New inequalities for F -convex functions pertaining generalized fractional integrals

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ABSTRACT. In this paper, the authors, utilizing F -convex functions which are defined by B. Samet, establish some new Hermite-Hadamard type inequalities via generalized fractional integrals. Some special cases of our main results recaptured the well-known earlier works.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite-Hadamard inequality, holds [17]:

$$(1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

Both inequalities in (1) hold in the reversed direction if f is concave.

Over the last decade, this classical double inequality has been improved and generalized in a number of ways, see [5, 7, 8, 13, 18], [23]–[25] and the references therein. Also, many types of convexities have been defined, such as quasi-convex in [6], pseudo-convex in [14], strongly convex in [20], ε -convex in [11], s -convex in [10], h -convex in [28], etc. Recently, Samet in [21], has defined a new concept of convexity that depends on a certain function satisfying some axioms, that generalizes different types of convexity.

Recall the family $\mathcal{F}$ of mappings $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ satisfying the following axioms:

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(A1) If $e_i \in L^1(0, 1)$, $i = 1, 2, 3$, then for every $\lambda \in [0, 1]$, we have

$$\int_0^1 F(e_1(t), e_2(t), e_3(t), \lambda) dt = F\left(\int_0^1 e_1(t) dt, \int_0^1 e_2(t) dt, \int_0^1 e_3(t) dt, \lambda\right);$$

(A2) For every $u \in L^1(0, 1)$, $w \in L^\infty(0, 1)$ and $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2, t) dt = T_{F,w}\left(\int_0^1 w(t)u(t) dt, z_1, z_2\right),$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a function that depends on (F, w) , and it is nondecreasing with respect to the first variable;

(A3) For any $(w, e_1, e_2, e_3) \in \mathbb{R}^4$, $e_4 \in [0, 1]$, we have

$$wF(e_1, e_2, e_3, e_4) = F(we_1, we_2, we_3, e_4) + L_w,$$

where $L_w \in \mathbb{R}$ is a constant that depends only on w .

Definition 1. Let $f : [a, b] \rightarrow \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be a given function. We say that f is a convex function with respect to some $F \in \mathcal{F}$ (or F -convex function), if and only if:

$$F(f(tx + (1-t)y), f(x), f(y), t) \leq 0, \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].$$

Remark 1. 1) Let $\varepsilon \geq 0$, and let $f : [a, b] \rightarrow \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an ε -convex function, see [11], that is

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon, \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ by

$$(2) \quad F(e_1, e_2, e_3, e_4) = e_1 - e_4 e_2 - (1 - e_4) e_3 - \varepsilon$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$(3) \quad T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 tw(t) dt\right) e_2 - \left(\int_0^1 (1-t)w(t) dt\right) e_3 - \varepsilon.$$

For

$$(4) \quad L_w = (1-w)\varepsilon,$$

it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y) - \varepsilon \leq 0,$$

that is f is an F -convex function. Particularly, taking $\varepsilon = 0$, we show that if f is a convex function then f is an F -convex function with respect to F defined above.

2) Let $h : J \rightarrow [0, +\infty)$ be a given function which is not identical to 0, where J is an interval in $\mathbb{R}$ such that $(0, 1) \subseteq J$. Let $f : [a, b] \rightarrow [0, +\infty)$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an h -convex function, see [28], that is

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y), \quad (x, y, t) \in [a, b] \times [a, b] \times [0, 1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ by

$$(5) \quad F(e_1, e_2, e_3, e_4) = e_1 - h(e_4)e_2 - h(1-e_4)e_3$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$(6) \quad T_{F,w}(e_1, e_2, e_3) = e_1 - \left(\int_0^1 h(t)w(t)dt \right) e_2 - \left(\int_0^1 h(1-t)w(t)dt \right) e_3.$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx+(1-t)y), f(x), f(y), t) = f(tx+(1-t)y) - h(t)f(x) - h(1-t)f(y) \leq 0,$$

that is, f is an F -convex function.

Samet in [21], established the following Hermite–Hadamard type inequalities using the new convexity concept:

Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an F -convex function, for some $F \in \mathcal{F}$. Suppose that $f \in L^1[a, b]$. Then

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{b-a} \int_a^b f(x)dx, \frac{1}{b-a} \int_a^b f(x)dx, \frac{1}{2}\right) \leq 0,$$

$$T_{F,1}\left(\frac{1}{b-a} \int_a^b f(x)dx, f(a), f(b)\right) \leq 0.$$

Definition 2. Let $f \in L^1[a, b]$. The Riemann–Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and

$$J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x).$$

Definition 3. Let $f \in L^1[a, b]$. Then k -fractional integrals of order $\alpha, k > 0$ are defined by

$$I_{a+,k}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a,$$

and

$$(7) \quad I_{b-}^{\alpha, k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t) dt, \quad b > x,$$

where $\Gamma_k(\cdot)$ stands for the k -gamma function. For $k = 1$, the k -fractional integrals yield Riemann–Liouville integrals. For $\alpha = k = 1$, the k -fractional integrals yield classical integrals. For more details, see [9, 12, 15, 19].

It is remarkable that Sarikaya et al. in [26], first give the following interesting integral inequalities of Hermite–Hadamard type involving Riemann–Liouville fractional integrals.

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L^1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$(8) \quad f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2},$$

with $\alpha > 0$.

Budak et al. in [1], prove the following Hermite–Hadamard type inequalities for F -convex functions via fractional integrals:

Theorem 3. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a mapping on I° , $a, b \in I^\circ$, $a < b$. If f is F -convex on $[a, b]$ for some $F \in \mathcal{F}$, then we have*

$$(9) \quad F\left(f\left(\frac{a+b}{2}\right), \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{a+}^\alpha f(b), \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{b-}^\alpha f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0,$$

and

$$(10) \quad T_{F,w}\left(\frac{\Gamma(\alpha+1)}{(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)], f(a) + f(b), f(a) + f(b)\right) + \int_0^1 L_{w(t)} dt \leq 0,$$

where $w(t) = \alpha t^{\alpha-1}$.

For other papers involving F -convex functions, see [1]–[4], [16, 27].

Now we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [22].

Let's define a function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the following conditions:

$$(11) \quad \int_0^1 \frac{\varphi(t)}{t} dt < +\infty,$$

$$(12) \quad \frac{1}{A_1} \leq \frac{\varphi(v)}{\phi(u)} \leq A_1 \quad \text{for} \quad \frac{1}{2} \leq \frac{v}{u} \leq 2,$$

$$(13) \quad \frac{\varphi(u)}{u^2} \leq A_2 \frac{\varphi(v)}{v^2} \quad \text{for} \quad v \leq u,$$

$$(14) \quad \left| \frac{\varphi(u)}{u^2} - \frac{\varphi(v)}{v^2} \right| \leq A_3 |u - v| \frac{\varphi(u)}{u^2} \quad \text{for} \quad \frac{1}{2} \leq \frac{v}{u} \leq 2,$$

where $A_1, A_2, A_3 > 0$ are independent of $u, v > 0$. If $\varphi(u)u^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(u)}{u^\beta}$ is decreasing for some $\beta \geq 0$, then φ satisfies the above conditions.

The following left-sided and right-sided generalized fractional integral operators are defined respectively, as follows:

$$(15) \quad {}_{a+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t) dt, \quad x > a,$$

$$(16) \quad {}_{b-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t) dt, \quad x < b.$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integral, k –Riemann–Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc.

Sarikaya and Ertuğral in [22], establish the following Hermite–Hadamard inequality and lemmas for the generalized fractional integral operators:

Theorem 4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$ with $a < b$, then the following inequalities for fractional integral operators hold:*

$$(17) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Psi(1)} [{}_{a+}I_\varphi f(b) + {}_{b-}I_\varphi f(a)] \leq \frac{f(a) + f(b)}{2},$$

where the mapping $\Lambda : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$\Psi(x) = \int_0^x \frac{\varphi((b-a)t)}{t} dt.$$

Budak et al. prove the following Hermite Hadamard type inequalities for F -convex functions.

Theorem 5 ([4]). *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a mapping on I° , $a, b \in I^\circ$, $a < b$. If f is F -convex on $[a, b]$ for some $F \in \mathcal{F}$, then we have*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Psi(1)} {}_{a+}I_\varphi f(b), \frac{1}{\Psi(1)} {}_{b-}I_\varphi f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0,$$

and

$$T_{F,w} \left(\frac{1}{\Psi(1)} [{}_a I_{\varphi} f(b) + {}_b I_{\varphi} f(a)], f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{w(t)} dt \leq 0,$$

where $w(t) = \frac{\varphi((b-a)t)}{t\Psi(1)}$.

Motivated by the above literatures, the main objective of this article is to establish some new Hermite–Hadamard type inequalities via generalized fractional integrals utilizing F -convex functions. Some special cases of our main results recaptured the well-known earlier works. At the end, a briefly conclusion will be given as well.

2. MAIN RESULTS

In this section, we establish some inequalities of Hermite–Hadamard type including generalized fractional integrals via F -convex functions.

Theorem 6. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^{\circ} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a mapping on I° , $a, b \in I^{\circ}$, $a < b$ and let F be linear with respect to the first three variables. If f is F -convex on $[a, b]$ for some $F \in \mathcal{F}$, then we have*

$$(18) \quad F \left(f \left(\frac{a+b}{2} \right), \frac{1}{\Lambda(1)} ({}_{\frac{a+b}{2}} I_{\varphi} f(b), \frac{1}{\Lambda(1)} ({}_{\frac{a+b}{2}} I_{\varphi} f(a), \frac{1}{2} \right) + \int_0^1 L_{w(t)} dt \leq 0,$$

and

$$(19) \quad T_{F,w} \left(\frac{1}{\Lambda(1)} \left[({}_{\frac{a+b}{2}} I_{\varphi} f(b) + ({}_{\frac{a+b}{2}} I_{\varphi} f(a) \right], f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{w(t)} dt \leq 0,$$

where $w(t) = \frac{\varphi((\frac{b-a}{2})t)}{t\Lambda(1)}$ and the function $\Lambda : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$\Lambda(x) = \int_0^x \frac{\varphi(\frac{b-a}{2}t)}{t} dt.$$

Kanıt. Since f is F -convex, we have

$$F \left(f \left(\frac{x+y}{2} \right), f(x), f(y), \frac{1}{2} \right) \leq 0, \quad \forall x, y \in [a, b].$$

For

$$x = \frac{t}{2}a + \left(\frac{2-t}{2}\right)b \text{ and } y = \left(\frac{2-t}{2}\right)a + \frac{t}{2}b,$$

we have

$$F\left(f\left(\frac{a+b}{2}\right), f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right), f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right), \frac{1}{2}\right) \leq 0.$$

for all $t \in [0, 1]$. Multiplying this inequality by $w(t) = \frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}$ and using axiom (A3), we get

$$F\left(\frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\frac{a+b}{2}\right), \frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right), \frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t\Lambda(1)}f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right), \frac{1}{2}\right) + L_{w(t)} \leq 0$$

for all $t \in (0, 1)$. Integrating over $(0, 1)$ with respect to the variable t and using axiom (A1), we obtain

$$F\left(\frac{f\left(\frac{a+b}{2}\right)}{\Lambda(1)} \int_0^1 \frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t} dt, \frac{1}{\Lambda(1)} \int_0^1 \frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t} f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) dt, \frac{1}{\Lambda(1)} \int_0^1 \frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t} f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right) dt, \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0.$$

Using the facts that

$$\begin{aligned} & \int_0^1 \frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t} f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) dt \\ &= \int_{\frac{a+b}{2}}^b \frac{\varphi(b-x)}{b-x} f(x) dx = {}_{\left(\frac{a+b}{2}\right)^+} I_{\varphi} f(b) \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \frac{\varphi\left(\left(\frac{b-a}{2}\right)t\right)}{t} f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right) dt \\ &= \int_a^{\frac{a+b}{2}} \frac{\varphi(x-a)}{x-a} f(x) dx = {}_{\left(\frac{a+b}{2}\right)^-} I_{\varphi} f(a), \end{aligned}$$

we obtain

$$\begin{aligned} & F\left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)} {}_{\left(\frac{a+b}{2}\right)^+} I_{\varphi} f(b), \frac{1}{\Lambda(1)} {}_{\left(\frac{a+b}{2}\right)^-} I_{\varphi} f(a), \frac{1}{2}\right) \\ &+ \int_0^1 L_{w(t)} dt \leq 0, \end{aligned}$$

which gives (18).

On the other hand, since f is F -convex, we have

$$F\left(f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right), f(a), f(b), t\right) \leq 0, \quad \forall t \in [0, 1],$$

and

$$F\left(f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right), f(a), f(b), t\right) \leq 0, \quad \forall t \in [0, 1].$$

Using the linearity of F , we get

$$F\left(f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) + f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right), f(a) + f(b), f(a) + f(b), t\right) \leq 0,$$

for all $t \in [0, 1]$. Applying the axiom (A3) for $w(t) = \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}$, we obtain

$$F\left(\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) + f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right)\right], \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}[f(a) + f(b)], \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}[f(a) + f(b)], t\right) + L_{w(t)} \leq 0,$$

for all $t \in (0, 1)$. Integrating over $(0, 1)$ and using axiom (A2), we have

$$T_{F,w}\left(\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}\left[f\left(\frac{t}{2}a + \left(\frac{2-t}{2}\right)b\right) + f\left(\left(\frac{2-t}{2}\right)a + \frac{t}{2}b\right)\right] dt, f(a) + f(b), f(a) + f(b)\right) + \int_0^1 L_{w(t)} dt \leq 0,$$

that is

$$T_{F,w}\left(\frac{1}{\Lambda(1)}\left[\left(\frac{a+b}{2}\right)^+ I_{\varphi} f(b) + \left(\frac{a+b}{2}\right)^- I_{\varphi} f(a)\right], f(a) + f(b), f(a) + f(b)\right) + \int_0^1 L_{w(t)} dt \leq 0.$$

The proof of Theorem 6 is completed. $\square$

Remark 2. If we choose $\varphi(t) = t$ in Theorem 6, then we have the following inequalities

$$(20) \quad F \left(f \left(\frac{a+b}{2} \right), \frac{2}{b-a} \int_{\frac{a+b}{2}}^b f(t) dt, \frac{2}{b-a} \int_a^{\frac{a+b}{2}} f(t) dt, \frac{1}{2} \right) + \int_0^1 L_{w(t)} dt \leq 0,$$

and

$$(21) \quad T_{F,w} \left(\frac{2}{b-a} \int_a^b f(t) dt, f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{w(t)} dt \leq 0,$$

where $w(t) = 1$.

Remark 3. If we choose $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 6, then we have the following inequalities for Riemann-Liouville fractional integrals

$$F \left(f \left(\frac{a+b}{2} \right), \frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b), \frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a), \frac{1}{2} \right) + \int_0^1 L_{w(t)} dt \leq 0,$$

and

$$T_{F,w} \left(\frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right], f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{w(t)} dt \leq 0,$$

where $w(t) = \alpha t^{\alpha-1}$ which is given by Budak et al. in [5].

Corollary 1. If we take $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 6, then we have the following inequalities for k -Riemann-Liouville fractional integrals

$$F \left(f \left(\frac{a+b}{2} \right), \frac{2^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} I_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b), \frac{2^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} I_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a), \frac{1}{2} \right) + \int_0^1 L_{w(t)} dt \leq 0,$$

and

$$T_{F,w} \left(\frac{2^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} \left[I_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) + I_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right], f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{w(t)} dt \leq 0,$$

$$f(a) + f(b), f(a) + f(b) \Big) + \int_0^1 L_{w(t)} dt \leq 0,$$

where $w(t) = \frac{\alpha}{k} t^{\frac{\alpha}{k}-1}$.

Theorem 7. Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a mapping on I° , $a, b \in I^\circ$, $a < b$ and let F be linear with respect to the first three variables. If f is F -convex on $[a, b]$ for some $F \in \mathcal{F}$, then we have

$$(22) \quad \begin{aligned} & F \left(f \left(\frac{a+b}{2} \right), \frac{1}{\Lambda(1)} {}_{b-I_\varphi} f \left(\frac{a+b}{2} \right), \right. \\ & \left. \frac{1}{\Lambda(1)} {}_{a+I_\varphi} f \left(\frac{a+b}{2} \right), \frac{1}{2} \right) + \int_0^1 L_{w(t)} dt \leq 0, \end{aligned}$$

and

$$(23) \quad \begin{aligned} & T_{F,w} \left(\frac{1}{\Lambda(1)} \left[{}_{a+I_\varphi} f \left(\frac{a+b}{2} \right) + {}_{b-I_\varphi} f \left(\frac{a+b}{2} \right) \right], \right. \\ & \left. f(a) + f(b), f(a) + f(b) \right) + \int_0^1 L_{w(t)} dt \leq 0, \end{aligned}$$

where $w(t) = \frac{\varphi((\frac{b-a}{2})t)}{t\Lambda(1)}$.

Kanıt. Since f is F -convex, we have

$$F \left(f \left(\frac{x+y}{2} \right), f(x), f(y), \frac{1}{2} \right) \leq 0, \quad \forall x, y \in [a, b].$$

For

$$x = \left(\frac{1-t}{2} \right) a + \left(\frac{1+t}{2} \right) b \quad \text{and} \quad y = \left(\frac{1+t}{2} \right) a + \left(\frac{1-t}{2} \right) b,$$

we have

$$\begin{aligned} & F \left(f \left(\frac{a+b}{2} \right), f \left(\left(\frac{1-t}{2} \right) a + \left(\frac{1+t}{2} \right) b \right), \right. \\ & \left. f \left(\left(\frac{1+t}{2} \right) a + \left(\frac{1-t}{2} \right) b \right), \frac{1}{2} \right) \leq 0, \end{aligned}$$

for all $t \in [0, 1]$. Multiplying this inequality by $w(t) = \frac{\varphi((\frac{b-a}{2})t)}{t\Lambda(1)}$ and using axiom (A3), we get

$$\begin{aligned} & F \left(\frac{\varphi((\frac{b-a}{2})t)}{t\Lambda(1)} f \left(\frac{a+b}{2} \right), \frac{\varphi((\frac{b-a}{2})t)}{t\Lambda(1)} f \left(\left(\frac{1-t}{2} \right) a + \left(\frac{1+t}{2} \right) b \right), \right. \\ & \left. \frac{\varphi((\frac{b-a}{2})t)}{t\Lambda(1)} f \left(\left(\frac{1+t}{2} \right) a + \left(\frac{1-t}{2} \right) b \right), \frac{1}{2} \right) + L_{w(t)} \leq 0, \end{aligned}$$

for all $t \in (0, 1)$. Integrating over $(0, 1)$ with respect to the variable t and using axiom (A1), we obtain

$$\begin{aligned}
 & F \left(\frac{f\left(\frac{a+b}{2}\right)}{\Lambda(1)} \int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t} dt, \right. \\
 & \quad \frac{1}{\Lambda(1)} \int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t} f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) dt, \\
 & \quad \left. \frac{1}{\Lambda(1)} \int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t} f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right) dt, \frac{1}{2} \right) \\
 & + \int_0^1 L_{w(t)} dt \leq 0.
 \end{aligned}$$

Using the facts that

$$\begin{aligned}
 & \int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t} f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) dt \\
 & = \int_{\frac{a+b}{2}}^b \frac{\varphi\left(x - \frac{a+b}{2}\right)}{x - \frac{a+b}{2}} f(x) dx \\
 & = {}_{b-}I_{\varphi} f\left(\frac{a+b}{2}\right),
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t} f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right) dt \\
 & = \int_a^{\frac{a+b}{2}} \frac{\varphi\left(\frac{a+b}{2} - x\right)}{\frac{a+b}{2} - x} f(x) dx \\
 & = {}_{a+}I_{\varphi} f\left(\frac{a+b}{2}\right),
 \end{aligned}$$

we obtain

$$\begin{aligned}
 & F \left(f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)} {}_{b-}I_{\varphi} f\left(\frac{a+b}{2}\right), \frac{1}{\Lambda(1)} {}_{a+}I_{\varphi} f\left(\frac{a+b}{2}\right), \frac{1}{2} \right) \\
 & + \int_0^1 L_{w(t)} dt \leq 0,
 \end{aligned}$$

which gives (22).

On the other hand, since f is F -convex, we have

$$F \left(f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right), f(a), f(b), t \right) \leq 0, \quad \forall t \in [0, 1],$$

and

$$F\left(f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right), f(a), f(b), t\right) \leq 0, \quad \forall t \in [0, 1].$$

Using the linearity of F , we get

$$F\left(f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right) + f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right), f(a) + f(b), f(a) + f(b), t\right) \leq 0, \quad \forall t \in [0, 1].$$

Applying the axiom (A3) for $w(t) = \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)}$, we obtain

$$\begin{aligned} & F\left(\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)} \times \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right) + \right. \\ & \quad \left. f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right], \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)} [f(a) + f(b)], \\ & \quad \left.\frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)} [f(a) + f(b)], t\right) + L_{w(t)} \leq 0, \end{aligned}$$

for all $t \in (0, 1)$. Integrating over $(0, 1)$ and using axiom (A2), we have

$$\begin{aligned} & T_{F,w}\left(\int_0^1 \frac{\varphi\left(\frac{(b-a)}{2}t\right)}{t\Lambda(1)} \times \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right) + \right. \\ & \quad \left. f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right] dt, \\ & \quad \left. f(a) + f(b), f(a) + f(b)\right) + \int_0^1 L_{w(t)} dt \leq 0, \end{aligned}$$

that is

$$\begin{aligned} & T_{F,w}\left(\frac{1}{\Lambda(1)} \left[{}_{a+}I_{\varphi}f\left(\frac{a+b}{2}\right) + {}_{b-}I_{\varphi}f\left(\frac{a+b}{2}\right) \right], \right. \\ & \quad \left. f(a) + f(b), f(a) + f(b)\right) + \int_0^1 L_{w(t)} dt \leq 0. \end{aligned}$$

The proof of Theorem 7 is completed. $\square$

Remark 4. If we take $\varphi(t) = t$ in Theorem 7, then the inequalities (22) and (23) reduce to the inequalities (20) and (21)

Remark 5. If we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 7, then we have the following inequalities for Riemann-Liouville fractional integrals

$$F\left(f\left(\frac{a+b}{2}\right), \frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} J_{b^-}^\alpha f\left(\frac{a+b}{2}\right), \frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} J_{a^+}^\alpha f\left(\frac{a+b}{2}\right), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0,$$

and

$$T_{F,w}\left(\frac{2^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{b^-}^\alpha f\left(\frac{a+b}{2}\right) + J_{a^+}^\alpha f\left(\frac{a+b}{2}\right) \right], f(a) + f(b), f(a) + f(b)\right) + \int_0^1 L_{w(t)} dt \leq 0,$$

where $w(t) = \alpha t^{\alpha-1}$ which is given by Budak et al. in [5].

Corollary 2. If we take $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 7, then we have the following inequalities for k -Riemann-Liouville fractional integrals:

$$F\left(f\left(\frac{a+b}{2}\right), \frac{2^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} I_{b^-}^{\alpha, k} f\left(\frac{a+b}{2}\right), \frac{2^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} I_{a^+}^{\alpha, k} f\left(\frac{a+b}{2}\right), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0,$$

and

$$T_{F,w}\left(\frac{2^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} \left[I_{a^+}^{\alpha, k} f\left(\frac{a+b}{2}\right) + I_{b^-}^{\alpha, k} f\left(\frac{a+b}{2}\right) \right], f(a) + f(b), f(a) + f(b)\right) + \int_0^1 L_{w(t)} dt \leq 0,$$

where $w(t) = \frac{\alpha}{k} t^{\frac{\alpha}{k}-1}$.

Remark 6. One can obtain several results for convexity, ε -convexity, h -convexity, etc by special choice of the function F in Theorems 6 and 7.

3. CONCLUSION

In the development of this work, using the definition of F -convex functions some new Hermite-Hadamard type inequalities via generalized fractional integrals have been deduced. We also give several results capturing Riemann-Liouville fractional integrals and k -Riemann-Liouville fractional integrals as special cases. The authors hope that these results will serve as a motivation for future work in this fascinating area.

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