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New family of special numbers associated with finite operator

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Abstract. Using the notion of the generating function of a function, we define an operator with whom we manage to build a large family of numbers and polynomials. This technique permits to give the closed formulae and interesting combinatorial identities. Among others, these polynomials are a generalization of the Fubini numbers and polynomials.

1. Introduction

New families of polynomials, numbers and operators are widely used in mathematics. With the help of an operator, we construct a family of numbers and polynomials, we give their explicit formulae, state combinatorial identities and derive some identities from the generating functions in terms of continued fractions. To do this, we come back to the notion of the generating function of a function, in order to introduce the desired operator. The result obtained has many applications in pure and applied mathematics.

Given a function $f \in \mathcal{F}(\mathbb{C}, \mathbb{C})$ such that $f(t) - f(0) - 1 \neq 0$, we define the operator $\mathcal{C}$ by the following relation

$$\mathcal{C}[f](t) = \frac{1}{1 + f(0) - f(t)}.$$

$\mathcal{C}$ satisfies the identity

$$\frac{1}{\mathcal{C}[f+g]} = \frac{1}{2\mathcal{C}[2f]} + \frac{1}{2\mathcal{C}[2g]}.$$

This operator is defined in a different ways from the known operators, in which we give three similar - but not the same - example. First one is those given by Simsek [20] extracted mainly from the operator

$$E^a[f](t) = f(t + a).$$

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Operator $P_n(f,t)$ presented by Cheney and Sharma [5, Identity 4], which is defined by

$$P_n(f,t) = (1-t)^{n+1} \exp\left(\frac{xt}{1-t}\right) \sum_{v\geq 0} f\left(\frac{v}{v+n}\right) L_v^{(n)}(x) t^v,$$

where $x \leq 0$ and $L_v^{(n)}(x)$ denotes the Laguerre polynomial. Finally operator $\mathcal{M}_n(f,x)$ introduced in [16] by the following expression

$$\mathcal{M}_n(f,x) = \frac{1}{A(g(1))\, B(ng(1)x)} \sum_{v\geq 0} p_v(nx) f\left(\frac{v}{n}\right),$$

where $p_v(x)$ are generalized Appell polynomials defined by the generating function

$$A(g(t))\, B(xg(t)) = \sum_{n\geq 0} p_v(x) t^v \tag{1}$$

and A, B, g are generating functions such that

$$\begin{aligned} A(t) &= \sum_{n\geq 0} a_n t^n, \quad a_0 \neq 0, \\ B(t) &= \sum_{n\geq 0} b_n t^n, \quad b_n \neq 0, \\ g(t) &= \sum_{n\geq 1} g_n t^n, \quad g_1 \neq 0. \end{aligned}$$

We recall the definition of the generating function of functions already given in the article [12]. The sequence of functions f_n admit a generating function if and only if there exists a function $F(t,y)$ such that

$$F(t,y) = \sum_{n\geq 0} f_n(t) y^n$$

on at least one non-empty interval I centered in zero. The convergence of the series to the left of the preceding equality is ensured once the sequence of functions f_n is bounded and $|y| < 1$. Throughout this paper we consider the family of functions f such that $|f(t) - f(0)| < 1$ on a non-empty interval $I \subset \mathbb{R}$ centered in zero. The series of functions $\sum_{n\geq 0} y^n (f(t) - f(0))^n$ is a convergent geometric series for $|y| \leq 1$. Then the generating function of $g(t) = f(t) - f(0)$ is the function $\frac{1}{1-yg(t)}$. The usual successive derivatives of $g(t)$ allow to write

$$\frac{\partial^n}{\partial t^n} \frac{1}{1-yg(t)} = \sum_{j\geq 0} \left(\frac{d^n}{dt^n} g^j(t)\right) y^j.$$

In the case $y = 1$, we will have

$$\mathcal{C}[g](t) = \mathcal{C}[f](t) = \sum_{j\geq 0} (f(t) - f(0))^j.$$

Applying the derivative operator to above sequence we get

$$\frac{d^n}{dt^n}\mathcal{C}[f](t) = \sum_{j\geq 0} \frac{d^n}{dt^n}\left((f(t) - f(0)\right)^j.$$

The computation of $\underbrace{\mathcal{C}\circ\mathcal{C}\cdots\circ\mathcal{C}}_{m \text{ times}}$ permits to introduce a large family of functions defined recursively by

$$\mathcal{C}^m[f](t) = \frac{1}{1 + \mathcal{C}^{m-1}[f](0) - \mathcal{C}^{m-1}[f](t)}; \quad m \geq 1. \tag{2}$$

with the initial term $\mathcal{C}^0[f](t) = f(t)$. If $f(t) = \sum_{n\geq 0} a_n t^n$ is a generating function; $\mathcal{C}^m[f](t)$ is a generating function too and we have

$$\mathcal{C}[f](t) = \frac{1}{1 + a_0 - f(t)}.$$

The higher iterations are given by the recursion (2), with $\mathcal{C}^m[f](0) = 1, m \geq 1$. For more information about generating function theory and computational methods for the sum of power series we refer to the book [22]. The operator $\mathcal{C}^m$ is a continued fraction, for example;

$$\mathcal{C}^5[f](t) = \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{1 + f(0) - f(t)}}}}}.$$

If $f(t) = 1 + t$ we conclude that

$$\mathcal{C}^5[f](t) = \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{1 - t}}}}}.$$

In this work we are interested in numbers generated by the operators $\mathcal{C}^m$ and the combinatorial identities that arise. At the end of this paper, we apply the operator method on Fubini numbers and polynomials to give their explicit formulae.

2. Statement of the main results

Given a generating function $f(t) = \sum_{n\geq 0} a_n t^n$, several types of continued fractions and their connections to generating functions have been studied. P. Flajolet (see [9]) investigated the Jacobi Type continued fraction (J-fraction) which is taken under the form:

$$J(X,t) = \frac{1}{1 - c_0 t - \frac{a_0 b_1 t^2}{1 - c_1 t - \frac{a_1 b_2 t^2}{\cdots}}}, \tag{3}$$

where $X = \{a_0, a_1, \ldots, b_0, b_1, \ldots, c_0, c_1, \ldots\}$. When we set formally the coefficients c_j to 0 and let $X' = \{a_0, a_1, \ldots, b_0, b_1, \ldots\}$, we obtain the Stieltjes type continued fraction defined as

$$S(X', t) = \cfrac{1}{1 - \cfrac{a_0 b_1 t^2}{1 - \cfrac{a_1 b_2 t^2}{\cdots}}}. \tag{4}$$

Each of these continued fractions has a power series expansion in t:

$$J(X, t) = \sum_{n\geq 0} R_n t^n \quad \text{and} \quad S(X', t) = \sum_{n\geq 0} R'_n t^n.$$

R_n and R'_n are polynomials in X and X' respectively; the first is Jacobi-Rogers polynomial, the second is Stieltjes-Rogers polynomial. Inspired from the recent work [15] of T. Komatsu we provide the continued fraction of operator $\mathcal{C}$. T. Komatsu considered that the continued T-fraction of f is written under the form

$$f(t) = 1 - \cfrac{h_1 t}{g_1 + h_1 t - \cfrac{g_1 h_2 t}{g_2 + h_2 t - \cfrac{g_2 h_3 t}{g_3 + h_3 t - \cfrac{g_3 h_4 t}{g_3 + h_4 t - \cdots}}}}.$$

Let $P_n(t)$ and $Q_n(t)$ be polynomials of degrees not exceeding n. The Padé approximants $R_n(t) = \frac{P_n(t)}{Q_n(t)}$ of f are defined (see [1]) with the property that

$$Q_n(t) f(t) - P_m(t) = O\left(t^{n+m+1}\right).$$

These polynomials are chosen so that the power-series expansion of $R_n(t)$ reproduces as many terms of the Taylor series of $f(t)$ as possible. The existence and convergence of rational functions $R_n(t)$ to $f(t)$ are established in the works [1, 2]. For more details on this approximation method we refer to the book [3] of G.A. Baker. Komatsu provided the following recurrence relations to calculate the polynomials $P_n(t)$ and $Q_n(t)$ according to the coefficients g_n and g_n of the continued T-fraction of f.

$$P_n(t) = (g_n + h_n t) P_{n-1}(t) - g_{n-1} h_n t P_{n-2}(t),$$

$$Q_n(t) = (g_n + h_n t) Q_{n-1}(t) - g_{n-1} h_n t Q_{n-2}(t),$$

with initial terms

$$P_{-1}(t) = 1, Q_{-1}(t) = 0 \quad \text{and} \quad P_0(t) = Q_0(t) = 1.$$

The closed formulae of $P_n(t)$ and $Q_n(t)$ are

$$P_n(t) = g_1 \cdots g_n \quad \text{and} \quad Q_n(t) = g_1 \cdots g_n \sum_{j=0}^{n} \frac{h_1 \cdots h_j}{g_1 \cdots g_j} t^j.$$

Then we have

$$f(t) = \lim_{n\to\infty} \frac{P_n(t)}{Q_n(t)} = \left(\sum_{j=0}^{\infty} \frac{h_1 \cdots h_j}{g_1 \cdots g_j} t^j \right)^{-1}.$$

According to this result we deduce that $\mathcal{C}^m[f]$ admit at least tow continued fraction expansions, the most important (at order 5) is

$$\mathcal{C}^5[f](t) = \cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{a_0-\cfrac{h_1t}{g_1+h_1t-\cfrac{g_1h_2t}{g_2+h_2t-\cfrac{g_2h_3t}{g_3+h_3t-\cfrac{g_3h_4t}{g_3+h_4t-\cdots}}}}}}}}}.$$

Throughout this paper we use the following notations and definitions:

$$0^n = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!},$$

the multinomial coefficient,

$$\binom{k}{k_1, \cdots, k_n} = \frac{k!}{k_1! \cdots k_n!}$$

and the set

$$\pi_n(k) = \left\{ (k_1, \ldots, k_n) \in \mathbb{N}^{n-k+1} \backslash \begin{array}{c} k_1 + \cdots + k_n = k, \\ k_1 + 2k_2 + \cdots + nk_n = n; \end{array} \right\}.$$

We complete the work of Komatsu [15] by the following theorem which shows the link between the coefficients g_n, h_n and the complex numbers a_n.

Theorem 1.

$$a_n = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} (-1)^k \binom{k}{k_1 \cdots k_n} a_0^{1-k} \prod_{r=1}^{n} \left(\frac{h_r}{g_r}\right)^{k_r}. \tag{5}$$

We note by $a_n^{(m)}$ the numbers generated by the function $\mathcal{C}^m[f](t)$. We compute the numbers $a_n^{(1)}$ in two different ways.This calculation allows us to find a combinatorial identity satisfied by the numbers a_n. More exactly we have the following result.

Theorem 2. *For $n \geq 1$ we have*

$$\sum_{i=1}^{n} \sum_{k=0}^{i} \sum_{(k_1,\cdots,k_i)\in\pi_i(k)} \binom{n-i}{k} \binom{k}{k_1 \cdots k_i} a_1^{n-i-k} \prod_{r=2}^{i+1} a_r^{k_{r-1}}$$
$$= \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{k}{k_1 \cdots k_n} \prod_{r=1}^{n} a_r^{k_r}. \tag{6}$$

Each part of the equality (6) is an expression of $a_n^{(1)}$ and we have

$$a_n^{(1)} = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{k}{k_1 \cdots k_n} \prod_{r=1}^{n} a_r^{k_r} \tag{7}$$

or

$$a_n^{(1)} = \sum_{i=1}^{n} \sum_{k=0}^{i} \sum_{(k_1,\cdots,k_i)\in\pi_i(k)} \binom{n-i}{k} \binom{k}{k_1 \cdots k_i} a_1^{n-i-k} \prod_{r=2}^{i+1} a_r^{k_{r-1}}. \tag{8}$$

If we consider $f(t) = \frac{1}{1-t}$, the following corollary holds true.

Corollary 1.

$$\sum_{k=0}^{n} \sum_{i=0}^{k} \sum_{(i_1,\cdots,i_k)\in\pi_k(i)} \binom{n-k}{i} \binom{i}{i_1 \cdots i_k} = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{k}{k_1 \cdots k_n}.$$

In addition the numbers $a_n^{(1)}$ admit a series expansion, for which the coefficients are products of powers of $a_i,\ 0 \le i \le n$. More precisely we have the following theorem.

Theorem 3.

$$a_n^{(1)} = \sum_{j\ge 0} \sum_{n,j} (-1)^{j-m} \binom{j}{m} \binom{m}{k} \binom{k}{k_1 \cdots k_n} a_0^{j-k} \prod_{r=1}^{n} a_r^{k_r}, \tag{9}$$

where

$$\sum_{n,j} = \sum_{m=0}^{j} \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)}.$$

According to identity (9), the following combinatorial identity holds.

Proposition 1. *For any complex number $a_0 \neq 0$ we have*

$$\sum_{j\ge 0} \sum_{m=0}^{j} (-1)^{j-m} \binom{j}{m} \binom{m}{k} a_0^{j-k} = 1. \tag{10}$$

If $a_0 = 0$ the last series reduces to the identity

$$\sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} \binom{m}{k} = 1.$$

If $a_0 = 1$, we will have

$$\sum_{j\ge 0} \sum_{m=0}^{j} (-1)^{j-m} \binom{j}{m} \binom{m}{k} = 1. \tag{11}$$

2.1. **Proof of Main results.** To prove the main results, we need the following lemma.

Lemma 1. *Let $\alpha \in \mathbb{C}\setminus\{0\}$ and $a_0 \neq 0$, then we have*

$$f^{\alpha}(t) = \sum_{n\geq 0}\sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\binom{\alpha}{k}\binom{k}{k_1\cdots k_n}a_0^{\alpha-k}\prod_{r=1}^{n}a_r^{k_r}t^n. \tag{12}$$

We reproduce here the proof given in [10]. If $f(t)$ and $g(t)$ are functions for which all the necessary derivatives are defined; Faà di Bruno (see [8]) provide the following formula for computing the successive derivatives of the composition $g\circ f(t)$.

$$(g\circ f)^{(n)}(t) = \sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\frac{n!}{k_1!\cdots k_n!}\left(g^{(k)}\circ f(t)\right)\prod_{i=1}^{n}\left(\frac{f^{(i)}(t)}{i!}\right)^{k_i}.$$

Let the auxiliary function $g(t) = t^{\alpha}$ then $g\circ f(t) = f^{\alpha}(t) = \sum_{n\geq 0} b_n t^n$ is a generating function and the derivative at order n in zero is

$$\frac{d^n f^{\alpha}(t)}{dt^n}|_{t=0} = n!b_n.$$

But from the Faà di Bruno formula we have

$$(g\circ f)^{(n)}(t) = \sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\frac{n!}{k_1!\cdots k_n!}(\alpha)_k f^{\alpha-k}(t)\prod_{i=1}^{n}\left(\frac{f^{(i)}(t)}{i!}\right)^{k_i}.$$

The evaluation of $g\circ f(t)$ in zero gives

$$(g\circ f)^{(n)}|_{t=0} = \sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\frac{n!}{k_1!\cdots k_n!}(\alpha)_k a_0^{\alpha-k}\prod_{i=1}^{n}a_i^{k_1},\ n\geq 1.$$

Finally $b_0 = (g\circ f)^{(0)}(0) = a_0^{\alpha}$ and

$$b_n = \sum_{k=0}^{n}\frac{1}{k!}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\binom{k}{k_1\cdots k_n}(\alpha)_k a_0^{\alpha-k}\prod_{r=1}^{n}a_r^{k_r},\ n\geq 1.$$

Using the property $\binom{-1}{k} = (-1)^k$ the following corollary is immediate.

Corollary 2. *For $a_0 \neq 0$, we have*

$$\frac{1}{f(t)} = \sum_{n\geq 0}\sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}(-1)^k\binom{k}{k_1\cdots k_n}a_0^{-1-k}\prod_{r=1}^{n}a_r^{k_r}t^n. \tag{13}$$

$a_0 \neq 0$ is a necessary and sufficient condition for the reciprocal to be a generating function, for the proof we refer to the Proposition in [22, §.2,p.31]. So expression (13) improves the result given in [4, Theorem 1] with the use of the determinants. If $\frac{1}{f(t)}$ is written in the form $\frac{1}{f(t)} = \sum_{n\geq 0} b_n t^n$. Then we have $b_n = (-1)^n \frac{D_n(a_r)}{a_0^{n+1}}$, where $D_n(a_r)$ is the determinant given by recurrence [4, Proposition 2.1]:

$$\sum_{i=0}^{n} (-a_0)^i a_i D_{n-i}(a_r) = 0.$$

We then deduce the explicit formula of $D_n(a_r)$:

$$D_n(a_r) = (-1)^n \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{k}{k_1 \cdots k_n} (-a_0)^{n-k} \prod_{r=1}^{n} a_r^{k_r}. \tag{14}$$

2.2. **Proof of Theorem 1.** In one hand we have $f(t) = \sum_{n\geq 0} a_n t^n$ and in another hand

$$f(t) = \left(\sum_{j=0}^{\infty} \frac{h_1 \cdots h_j}{g_1 \cdots g_j} t^j \right)^{-1}.$$

After computation and simplification of $f^{-1}(t)$ with the coefficient

$$a_n = \frac{h_1 \cdots h_n}{g_1 \cdots g_n}$$

as in the identity (13) Corollary 2, we will have the identity (5) Theorem 1.

2.3. **Proof of Theorem 2.** The function

$$\mathcal{C}[f](t) = \sum_{n\geq 0} a_n^{(1)} t^n$$

is written in two different ways. First from the expression

$$\mathcal{C}[f](t) = \sum_{j\geq 0} (f(t) - a_0)^j$$

we have

$$\mathcal{C}[f](t) = \sum_{j\geq 0} \left(\sum_{i\geq 0} a_{i+1} t^i \right)^j t^j.$$

But according to identity (12) Lemma 1 we have

$$\left(\sum_{i\geq 0} a_{i+1} t^i \right)^j = \sum_{i\geq 0} \sum_{k=0}^{i} \sum_{(k_1,\cdots,k_i)\in\pi_i(k)} \binom{j}{k} \binom{k}{k_1 \cdots k_i} a_1^{j-k} \prod_{r=2}^{i+1} a_r^{k_{r-1}} t^i$$

and

$$\mathcal{C}[f](t) = \sum_{j\geq 0} \sum_{i\geq 0} \sum_{k=0}^{i} \sum_{(k_1,\cdots,k_i)\in\pi_i(k)} \binom{j}{k} \binom{k}{k_1 \cdots k_i} a_1^{j-k} \prod_{r=2}^{i+1} a_r^{k_{r-1}} t^{i+j}.$$

Then we replace j by $n=j+i$, to deduce that

$$\mathcal{C}[f](t)=\sum_{n\geq 0}\sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{(k_1,\cdots,k_i)\in\pi_i(k)}\binom{n-i}{k}\binom{k}{k_1\cdots k_i}a_1^{n-i-k}\prod_{r=2}^{i+1}a_r^{k_{r-1}}t^n$$

and

$$a_n^{(1)}=\sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{(k_1,\cdots,k_i)\in\pi_i(k)}\binom{n-i}{k}\binom{k}{k_1\cdots k_i}a_1^{n-i-k}\prod_{r=2}^{i+1}a_r^{k_{r-1}}.$$

In another way we have

$$\mathcal{C}[f](t)=\frac{1}{1-\sum_{n\geq 1}a_nt^n}$$

but by means of identity (12) Lemma 1 we have

$$\left(1-\sum_{n\geq 1}a_nt^n\right)^{-1}=\sum_{n\geq 0}\sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}(-1)^k\binom{-1}{k}\binom{k}{k_1\cdots k_n}\prod_{r=1}^{n}a_r^{k_r}t^n$$

and $(-1)^k\binom{-1}{k}=1$. Then

$$a_n^{(1)}=\sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\binom{k}{k_1\cdots k_n}\prod_{r=1}^{n}a_r^{k_r}$$

and the combinatorial identity (6) follows.

2.4. **Proof of Theorem 3.** The proof of Theorem 3 consists in writing

$$\mathcal{C}[f](t)=\sum_{j\geq 0}(f(t)-a_0)^j=\sum_{j\geq 0}\sum_{m=0}^{j}\binom{j}{m}(-a_0)^{j-m}f^m(t).$$

With the use of identity (12) Lemma 1, we can show that

$$\mathcal{C}[f](t)=\sum_{n\geq 0}\sum_{j\geq 0}\sum_{m=0}^{j}\sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}(-1)^{j-m}\binom{j}{m}\binom{m}{k}\binom{k}{k_1\cdots k_n}a_0^{j-k}\prod_{r=1}^{n}a_r^{k_r}t^n$$

and the coefficient $a_n^{(1)}$ is deduced:

$$a_n^{(1)}=\sum_{j\geq 0}\sum_{m=0}^{j}\sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}(-1)^{j-m}\binom{j}{m}\binom{m}{k}\binom{k}{k_1\cdots k_n}a_0^{j-k}\prod_{r=1}^{n}a_r^{k_r}.$$

Finally the identity (10) Proposition 1 is derived from the identities (7) and (9).

2.5. **Numbers associated to $\mathcal{C}^m(f)$.** In the general case, what we hope is a few recurrence formulae satisfied by the numbers $a_n^{(m)}$ $m \geq 2$. First, we can derive from relation (9) Theorem 3 the following identity.

$$a_n^{(m)}(x) = \sum_{j\geq 0} \sum_{n,j} (-1)^{j-m} \binom{j}{m} \binom{m}{k} \prod_{r=1}^{n} \left(a_r^{(m-1)}\right)^{k_r}.$$

But the most elegant is given by the following theorem.

Theorem 4. *For $m \geq 2$ we have*

$$a_n^{(m)} = \frac{1}{2} \sum_{j=1}^{n} \binom{n}{j} a_{n-j}^{(m)} a_j^{(m-1)}. \tag{15}$$

The proof is to use the relation

$$\mathcal{C}^m[f](t) \left(2 - \mathcal{C}^{m-1}[f](t)\right) = 1.$$

So we will have

$$2\mathcal{C}^m[f](t) - \mathcal{C}^m[f](t)\mathcal{C}^{m-1}[f](t) = 1.$$

Returning to the generating functions we conclude that $a_0^{(m)} = 1$ and

$$2a_n^{(m)} - \sum_{j=0}^{n} \binom{n}{j} a_{n-j}^{(m)} a_j^{(m-1)} = 0.$$

Thus the desired result follows.

3. Generalization and Application

We can extend the operator $\mathcal{C}$ to the sequence $f^n(t) = (f(t) - f(0))^{\omega n}$ and we consider $f(y,t) = \sum_{n\geq n} y^{\omega n} (f(t) - f(0))^{\omega n}$ where ω is a positive integer. This series is convergent for $|y| \leq 1$ because we also have $|\,(f(t) - f(0))^{\omega}\,| < 1$ in the interval I. The operator $\mathcal{C}_{y,\omega}$ is defined by

$$\mathcal{C}_{y,\omega}[f](t) = \frac{1}{1 - y^{\omega} (f(t) - f(0))^{\omega}}.$$

We have $\mathcal{C}_{1,1} = \mathcal{C}$, the composition of $\mathcal{C}_{y,\omega}$ with itself m times gives the operator $\mathcal{C}_{y,\omega}^{(m)}$ obtained recursively by the formula

$$\mathcal{C}_{y,\omega}^{(m)}[f](t) = \frac{1}{1 - y^{\omega}\left(\mathcal{C}_{y,\omega}^{(m-1)}[f](t) - f(0)\right)^{\omega}}, \quad m \geq 1. \tag{16}$$

For $f = \sum_{n\geq 0} a_n t^n$ be a generating function and $a^{(m,\omega)}(y)$ the corresponding polynomials associated to the generating function $\mathcal{C}_{y,\omega}^{m}[f](t)$. It is obvious to remark that

$$\mathcal{C}_{y,\omega}[f](t) = \left(1 - \left(\sum_{n\geq 1} y a_n t^n\right)^{\omega}\right)^{-1}.$$

According to the identity (12) Lemma 1 we will have

$$\left(\sum_{n\geq 1} ya_nt^n\right)^{\omega} = \sum_{n\geq\omega}\sum_{(\omega_1,\cdots,\omega_n)\in\pi_n(\omega)}\binom{\omega}{\omega_1\cdots\omega_n}a_1^{\omega_1}\cdots a_n^{\omega_n}y^{\omega}t^n.$$

For the sake of simplifying the calculations, we write

$$1-\left(\sum_{n\geq 1} ya_nt^n\right)^{\omega} = \sum_{n\geq 0} a^*_{n,\omega}(y)t^n$$

with $a^*_{0,\omega}(y)=1$, $a^*_{n,\omega}(y)=0; 1\leq n\leq\omega-1$ and

$$a^*_{n,\omega}(y) = -\sum_{(\omega_1,\cdots,\omega_n)\in\pi_n(\omega)}\binom{\omega}{\omega_1\cdots\omega_n}a_1^{\omega_1}\cdots a_n^{\omega_n}y^{\omega}, \quad n\geq\omega.$$

Using the result (13) Corollary 2 we conclude that

$$\mathcal{C}_{y,\omega}[f](t) = \sum_{n\geq 0}\sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\binom{k}{k_1\cdots k_n}{a^*}^{k_1}_{1,\omega}(y)\cdots {a^*}^{k_n}_{n,\omega}(y)t^n$$

and

$$a_n^{(1,\omega)}(y) = \sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\binom{k}{k_1\cdots k_n}{a^*}^{k_1}_{1,\omega}(y)\cdots {a^*}^{k_n}_{n,\omega}(y).$$

The identity

$$\mathcal{C}_{y,\omega}[f](t)\left(1-y\left(\mathcal{C}^{(m-1)}_{y,\omega}[f](t)-1\right)\right)=1, \quad m\geq 2$$

implies that

$$(1+y)\sum_{n\geq 0}a_n^{(m,\omega)}(y)t^n - y\left(\sum_{n\geq 0}a_n^{(m,\omega)}(y)t^n\right)\left(\sum_{n\geq 0}a_n^{(m-1,\omega)}(y)t^n\right)=1.$$

Then we have $a_0^{(m,\omega)}(y)=1$ and

$$(1+y)\,a_n^{(m,\omega)}(y) = y\sum_{k=0}^{n}a_k^{(m,\omega)}(y)a_{n-k}^{(m-1,\omega)}(y).$$

Finally

$$a_n^{(m,\omega)}(y) = y\sum_{k=0}^{n-1}a_k^{(m,\omega)}(y)a_{n-k}^{(m-1)}(y).$$

So we have already proved the following theorem.

Theorem 5.

$$a_n^{(1,\omega)}(y) = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{k}{k_1\cdots k_n} {a^*}_{1,\omega}^{k_1}(y)\cdots {a^*}_{n,\omega}^{k_n}(y) \tag{17}$$

and

$$(1+y)\,a_n^{(m,\omega)}(y) = y\sum_{k=0}^{n-1} a_k^{(m,\omega)}(y)a_{n-k}^{(m-1)}(y), \quad m\geq 2. \tag{18}$$

Now for $m=\omega=1$, we note $a_n^{(1,1)}(y) = a_n^{(1)}(y)$ and ${a^*}_{n,1}(y) = -a_n y$. We deduce the following corollary.

Corollary 3. *For $n>0$ we have*

$$a_n^{(1)}(y) = \sum_{k=0}^{n} \sum_{(k_1,\cdots,k_n)\in\pi_n(k)} \binom{k}{k_1\cdots k_n} \prod_{r=1}^{n} a_r^{k_r}(-y)^k. \tag{19}$$

Substitute $y=1$ in the identity (17) Theorem 5, we get identity (7) in another way.

3.1. **Application to Fubini numbers and two variable Fubini polynomials.** The application of this operator on the exponential function, allows us to calculate the explicit formulas of the Fubini numbers and the two variables Fubini polynomials. The two variable Fubini or geometric polynomials (see [13]) are usually defined by means of the generating function

$$\frac{e^{xt}}{1-y\left(e^t-1\right)} = \sum_{n\geq 0} F_n(x,y)\frac{t^n}{n!}. \tag{20}$$

The case $x=0$ corresponds to Fubini polynomials $F_n(y)=F_n(0,y)$; for which the generating function is

$$\frac{1}{1-y\left(e^t-1\right)} = \sum_{n\geq 0} F_n(y)\frac{t^n}{n!}. \tag{21}$$

The case $(x,y)=(0,1)$ corresponds to the ordered Bell numbers; given by

$$\frac{1}{2-e^t} = \sum_{n\geq 0} F_n\frac{t^n}{n!}. \tag{22}$$

The function e^t respects the condition $|e^t-1|<1$ on a chosen nonempty interval I centered in 0. Thus the application of the operator $\mathcal{C}_{y,\omega}$ on the function e^t makes it possible to deduce that $e^{xt}\mathcal{C}_{y,1}[e](t)$ generates polynomials $F_n(x,y)$, $\mathcal{C}_{y,1}[e](t)$ generates polynomials $F_n(y)$ and $\mathcal{C}[e](t)$ generates numbers F_n.

In 1939 Sheffer (see [19]) initiated study of a class of polynomials which are known as Sheffer sequences. These sequences have been characterized in a variety of ways. We choose here to take the Sheffer sequences investigated in article [14]; a sequence $S_n(x)$ is called the Sheffer sequence for the Sheffer pair $(g(t), f(t))$, which is denoted by $S_n(g(t), f(t)) \sim (g(t), f(t))$ if and only if

$$\frac{e^{x\bar{f}(t)}}{g\left(\bar{f}(t)\right)} = \sum_{n\geq 0} S_n(x)\frac{t^n}{n!}, \tag{23}$$

where f, g two generating functions and $\bar{f}$ is the compositional inverse of f satisfying $f\left(\bar{f}(t)\right) = \bar{f}\left(f(t)\right) = t$. $S_n(x)$ satisfies the Sheffer identity (see [18]):

$$S_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} S_k(x)P_{n-k}(y),$$

where $P_n(x) = g(t)S_n(x) \sim (1, f(t))$. According to identity (23) we have for example

$$F_n(x, y) \sim \left(1 - y\left(e^t - 1\right), t\right).$$

The ω-torsion Fubini polynomial is the Sheffer sequence

$$F_{n,\omega}(x, y) \sim \left(1 - y^{\omega}\left(e^t - 1\right)^{\omega}, t\right)$$

and we have

$$\frac{e^{xt}}{1 - y^{\omega}\left(e^t - 1\right)^{\omega}} = \sum_{n\geq 0} F_{n,\omega}(x, y)\frac{t^n}{n!}. \tag{24}$$

In addition, more general Fubini polynomials than $F_{n,\omega}(x, y)$ have been studied in the literature, like r-Fubini polynomials $F_{n,r}(x)$, r-Whitney-Fubini polynomials $\mathcal{F}_{m,r}(n, x)$ and Eulerian-Fubini polynomials $\mathcal{A}_{m,r}(n, x)$. We recall respectively their generating functions; for $F_{n,r}(x)$ (see [17, Theorem 1, p.73]) we have

$$\frac{r!e^{rt}}{\left(1 - x\left(e^t - 1\right)\right)^{r+1}} = \sum_{n\geq 0} F_{n,r}(x)\frac{t^n}{n!}.$$

But for $\mathcal{F}_{m,r}(n, x)$ (see [7, Theorem 10 Identity 14]) we have

$$\frac{r!e^{rt}}{\left(1 - x\left(e^{mt} - 1\right)\right)^{r+1}} = \sum_{n\geq 0} \mathcal{F}_{m,r}(n, x)\frac{t^n}{n!}.$$

Finally for $\mathcal{A}_{m,r}(n, x)$ (see [7, Theorem 19 Identity 18])we have

$$\frac{r!(x-1)^{r+1}e^{r(x-1)t}}{\left(x - e^{m(x-1)t}\right)^{r+1}} = \sum_{n\geq 0} \mathcal{A}_{m,r}(n, x)\frac{t^n}{n!}.$$

These polynomials are related each other by the following connections:

$$\mathcal{F}_{m,r}(n, x) = x^n\mathcal{A}_{m,r}\left(n, \frac{x+1}{x}\right),$$

$$\mathcal{F}_{1,r}(n, x) = F_{n,r}(x)$$

and then

$$F_{n,r}(x) = x^n \mathcal{A}_{1,r}\left(n, \frac{x+1}{x}\right).$$

Of course, Fubini polynomials have been studied as ordered Bell polynomials too. Different of Bell polynomials $Bel_n(x)$ defined by means of the generating function $e^{x\left(e^t-1\right)}$ and Bell numbers $Bel_n = Bel_n(1)$ given by the generating function e^{e^t-1}. First we consider the Fubini polynomials $F_n(x)$ (see [6]). Many authors have been very interested in arithmetic properties of these polynomials. S.M. Tannay (see [21]) provided that the polynomials $F_n(y)$ admit the following configuration

$$F_n(y) = \sum_{k=0}^{n} S(n,k)\, k! y^k, \tag{25}$$

where $S(n,k)$ are the Stirling numbers of the second kind (see [6, Definition A §5.1]). From the generating function (21), he derived a remarkable representation of $F_n(x)$ for $x \neq -1$ as an infinite series:

$$F_n(y) = \frac{1}{1+y}\sum_{n\geq 0}\left(\frac{y}{1+y}\right)^k k^n, \tag{26}$$

and obtained the known identity

$$F_n = \frac{1}{2}\sum_{n\geq 0}\frac{k^n}{2^k}. \tag{27}$$

The next corollary states an improvement of the formula (25).

Corollary 4.

$$F_n(y) = n!\sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\binom{k}{k_1\cdots k_n}\left(\frac{1}{1!}\right)^{k_1}\cdots\left(\frac{1}{n!}\right)^{k_n} y^k. \tag{28}$$

After comparison between the two forms of $F_n(y)$ we conclude that

$$S(n,k) = (n!/k!)\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\binom{k}{k_1\cdots k_n}\left(\frac{1}{1!}\right)^{k_1}\cdots\left(\frac{1}{n!}\right)^{k_n}.$$

Also we have

$$\frac{1}{1-y^{\omega}\left(e^t-1\right)^{\omega}} = \sum_{n\geq 0} F_{n,\omega}(y)\frac{t^n}{n!}.$$

that is to say that

$$\mathcal{C}_{y,\omega}[f](t) = \sum_{n\geq 0} F_{n,\omega}(y)\frac{t^n}{n!},$$

and then

Corollary 5.

$$F_{n,\omega}(y)=n!\sum_{k=0}^{n}\sum_{(k_1,\cdots,k_n)\in\pi_n(k)}\binom{k}{k_1\cdots k_n}{a^*}_{1,\omega}^{k_1}(y)\cdots{a^*}_{n,\omega}^{k_n}(y), \tag{29}$$

with

$$a^*_{n,\omega}(y)=-\sum_{(\omega_1,\cdots,\omega_n)\in\pi_n(\omega)}\binom{\omega}{\omega_1\cdots\omega_n}\left(\frac{1}{1!}\right)^{\omega_1}\cdots\left(\frac{1}{n!}\right)^{\omega_n}y^{\omega},\quad n\geq\omega.$$

We end this work, by establishing the explicit formula of the ω-torsion Fubini polynomials. Using the Cauchy product (see [11]) of generating functions we will have

$$\frac{e^{xt}}{1-y^{\omega}\left(e^t-1\right)^{\omega}}=\left(\sum_{n\geq0}x^n\frac{t^n}{n!}\right)\left(\sum_{n\geq0}F_{n,\omega}(y)\frac{t^n}{n!}\right)$$

and

$$\sum_{n\geq0}F_{n,\omega}(x,y)\frac{t^n}{n!}=\sum_{n\geq0}\left(\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}F_{k,\omega}(y)x^{n-k}\right)t^n.$$

Furthermore

$$F_{n,\omega}(x,y)=\sum_{k=0}^{n}\binom{n}{k}F_{k,\omega}(y)x^{n-k}.$$

So the following identity is true

Corollary 6.

$$F_{n,\omega}(x,y)=\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{(j_1,\cdots,j_k)\in\pi_k(j)}\frac{n!}{(n-k)!}\binom{j}{j_1\cdots j_k}\prod_{r=1}^{k}{a^*}_{r,\omega}^{j_r}(y)x^{n-k}.$$

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