et al. [18, Theorem 2].

## 4. CONCLUSION

In the present work, we have introduced two new classes of analytic and  $m$ -fold symmetric bi-univalent functions in the open unit disk  $U$  associated with Mittag-Leffler function. We have then derived the initial coefficient estimations for functions belonging to these new classes. Further by specializing the parameters, several consequences of these new classes are mentioned.

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