Mathematica Moravica
Vol. **25**, No. **1** (2021), 31–45 doi: 10.5937/MatMor2101031G

# Fixed point theorems of generalized multi-valued mappings in cone $b$-metric spaces

Mani Gunaseelan, Lakshmi Narayan Mishra, Vishnu Narayan Mishra*

Abstract. The aim of this paper is to establish fixed points for multi-valued mappings, by adapting the ideas in [1] to the cone $b$-metric space setting.

## 1. Introduction and preliminaries

The well-known Banach contraction principle and its several generalization in the setting of metric spaces play a central role for solving many problems of nonlinear analysis. For example, see [3, 10, 12, 20, 21]. Several authors introduced some interesting concept, see [28, 29, 30, 31, 32]. In [4], Bakhtin introduced $b$-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in $b$-metric spaces that generalized the famous contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in $b$-metric spaces(see [6, 7, 11] and reference therein). In recent investigations, the fixed point in non-convex analysis, especially in an ordered normed space, occupies a prominent place in many aspects (see [14, 15, 18, 22]). The authors define an ordering by using a cone, which naturally induces a partial ordering in Banach spaces. In 2007, Huang and Zhang [14] introduced the concept of cone metric spaces as a generalization of metric spaces and establish some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors [2, 16, 23, 25] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space. Recently, Rezapour and Hamlbarani [23] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space. In 2011, Hussain and

2020 *Mathematics Subject Classification.* Primary: 47H10; Secondary: 54H25.
*Key words and phrases.* Cone $b$-metric space, Multi-valued mappings, Fixed point.
*Full paper.* Received 1 July 2020, revised 1 September 2020, accepted 1 October 2020, available online 16 March 2021.
*Corresponding author

Shah [15] introduced the concept of cone $b$-metric space as a generalization of $b$-metric space and cone metric spaces. They established some topological properties in such space and improved some recent results about KKM mappings in the setting of a cone $b$-metric space. In 2020, Wasfi Shatanawi, Zoran D. Mitrović, Nawab Hussain and Stojan Radenović [33] proved Generalized Hardy–Rogers Type $\alpha$-Admissible Mappings in Cone $b$-Metric Spaces over Banach Algebras. Krishnakumar and Marudai [1] proved the following fixed point theorems of multi-valued mappings in cone metric spaces.

**Theorem 1.** *Let $(X,d)$ be a complete cone metric space and the mapping $T\colon X \to CB(X)$ be multi-valued map satisfying for each $x, y \in X$,*

$$H(Tx,Ty) \leq a[d(x,Tx) + d(y,Ty)] + b[d(x,Ty) + d(Tx,y)]$$

*for all $x, y \in X$, and $a + b < \frac{1}{2}$, $a, b \in [0, \frac{1}{2})$. Then $T$ has a fixed point in $X$.*

**Theorem 2.** *Let $(X,d)$ be a complete cone metric space and the mapping $T\colon X \to CB(X)$ be multi-valued map satisfy the condition,*

$$H(Tx,Ty) \leq r \max\{d(x,y), d(x,Tx), d(y,Ty)\}$$

*for all $x, y \in X$, and $r \in [0,1)$. Then $T$ has a fixed point in $X$.*

**Theorem 3.** *Let $(X,d)$ be a complete cone metric space and $P$ a normal cone with normal constant $K$. Suppose the mapping $T\colon X \to CB(X)$ be multi valued mapping satisfying the condition*

$$H(Tx,Ty) \leq r \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

*for all $x, y \in X$, and $r \in [0,1)$. Then $T$ has a unique fixed point in $X$.*

**Definition 1** ([14]). Let $E$ be a real Banach space. A subset $P$ of $E$ is called a cone whenever the following conditions hold:

$(C_1)$ $P$ is closed, nonempty and $P \neq \{0\}$;
$(C_2)$ $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;

$(C_3)$ $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$, where $P^0$ stands for the interior of $P$. If $P^0 \neq \emptyset$ then $P$ is called a solid cone(see[23]).
There exist two kinds of cone-normal(with the normal constant $K$) and non-normal ones [12].
Let $E$ be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defined by $P$. Then $P$ is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \quad \text{imply} \quad \|x\| \leq K\|y\|, \tag{1}$$

or equivalently, if $(\forall n) x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \quad \text{imply} \quad \lim_{n\to\infty} y_n = x. \tag{2}$$

The least positive number $K$ satisfying (1) is called the normal constant of $P$.
The cone $P$ is called regular if every increasing sequence which is bounded above is convergent and every decreasing sequence which is bounded below is convergent.

**Example 1** (see [24]). Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E\colon x(t) \geq 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $0 \leq x_n \leq y_n$, and $\lim_{n\to\infty} y_n = 0$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence $x_n$ does not converge to zero. It follows by (2) that $P$ is a non-normal cone.

**Definition 2** ([14, 26]). Let $X$ be a nonempty set. Suppose that the mapping $d\colon X \times X \to E$ satisfies:

(d$_1$) $0 \leq d(x,y)$ for all $x, y \in X$ with $x \neq y$ and $d(x,y) = 0$ if and only if $x = y$;
(d$_2$) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(d$_3$) $d(x,y) \leq d(x,z) + d(z,y)$ $x, y, z \in X$.

Then $d$ is called a cone metric [14] or$K$-metric [26] on $X$ and $(X,d)$ is called a cone metric [14] or $K$-metric space [26].

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

**Example 2** (see [14]). Let $E = \mathbb{R}^2$, $P = \{(x,y) \in \mathbb{R}^2\colon x \geq 0, y \geq 0\}$, $X = \mathbb{R}$ and $d\colon X \times X \to E$ defined by $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then $(X,d)$ is a cone metric space with normal cone $P$ where $K = 1$.

**Example 3** (see [22]). Let $E = \ell^2$, $P = \{\{x_n\}_{n\geq} \in E\colon x_n \geq 0 forall n\}$, $(X,\rho)$ a metric space, and $d\colon X \times X \to E$ defined by $d(x,y) = \{\rho(x,y)/2^n\}_{n\geq 1}$. Then $(X,d)$ is a cone metric space.

Clearly, the above examples show that class of cone metric spaces contains the class of metric spaces.

**Definition 3** ([15]). Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A mapping $d\colon X \times X \to E$ is said to be cone $b$-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $0 \leq d(x,y)$ with $x \neq y$ and $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair $(X,d)$ is called a cone $b$-metric space.

**Remark 1.** The class of cone $b$-metric spaces is larger than the class of cone metric space since any cone metric spaces must be a cone $b$-metric spaces. Therefore, it is obvious that cone $b$-metric spaces generalize $b$-metric spaces and cone metric spaces.

We give some examples, which show that introducing a cone $b$-metric space instead of a cone metric space is meaningful since there exist cone $b$-metric space which are not cone metric space.

**Example 4** (see [13])**.** Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E \colon x \geq 0, y \geq 0\} \subset E$, $X = \mathbb{R}$ and $d \colon X \times X \to E$ defined by $d(x, y) = (|x - y|^p, \alpha|x - y|^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then $(X, d)$ is a cone $b$-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space.

**Example 5** (see [13])**.** Let $X = \ell^p$ with $0 < p < 1$, where $\ell^p = \{\{x_n\} \subset \mathbb{R} \colon \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d \colon X \times X \to \mathbb{R}_+$ defined $d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$, where $x = \{x_n\}, y = \{y_n\} \in \ell^p$. Then $(X, d)$ is a cone $b$-metric space with the coefficient $s = 2^p > 1$, but not a cone metric space.

**Example 6** (see [13])**.** Let $X = \{1, 2, 3, 4\}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E \colon x \geq 0, y \geq 0\}$. Define $d \colon X \times X \to E$ by

$$d(x, y) = \begin{cases} (|x - y|^{-1}, |x - y|^{-1}), & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases} \tag{3}$$

Then $(X, d)$ is a cone $b$-metric space with the coefficient $s = \frac{6}{5} > 1$. But it is not a cone metric space since the triangle inequality is not satisfied,

$$d(1, 2) > d(1, 4) + d(4, 2), \quad d(3, 4) > d(3, 1) + d(1, 4).$$

**Definition 4** ([14])**.** Let $(X, d)$ be a cone $b$-metric space, $x \in X$ and $\{x_n\}$ be a sequence in $X$. Then:

(i) $\{x_n\}$ is a Cauchy sequence whenever, if for every $c \in E$ with $0 \ll c$, then there is natural number $N$ such that for all $n, m \geq N$, $d(x_n, x_m) \ll c$;

(ii) $\{x_n\}$ converges to $x$ whenever, for every $c \in E$ with $0 \ll c$, then there is a natural number $N$ such that for all $n \geq N$, $d(x_n, x) \ll c$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(iii) $(X, d)$ is a complete cone $b$-metric space if every Cauchy sequence is convergent.

In the following $(X, d)$ will stand for a cone $b$-metric space with respect to a cone $P$ with $P^0 \neq \emptyset$ in a real Banach space $E$ and $\leq$ is partial ordering in $E$ with respect to $P$. The following lemmas are often used(in particular while dealing with cone metric spaces in which the cone need not be normal).

**Lemma 1** ([18]). *Let $P$ be a cone and $\{a_n\}$ be a sequence in $E$. If $c \in intP$ and $0 \leq a_n \to 0$ as $n \to \infty$, then there exists $N$ such that for all $n > N$, we have $a_n \ll c$.*

**Lemma 2** ([18]). *Let $x, y, z \in E$, if $x \leq y$ and $y \ll z$, then $x \ll z$.*

**Lemma 3** ([15]). *Let $P$ be a cone and if $0 \leq u \ll c$ for each $c \in intP$, then $u = 0$.*

**Lemma 4** ([9]). *Let $P$ be a cone, if $u \in P$ and $u \leq ku$ for some $0 \leq k < 1$, then $u = 0$.*

**Lemma 5** ([18]). *Let $P$ be a cone and $a \leq b + c$ for each $c \in intP$, $a \leq b$.*

Let $(X, d)$ be a metric space. We denote by $CB(X)$ the family of nonempty closed bounded subset of $X$. Let $H$ be the Hausdorff distance on $CB(X)$. That is, for $A, B \in CB(X)$,

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from the point a to the subset $B$. An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T\colon X \to 2^X$ if $x \in T(X)$.

In this paper, we study the existence of fixed points for multi-valued mappings by adapting the ideas in [1] to the cone $b$-metric spaces setting.

## 2. Main results

**Theorem 4.** *Let $(X, d)$ be a complete cone $b$-metric space with the coefficient $s \geq 1$ and the mapping $T\colon X \to CB(X)$ be multi-valued map satisfying for each $x, y \in X$*

$$H(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + b[d(x, Ty) + d(Tx, y)]$$

*for all $x, y \in X$, and $a, b \in [0, 1)$ are constants such that $2a + 2bs < 1$. Then $T$ has a fixed point in $X$.*

*Proof.* For every $x_0 \in X$ and $n \geq 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$. We have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq H(Tx_n, Tx_{n-1}) \\ &\leq a[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] \\ &\quad + b[d(x_n, Tx_{n-1}) + d(Tx_n, x_{n-1})] \\ &\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_n, x_n) + d(x_{n+1}, x_{n-1})] \\ &\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_{n+1}, x_{n-1})] \\ &\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &\quad + bs[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] \\ &\leq (a + bs)[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)], \end{aligned}$$

$$d(x_{n+1}, x_n) \leq Ld(x_n, x_{n-1}), \tag{4}$$

where $L = \frac{a+bs}{1-(a+bs)}$. As $2a + 2bs < 1$, we obtain that $L < 1$. Similarly, we obtain

$$d(x_n, x_{n+1}) \leq Ld(x_{n-1}, x_{n-2}). \tag{5}$$

Using (5) in (4), we get

$$d(x_{n+1}, x_n) \leq L^2 d(x_n, x_{n-1}).$$

Continuing this process, we obtain

$$d(x_{n+1}, x_n) \leq L^n d(x_1, x_0).$$

For any $m \geq 1$, $p \geq 1$, we have

$$\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\
&+ \cdots + s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\
&\leq sL^m d(x_1, x_0) + s^2 L^{m+1} d(x_1, x_0) + s^3 L^{m+2} d(x_1, x_0) \\
&+ \cdots + s^{p-1} L^{m+p-2} d(x_1, x_0) + s^p L^{m+p-1} d(x_1, x_0) \\
&= sL^m [1 + sL + s^2 L^2 + s^3 L^3 + \cdots + (sL)^{p-1}] d(x_1, x_0) \\
&\leq (\frac{sL^m}{1 - sL}) d(x_1, x_0).
\end{aligned}$$

Let $0 \ll r$ be given. Note that $(\frac{sL^m}{1-sL})d(x_1, x_0) \to 0$ as $m \to \infty$ for any $p$. Making full use of ([13], Lemma 1.8), we find $m_0 \in \mathbb{N}$ such that

$$\left(\frac{sL^m}{1 - sL}\right) d(x_1, x_0) \ll r$$

for each $m > m_0$. Thus,

$$d(x_m, x_{m+p}) \leq \left(\frac{sL^m}{1 - sL}\right) d(x_1, x_0) \ll r$$

for all $m \geq 1$, $p \geq 1$. Therefore,$\{x_n\}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete cone $b$-metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Take $n_0 \in \mathbb{N}$ such that $d(x_n, z) \ll r\frac{1-as-bs}{s(1+b)}$ for all $n > n_0$. Hence,

$$\begin{aligned}
d(z, Tz) &\leq s[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\
&\leq sd(z, Tx_n) + sH(Tx_n, Tz) \\
&\leq sd(z, x_{n+1}) + s[a(d(x_n, Tx_n) + d(z, Tz)) \\
&+ b(d(x_n, Tz) + d(Tx_n, z))]
\end{aligned}$$

$$\leq sd(z, x_{n+1}) + s[a(d(x_n, x_{n+1}) + d(z, T(z))) + b(d(x_n, Tz) + d(x_{n+1}, z))].$$

This implies that

$$d(z, Tz) \leq \left(\frac{s(1+b)}{1-as-bs}\right) d(x_n, z) \ll r,$$

for $n > n_0$. Then, by Lemma (1.10), we deduce that $d(Tz, z) = 0$, that is $Tz = z$. □

**Example 7.** Let $X = [0,1]$ endowed with the standard order and $E = C^1_R[0,1]$ with $\|u\| = \|u\|_\infty + \|u'\|_\infty$, $u \in E$ and let $P = \{u \in E : u(t) \geq 0 \quad \text{on} \quad [0,1]\}$. It is well known that this cone is solid, but it is not normal. Define a cone $b$-metric $d : X \times X \to E$ by $d(x,y)(t) = |x-y|^2 \exp^t$. Then $(X, d)$ is a complete cone $b$-metric space with the coefficient $s = 2$. Define $T : X \to CB(X)$ by

$$T(x) = \begin{cases} \{\frac{1}{3}, \frac{2}{3}\}, & \text{if} \quad 0 \leq x < 1, \\ \{\frac{1}{3}\}, & \text{if} \quad x = 1. \end{cases} \tag{6}$$

Let $x, y \in X$. Without loss of generality, take $x \leq y$.
If $x = y$ or $x, y < 1$, then $Tx = Ty$. Hence $H(Tx, Ty) = 0$.
If $x < 1$ and $y = 1$, then

$$\begin{aligned} H(Tx, Ty) &= \frac{1}{9} \exp^t \\ &\leq \frac{4}{27} \exp^t \\ &= \frac{1}{3} . \frac{4}{9} \exp^t \\ &= \frac{1}{3}(d(x, Tx) + d(y, Ty)) \\ &\leq b(d(x, Tx) + d(y, Ty)) \end{aligned}$$

where $b = \frac{1}{3} \in [0, 1)$ and $a = 0$. So all the conditions of Theorem 2.1 are satisfied. Moreover, $\frac{1}{3}$ and $\frac{2}{3}$ are the two fixed points of $T$.

**Corollary 1.** *Let $(X, d)$ be a complete cone b-metric space with the coefficient $s \geq 1$ and the mapping $T \colon X \to CB(X)$ be multi valued map satisfies condition*

$$d(Tx, Ty) \leq b(d(x, Ty) + d(x, Ty))$$

*for all $x, y \in X$, where $b \in [0, \frac{1}{2s})$ is a constant. Then $T$ has a fixed point in $X$.*

*Proof.* The proof of the corollary immediately follows by putting $a = 0$ in the previous theorem. □

**Corollary 2.** *Let $(X, d)$ be a complete cone b-metric space with the coefficient $s \geq 1$ and the mapping $T\colon X \to CB(X)$ be multi valued map satisfies condition*

$$d(Tx, Ty) \leq a(d(x, Tx) + d(y, Ty))$$

*for all $x, y \in X$, where $a \in [0, \frac{1}{2s})$ is a constant. Then $T$ has a fixed point in $X$.*

*Proof.* The proof of the corollary immediately follows by putting $b = 0$ in the previous theorem. □

**Theorem 5.** *Let $(X, d)$ be a complete cone b-metric space with the coefficient $s \geq 1$ and the mapping $T\colon X \to CB(X)$ be multi valued map satisfy the condition, $H(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ for all $x, y \in X$, and $r \in [0, 1)$. Then $T$ has a unique fixed point in $X$.*

*Proof.* For every $x_0 \in X$ and $n \geq 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$

$$\begin{aligned} d(x_{n+1}, x_n) &\leq H(Tx_n, Tx_{n-1}) \\ &\leq r \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} \\ &\leq r \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &\leq r d(x_{n-1}, x_n) \\ &\leq r^n d(x_1, x_0) \end{aligned}$$

For any $m \geq 1$, $p \geq 1$, we have

$$\begin{aligned} d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\ &= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\ &\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\ &= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\ &\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\ &+ \cdots + s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\ &\leq sr^m d(x_1, x_0) + s^2 r^{m+1} d(x_1, x_0) + s^3 r^{m+2} d(x_1, x_0) \\ &+ \cdots + s^{p-1} r^{m+p-2} d(x_1, x_0) + s^p r^{m+p-1} d(x_1, x_0) \\ &= sr^m [1 + sr + s^2 r^2 + s^3 r^3 + \cdots + (sr)^{p-1}] d(x_1, x_0) \\ &\leq \left( \frac{sr^m}{1 - sr} \right) d(x_1, x_0). \end{aligned}$$

Let $0 \ll r$ be given. Note that $(\frac{sr^m}{1-sr}) d(x_1, x_0) \to 0$ as $m \to \infty$ for any $p$. Making full use of ([13], Lemma 1.8), we find $m_0 \in \mathbb{N}$ such that

$$\left( \frac{sr^m}{1 - sr} \right) d(x_1, x_0) \ll c$$

for each $m > m_0$. Thus,

$$d(x_m, x_{m+p}) \leq \left(\frac{sr^m}{1-sr}\right) d(x_1, x_0) \ll c$$

for all $m \geq 1$, $p \geq 1$. Therefore, $\{x_n\}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete cone $b$-metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Take $n_0 \in \mathbb{N}$ such that $d(x_n, z) \ll c\frac{1-s}{s}$ for all $n > n_0$. Hence,

$$\begin{aligned} d(z, Tz) &\leq s[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\ &\leq sd(z, Tx_n) + sH(Tx_n, Tz) \\ &\leq sd(z, x_{n+1}) + s[\max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz)\}] \\ &\leq sd(z, x_{n+1}) + s[\max\{0, d(x_n, x_{n+1}), d(z, Tz)\}] \\ &\leq sd(z, x_{n+1}) + s[\max\{0, 0, d(z, Tz)\}] \\ &\leq sd(z, x_n) + sd(z, Tz). \end{aligned}$$

This implies that

$$d(z, Tz) \leq \left(\frac{s}{1-s}\right) d(x_n, z) \ll c,$$

for $n > n_0$. Then, by Lemma (1.10), we deduce that $d(Tz, z) = 0$,
that is $Tz = z$.
Assume that there is another fixed point $q$ in $X$ such that $Tq = q$.

$$\begin{aligned} \therefore d(z, q) &\leq H(Tz, Tq) \\ &\leq r \max\{d(z, q), d(z, Tz), d(q, Tq), d(z, Tq), d(q, Tz)\} \\ &\leq r \max\{d(z, q), d(z, z), d(q, q), d(z, q), d(q, z)\} \\ &\leq rd(z, q) \end{aligned}$$

This is contradiction and hence $T$ has a unique fixed point in $X$. □

**Example 8.** Let $X = [0, \infty)$ endowed with the standard order and $E = C^1_R[0, 1]$ with $\|u\| = \|u\|_\infty + \|u'\|_\infty$, $u \in E$ and let $P = \{u \in E : u(t) \geq 0 on[0, 1]\}$. It is well known that this cone is solid, but it is not normal. Define a cone metric $d : X \times X \to E$ by $d(x, y)(t) = |x - y|^2 \exp^t$. Then $(X, d)$ is a complete cone $b$-metric space with the coefficient $s = 2$. Define $T : X \to CB(X)$ by

$$T(x) = \begin{cases} \{\frac{2}{3}\}, & \text{if } \quad 0 \leq x < 1, \\ \{\frac{1}{3}\}, & \text{if } \quad x > 1. \end{cases} \tag{7}$$

Let $x, y \in X$. Without loss of generality, take $x \leq y$.
If $x = y$ or $x, y < 1$, then $Tx = Ty$. Hence $H(Tx, Ty) = 0$.
If $x < 1$ and $y = 1$, then

$$\begin{aligned} H(Tx, Ty) &= \frac{1}{9}\exp^t \\ &\leq \frac{4}{27}\exp^t \\ &= \frac{1}{3}\cdot\frac{4}{9}\exp^t \\ &= \frac{1}{3}d(y, Ty) \\ &\leq r\max\{d(x,y), d(x,Tx), d(y,Ty)\} \end{aligned}$$

where $r = \frac{1}{3} \in [0,1)$. So all the conditions of Theorem 2.5 are satisfied. Moreover, 0 is a unique fixed point of $T$.

**Corollary 3.** *Let $(X, d)$ be a complete cone b-metric space with the coefficient $s \geq 1$ and the mapping $T\colon X \to CB(X)$ be multi valued mapping satisfy the condition*

$$H(Tx, Ty) \leq kd(x, y) \tag{8}$$

*for all $x, y \in X$ where $k \in [0, \frac{1}{2s})$ is a constant. Then $T$ has a unique fixed point in $X$.*

*Proof.* The proof of the corollary immediately follows by taking $d(x, y)$ as maximum value in the previous theorem. □

We prove the above theorems in the setting of $P$ is a normal cone with normal constant $K$.

**Theorem 6.** *Let $(X, d)$ be a complete cone b-metric space with the coefficient $s \geq 1$ and $P$ a normal cone with normal constant $K$. Suppose the mapping $T\colon X \to CB(X)$ be multi valued mapping satisfying the condition, $H(Tx, Ty) \leq r\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$ for all $x, y \in X$, and $r \in [0, 1), 2sr < 1$. Then $T$ has a unique fixed point in $X$.*

*Proof.* For every $x_0 \in X$ and $n \geq 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$,

$$\begin{aligned} d(x_{n+1}, x_n) &\leq H(Tx_n, Tx_{n-1}) \\ &\leq r\max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \\ &\qquad d(x_n, Tx_{n-1}), d(x_{n-1}, Tx_n)\} \\ &\leq r\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), \\ &\qquad d(x_{n+1}, x_{n-1})\} \\ &\leq r\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n+1}, x_{n-1})\}. \end{aligned}$$

**Case (i)**

If $d(x_{n+1}, x_n) \leq rd(x_n, x_{n-1})$ then we get, $d(x_{n+1}, x_n) \leq r^n d(x_1, x_0)$.
For any $m \geq 1$, $p \geq 1$, we have

$$\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\
&\quad + \cdots + s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\
&\leq sr^m d(x_1, x_0) + s^2 r^{m+1} d(x_1, x_0) + s^3 r^{m+2} d(x_1, x_0) \\
&\quad + \cdots + s^{p-1} r^{m+p-2} d(x_1, x_0) + s^p r^{m+p-1} d(x_1, x_0) \\
&= sr^m [1 + sr + s^2 r^2 + s^3 r^3 + \cdots + (sr)^{p-1}] d(x_1, x_0) \\
&\leq \left( \frac{sr^m}{1 - sr} \right) d(x_1, x_0).
\end{aligned}$$

We get $\|d(x_m, x_m + p)\| \leq K(\frac{sr^m}{1-sr})\|d(x_1, x_0)\|.d(x_m, x_m + p) \to 0$ as $p, m \to \infty$. Hence $\{x_m\}$ is a Cauchy sequence. By the completeness of $X$, there is $z \in X$ such that $x_m \to z$ as $m \to \infty$.

$$\begin{aligned}
d(z, Tz) &\leq s[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\
&\leq sd(z, Tx_n) + sH(Tx_n, Tz) \\
&\leq sd(z, x_{n+1}) + s[r \max\{d(x_n, z), d(x_n, Tx_n), \\
&\qquad\qquad d(z, Tz), d(x_n, Tz), d(z, Tx_n)\}] \\
&\leq sd(z, x_{n+1}) + s[r \max\{0, d(x_n, x_{n+1}), d(z, Tz), \\
&\qquad\qquad d(x_n, Tz), d(z, x_{n+1})\}] \\
&\leq sd(z, x_{n+1}) + s[r \max\{0, 0, d(z, Tz)\}] \\
&\leq sd(z, x_n) + srd(z, Tz) \\
&\leq srd(z, Tz),
\end{aligned}$$

which implies that $d(Tz, z) = 0$. Hence $z \in Tz$.

**Case (ii)**

If $d(x_{n+1}, x_n) \leq rd(x_{n+1}, x_{n-1})$ then we get

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq rs[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] \\
&\leq \frac{sr}{1 - sr} d(x_n, x_{n-1}) \\
&\leq hd(x_n, x_{n-1}), \quad \text{where} \quad h = \frac{sr}{1 - sr} < 1.
\end{aligned}$$

For any $m \geq 1$, $p \geq 1$, we have

$$\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})]
\end{aligned}$$

$$
\begin{aligned}
&= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\
&\le sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\
&\quad + \cdots + s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\
&\le sh^m d(x_1, x_0) + s^2 h^{m+1} d(x_1, x_0) + s^3 h^{m+2} d(x_1, x_0) \\
&\quad + \cdots + s^{p-1} h^{m+p-2} d(x_1, x_0) + s^p h^{m+p-1} d(x_1, x_0) \\
&= sh^m [1 + sh + s^2 h^2 + s^3 h^3 + \cdots + (sh)^{p-1}] d(x_1, x_0) \\
&\le \left( \frac{sh^m}{1 - sh} \right) d(x_1, x_0).
\end{aligned}
$$

We get $\|d(x_m, x_m + p)\| \le K(\frac{sh^m}{1-sh})\|d(x_1, x_0)\|$. $d(x_m, x_m + p) \to 0$ as $p, m \to \infty$. Hence $\{x_m\}$ is a Cauchy sequence. By the completeness of $X$, there is $z \in X$ such that $x_m \to z$ as $m \to \infty$.

$$
\begin{aligned}
d(z, Tz) &\le s[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\
&\le sd(z, Tx_n) + sH(Tx_n, Tz) \\
&\le sd(z, x_{n+1}) + s[r \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \\
&\qquad\qquad d(x_n, Tz), d(z, Tx_n)\}] \\
&\le sd(z, x_{n+1}) + s[r \max\{0, d(x_n, x_{n+1}), d(z, Tz), d(x_n, Tz), \\
&\qquad\qquad d(z, x_{n+1})\}] \\
&\le sd(z, x_{n+1}) + s[r \max\{0, 0, d(z, Tz)\}] \\
&\le sd(z, x_n) + srd(z, Tz) \\
&\le srd(z, Tz) \\
d(Tz, z) &= 0.
\end{aligned}
$$

Hence $z \in Tz$.
Assume that there is another fixed point $q$ in $X$ such that $Tq = q$.

$$
\begin{aligned}
\therefore d(z, q) &\le H(Tz, Tq) \\
&\le r \max\{d(z, q), d(z, Tz), d(q, Tq), d(z, Tq), d(q, Tz)\} \\
&\le r \max\{d(z, q), d(z, z), d(q, q), d(z, q), d(q, z)\} \\
&\le rd(z, q)
\end{aligned}
$$

This is contradiction and hence $T$ has a unique fixed point in $X$. □

**Example 9.** Let $X = [0, 1]$, $E = \mathbb{R}^2$. Take $P = \{(x, y) \in E : x, y \ge 0\}$. We define $d\colon X \times X \to E$ as $d(x, y) = (|x - y|^2, |x - y|^2)$ for all $x, y \in X$. Then $(X, d)$ is a complete cone $b$-metric. Let us define $T\colon X \to CB(X)$ as

$$
T(x) = \begin{cases} \{\frac{2}{5}\}, & \text{if } \ 0 \le x < 1, \\ \{\frac{1}{5}\}, & \text{if } \ x = 1. \end{cases} \tag{9}
$$

Let $x, y \in X$. Without loss of generality, take $x \le y$.

If $x = y$ or $x, y < 1$, then $Tx = Ty$. Hence $H(Tx, Ty) = 0$.
If $x < 1$ and $y = 1$, then

$$\begin{aligned} H(Tx, Ty) &= (\tfrac{1}{25}, \tfrac{1}{25}) \leq (\tfrac{16}{125}, \tfrac{16}{125}) = \tfrac{1}{5}(\tfrac{1}{25}, \tfrac{1}{25}) = \tfrac{1}{5}d(y, Ty) \\ &\leq r \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \end{aligned}$$

where $r = \frac{1}{5} \in [0, 1)$. So all the conditions of Theorem 2.8 are satisfied. Moreover, $\frac{2}{5}$ is a unique fixed point of $T$.

**Corollary 4.** *Let $(X, d)$ be a complete cone b-metric space with the coefficient $s \geq 1$ and $P$ a normal cone with normal constant $K$. Suppose the mapping $T\colon X \to CB(X)$ be multi-valued mapping satisfies the condition, $H(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ for all $x, y \in X$, and $r \in [0, 1)$. Then $T$ has a unique fixed point in $X$.*

*Proof.* The proof of the corollary immediately follows since

$$\max\{d(x, y), d(x, Tx), d(y, Ty)\} \leq \\ \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$ □

## References


[1] R. Krishnakumar and M. Marudai, *Fixed point theorems of multivalued mappings in cone metric spaces*, International Journal of Contemporary Mathematical Sciences, 5 (31) (2010), 1533-1540.

[2] M. Abbas and G. Jungck, *Common fixed point results for non commuting mappings without continuity in cone metric spaces*, Journal of Mathematical Analysis and Applications, 341 (2008), 416-420.

[3] R.P. Agarwal, D. O'Regan and R. Precup, *Domain invariance theorems for contractive type maps*, Dynamic Systems and Applications, 16 (3) (2007), 579-586.

[4] S. Banach, *Surles operation dans les ensembles abstraits et leur application aux equation integrals*, Fundamenta Mathematica, 3 (1922), 133-181.

[5] I.A. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Functional Analysis (Unianowsk Gos. Ped. Inst.), 30 (1989), 26-37. (Russian)

[6] M. Boriceanu, M. Bota and A. Petrusel, *A multivalued fractals in b-metric spaces*, Central European Journal of Mathematics, 8 (2) (2010), 367-377.

[7] M. Bota, A. Moinar and V. Csaba, *On Ekeland's variational principle in b-metric spaces*, Fixed Point Theory, 12 (2011), 21-28.

[8] S.K. Chatterjae, *Fixed point theorems compactes*, Comptes Rendus de l'Academie Bulgare des Sciences, 25 (1972), 727-730.

[9] S-H Cho and J-S Bae, *Common fixed point theorems for mappings satisfying property(E.A) on cone metric space*, Mathematical and Computer Modelling, 53 (2011), 945-951.

[10] L.B. Ćirić, *Generalized contractions and fixed point theorems*, Publications de l'Institut Mathematique (Nouvelle Serie), 12 (26) (1971), 19-26.

[11] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric*, Atti del Seminario Matematico e Fisico dell'Universitá di Modena, 46 (1998), 263-276.

[12] K. Deimling, Nonlinear Functional Analysis, Springer, Berlin, Germany, 1985.

[13] H. Huang and S. Xu, *Fixed point theorems of contractive mappings in cone b-metric spaces and applications*, Fixed Point Theory and Applications, 2013 (2013), Article ID: 112, 10 pages.

[14] L-G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications, 332 (2) (2007), 1468-1476.

[15] N. Hussain and MH. Shah, *KKM mappings in cone b-metric spaces*, Computers & Mathematics with Applications, 62 (2011), 1677-1684.

[16] D. Ilic and V. Rakočević, *Common fixed points for maps on cone metric space*, Journal of Mathematical Analysis and Applications, 341 (2008), 876-882.

[17] D. Ilic and V. Rakočević, *Quasi-contraction on a cone metric space*, Applied Mathematics Letters, 22 (5) (2009), 728-731.

[18] S. Janković, Z. Kadelburg and S. Radenović, *On cone metric spaces: a survey*, Nonlinear Analysis, 4 (7) (2011), 2591-2601.

[19] R. Kannan, *Some results on fixed point theorems*, Bulletin of the Calcutta Mathematical Society, 60 (1969), 71-78.

[20] G. Petruşel and A. Petruşel, *Multivalued contractions of Feng-Liu type in complete gauge spaces*, Carpthian Journal of Mathematics, 24 (3) (2008), 392-396.

[21] A. Petrusel, *Generalized multivalued contractions*, Nonlinear Analysis (TMA), 47 (1) (2001), 649-659.

[22] Sh. Rezapour, *A review on topological properties of cone metric spaces*, in Proceedings of the International Conference on Analysis, Topology and Applications (ATA 08), Vrinjačka Banja, Serbia, May-June 2008.

[23] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper cone metric spaces and fixed point theorems of contractive mappings*, Journal of Mathematical Analysis and Applications, 345 (2) (2008), 719-724.

[24] J. Vandergraft, *Newton method for convex operators in partially ordered spaces*, SIAM Journal on Numerical Analysis, 4 (1967), 406-432.

[25] P. Vetro, *Common fixed points in cone metric spaces*, Rendiconti del Circolo Matematico di Palermo Series 2, 56 (3) (2007), 464-468.

[26] P.P. Zabrejko, *$K$-metric and $K$-normal linear spaces: survey*, Collectanea Mathematica, 48 (4-6) (1997), 825-859.

[27] G. S Saluja, *Some fixed point theorems of contractive type conditions in cone b-metric spaces*, Nonlinear Analysis Forum, 20 (8) (2015), 241-255.

[28] R. Dubey, Deepmala, V.N. Mishra, *Higher-order symmetric duality in nondifferentiable multiobjective fractional programming problem over cone constraints*, Statistics, Optimization & Information Computing, 8 (2020), 187–205.

[29] L.N. Mishra, V. Dewangan, V.N. Mishra, S. Karateke, *Best proximity points of admissible almost generalized weakly contractive mappings with rational expressions on b-metric spaces*, Journal of Mathematics and Computer Science, 22 (2) (2021), 97–109.

[30] L.N. Mishra, S.K. Tiwari, V.N. Mishra, I.A. Khan, *Unique fixed point theorems for generalized contractive mappings in partial metric spaces*, Journal of Function Spaces, 2015 (2015), Article ID: 960827, 8 pages.

[31] L. N. Mishra, S.K. Tiwari, V.N. Mishra, *Fixed point theorems for generalized weakly $S$-contractive mappings in partial metric spaces*, Journal of Applied Analysis and Computation, 5 (4) (2015), 600-612.

[32] X. Liu, M. Zhou, L.N. Mishra, V.N. Mishra, B. Damjanović, *Common fixed point theorem of six self-mappings in Menger spaces using $(CLR_{ST})$ property*, Open Mathematics, 16 (2018), 1423–1434.

[33] Wasfi Shatanawi, Zoran D. Mitrović, Nawab Hussain and Stojan Radenović, *On generalized Hardy–Rogers type $\alpha$-admissible mappings in cone b-metric spaces over banach algebras,* Symmetry, 12 (1) (2020), Article ID: 81, 12 pages.

**Mani Gunaseelan**
Department of Mathematics,
Sri Sankara Arts and Science College(Autonomous)
Affiliated to Madras University
Enathur, Kanchipuram, Tamil Nadu
India 631 561
*E-mail address*: mathsguna@yahoo.com

**Lakshmi Narayan Mishra**
Department of Mathematics
School of Advanced Sciences
Vellore Institute of Technology (VIT) University
Vellore 632 014, Tamil Nadu
India
*E-mail address*: lakshminarayanmishra04@gmail.com

**Vishnu Narayan Mishra**
Department of Mathematics
Indira Gandhi National Tribal University
Laipur, Amarkantak, Anuppur
Madhya Pradesh
India 484887
*E-mail address*: vishnunarayanmishra@gmail.com