Fixed point theorems on a closed ball

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ABSTRACT. The aim of the paper is to obtain some fixed point theorems for extended (φ, F) -weak type contraction on a closed ball in metric spaces. Our results generalize some recently established results.

1. INTRODUCTION

In 2012, Samet et al. [8] introduced a class of α -admissible mapping.

Definition 1 ([8]). Let $T : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, +\infty)$. We say that T is α -admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.

Definition 2 ([7]). Let $T : X \rightarrow X$ and $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ be two functions. We say that T is α -admissible mapping with respect to η if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies that $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$.

If $\eta(x, y) = 1$, then Definition 2 reduces to Definition 1. If $\alpha(x, y) = 1$, then T is called an η -subadmissible mapping.

Definition 3 ([4]). Let (X, d) be a metric space. Let $T : X \rightarrow X$ and $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ be two functions. We say that T is α - η -continuous mapping on (X, d) if for given $x \in X$ and sequence $\{x_n\}$ with

$$x_n \rightarrow x \text{ as } n \rightarrow \infty; \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \quad \text{for all } n \in \mathbb{N} \Rightarrow Tx_n \rightarrow Tx.$$

Definition 4 ([6]). Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be an F -contraction if there exists $\tau > 0$ such that

$$(1) \quad \forall x, y \in X, d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a mapping satisfying the following conditions:

- (F₁) F is strictly increasing, i.e., for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;
- (F₂) For each sequence $\{\alpha_n\}_{n=1}^\infty$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;

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(F₃) There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

We denote by Δ_F , the set of all functions satisfying the conditions (F₁)–(F₃).

Wardowski [9] modified Banach contraction principle for F -contraction as follows.

Theorem 1. *Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be an F -contraction. Then T has a unique fixed point $z \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n \in \mathbb{N}}$ converges to z .*

Hussain et al. [4] introduced the following family of new functions. Let Δ_G denote the set of all functions $G : \mathbb{R}_+^4 \rightarrow \mathbb{R}_+$ satisfying:

(G) for all $t_1, t_2, t_3, t_4 \in \mathbb{R}_+$ with $t_1 t_2 t_3 t_4 = 0$, there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

Definition 5 ([4]). Let (X, d) be a metric space and T be a self-mapping on X . Also, suppose that $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ are two function. We say that T is α - η -GF-contraction, if for $x, y \in X$ with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$, we have

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $G \in \Delta_G$ and $F \in \Delta_F$.

For $x \in X$ and $\epsilon > 0$, $\overline{B(x, \epsilon)} = \{y \in X : d(x, y) \leq \epsilon\}$ is a closed ball in (X, d) . The following result, regarding the existence of the fixed point of the mapping satisfying a contractive condition on the closed ball, was given in [5]. The result is very useful in the sense that it requires the contraction condition only on a closed ball, instead of on the whole space.

Theorem 2 ([5]). *Let (X, d) be a complete metric space, $T : X \rightarrow X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X . Suppose there exists $k \in [0, 1)$ with*

$$d(Tx, Ty) \leq k d(x, y), \quad \text{for all } x, y \in Y = \overline{B(x_0, r)},$$

and $d(x_0, Tx_0) < (1 - k)r$. Then there exists a unique point x^ in $\overline{B(x_0, r)}$ such that $x^* = Tx^*$.*

Recently, in 2019, Hussain [3] introduced the Ćirić type modified F -contraction on a closed ball in a complete metric space.

Definition 6 ([3]). Let (X, d) be a metric space. A self-mapping $T : X \rightarrow X$ is said to be a modified F -contraction via α -admissible mappings if there exists $\tau > 0$ such that

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(\alpha(x, y)d(Tx, Ty)) \leq F(\psi(M(x, y))),$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}$$

for all $x, y \in \overline{B(x_0, r)} \subseteq X$; where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a mapping satisfying $(F1) - (F3)$ and $\psi \in \Psi$.

In Definition 6, Ψ be the family of functions of self-mappings on $[0, \infty)$ satisfying:

- (i) ψ is nondecreasing.
- (ii) $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$, for each $t > 0$.

Remark 1. If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.

Using Definition 6, Hussain [3] obtained the following result.

Theorem 3 ([3]). *Let (X, d) be a complete metric space. Let $T : X \rightarrow X$ be a modified F -contraction via α -admissible mappings and x_0 be an arbitrary point in X . Assume that*

$$(2) \quad x, y \in \overline{B(x_0, r)}, \quad \tau + F(\alpha(x, y)d(Tx, Ty)) \leq F(\psi(M(x, y))),$$

where $\tau > 0$. Moreover

$$\sum_{j=0}^N d(x_0, Tx_0) \leq r, \text{ for all } j \in \mathbb{N} \text{ and } r > 0.$$

Suppose that the following assertions hold:

- (i) T is an α -admissible mapping;
- (ii) there exist a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
- (iii) T is continuous.

Then there exist a point x^* in $\overline{B(x_0, r)}$ such that $Tx^* = x^*$.

In this paper, we obtain some fixed point results which generalize the results of Dey et al [1], Dung and Hang [2] and Hussain [3] on a closed ball in a complete metric space.

2. MAIN RESULTS

Now, we introduce the following definition.

Definition 7. Let (X, d) be a metric space. A self-mapping $T : X \rightarrow X$ is said to be a modified F -contraction II via α -admissible mappings if there exists $\tau > 0$ such that

$$(3) \quad d(Tx, Ty) > 0 \Rightarrow \tau + F(\alpha(x, y)d(Tx, Ty)) \leq F(\psi(M(x, y)));$$

where,

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}, \right. \\ \left. \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty) \right\},$$

for all $x, y \in \overline{B(x_0, r)} \subseteq X$, and $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a mapping satisfying $(F_1) - (F_3)$ and $\psi \in \Psi$, where Ψ is defined as the same in Definition 6.

Theorem 4. Let (X, d) be a complete metric space and $T : X \rightarrow X$ a modified F -contraction II via α -admissible mappings and x_0 be an arbitrary point in X . Assume that

$$(4) \quad x, y \in \overline{B(x_0, r)}, \quad \tau + F(\alpha(x, y)d(Tx, Ty)) \leq F(\psi(M(x, y))),$$

where $\tau > 0$. Moreover,

$$\Sigma_{j=0}^N d(x_j, Tx_j) \leq r, \quad \forall j \in \mathbb{N} \text{ and } r > 0.$$

Suppose that the following assertions hold:

- (i) T is an α -admissible mapping;
- (ii) there exist a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
- (iii) T is continuous.

Then there exist a point x^* in $\overline{B(x_0, r)}$ such that $Tx^* = x^*$.

Proof. Due to assumption (ii), there exist a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Now, we construct a sequence $\{x_n\}_{n \geq 0}$ in X such that $x_{n+1} = Tx_n$. $\{x_n\}$ is a non-increasing sequence. If we assume that $x_n = x_{n+1}$ for some $n \geq 0$, then the proof is complete obviously. So, we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Since $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$ and T is α -admissible, we have

$$(5) \quad \alpha(x_n, x_{n+1}) \geq 1, \quad \forall n \geq 0.$$

Firstly, we show that $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$. For this, consider $d(x_0, x_1) = d(x_0, Tx_0) \leq r$. Thus $x_1 \in \overline{B(x_0, r)}$. Suppose that $x_2, \dots, x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$, then from (4),

$$\begin{aligned} F(\alpha(x_{j-1}, x_j)d(Tx_{j-1}, Tx_j)) &\leq F(\psi(M(x_{j-1}, x_j)) - \tau \\ &\Rightarrow d(x_j, x_{j+1}) < \psi((M(x_{j-1}, x_j))) < M(x_{j-1}, x_j) \end{aligned}$$

where

$$\begin{aligned} M(x_{j-1}, x_j) &= \max \left\{ d(x_{j-1}, x_j), d(x_{j-1}, x_j), d(x_j, x_{j+1}), \right. \\ &\quad \left. \frac{d(x_{j-1}, x_{j+1}) + d(x_j, x_j)}{2}, \frac{d(x_{j+1}, x_{j-1}) + d(x_{j+1}, x_{j+1})}{2}, \right. \\ &\quad \left. d(x_{j+1}, x_j), d(x_{j+1}, x_j), d(x_{j+1}, x_{j+1}) \right\} \\ &= \max \{ d(x_{j-1}, x_j), d(x_j, x_{j+1}) \}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} F(d(x_j, x_{j+1})) &\leq F(\alpha(x_{j-1}, x_j)d(Tx_{j-1}, Tx_j)) \\ &\leq F(\max\{d(x_{j-1}, x_j), d(x_j, x_{j+1})\}) - \tau. \end{aligned}$$

If $\max\{d(x_{j-1}, x_j), d(x_j, x_{j+1})\} = d(x_j, x_{j+1})$, then

$$\Rightarrow F(d(x_j, x_{j+1})) \leq F(d(x_j, x_{j+1})) - \tau.$$

This gives $\tau \leq 0$, a contradiction. Hence, $\max\{d(x_{j-1}, x_j), d(x_j, x_{j+1})\} = d(x_{j-1}, x_j)$. Now,

$$\begin{aligned} d(x_0, x_{j+1}) &\leq d(x_0, x_1) + \dots + d(x_j, x_{j+1}) \\ &= \sum_{j=0}^N d(x_j, Tx_j) \leq r. \end{aligned}$$

Therefore, $x_{j+1} \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$. Continuing this process, we get

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau \\ &= F(d(Tx_{n-2}, Tx_{n-1})) - \tau \\ &\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\ &\vdots \\ &\leq F(d(x_0, x_1)) - n\tau. \end{aligned}$$

This implies

$$(6) \quad F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau.$$

Taking limit we get, $\lim_{n \rightarrow \infty} F(d(x_n, x_{n+1})) = -\infty$. So, we have

$$(7) \quad d(x_n, x_{n+1}) = 0.$$

From (F3), there exists $k \in (0, 1)$ such that

$$(8) \quad \lim_{n \rightarrow \infty} (d(x_n, x_{n+1}))^k F(d(x_n, x_{n+1})) = 0.$$

From (6), for all $n \in \mathbb{N}$, we obtain

$$(9) \quad \begin{aligned} (d(x_n, x_{n+1}))^k (F(d(x_n, x_{n+1})) - F(d(x_0, x_1))) &\leq \\ - (d(x_n, x_{n+1}))^k n\tau &\leq 0. \end{aligned}$$

By using (7), (8) and letting $n \rightarrow \infty$ in (9), we have

$$(10) \quad \lim_{n \rightarrow \infty} (n(d(x_n, x_{n+1}))^k) = 0.$$

We observe that from (10), there exist $n_1 \in \mathbb{N}$ such that $n(d(x_n, x_{n+1}))^k \leq 1$ for all $n \geq n_1$, we get

$$(11) \quad d(x_n, x_{n+1}) \leq \frac{1}{n^k}, \quad \forall n \geq n_1.$$

Now $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Then by triangle inequality and from (11), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ (12) \quad &\leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The series $\frac{1}{i^k}$ is convergent. Taking the limit as $n \rightarrow \infty$, in (12), we have $\lim_{n,m \rightarrow \infty} d(x_n, x_m) = 0$. Hence x_n is a Cauchy sequence. Since, X is a complete metric space there exists an $x^* \in \overline{B(x_0, r)}$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. T is a continuous then $x_{n+1} = Tx_n \rightarrow Tx^*$ as $n \rightarrow \infty$. That is, $x^* = Tx^*$. Hence x^* is a fixed point of T . $\square$

Motivating by the paper [1], we introduce the following definition.

Definition 8. Let (X, d) be a metric space. A self-mapping $T : X \rightarrow X$ is said to be a modified F -contraction III via α -admissible mappings if there exists $\tau > 0$ such that

$$(13) \quad d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F(\alpha(x, y)d(Tx, Ty)) \leq F(\psi(M'(x, y))),$$

where

$$M'(x, y) = \max \left\{ d(x, y), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, \right. \\ \left. d(T^2x, Tx), d(T^2x, y), d(Tx, y) + d(y, Ty), d(T^2x, Ty) + d(x, Tx) \right\},$$

for all $x, y \in \overline{B(x_0, r)} \subseteq X$, and $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a mapping satisfying (F_1) -(F_3) and $\psi \in \Psi$, where Ψ is defined as the same in Definition 6.

Remark 2. Every modified F -contraction III is a modified F -contraction II via α -admissible mapping. The reverse implications do not hold.

Now we obtain the following result which is a generalization of Theorem 4.

Theorem 5. Let (X, d) be a complete metric space and $T : X \rightarrow X$ a modified F -contraction III via α -admissible mappings and x_0 be an arbitrary point in X . Assume that

$$(14) \quad x, y \in \overline{B(x_0, r)}, \quad \tau + F(\alpha(x, y)d(Tx, Ty)) \leq F(\psi(M'(x, y))),$$

where $\tau > 0$. Moreover,

$$\sum_{j=0}^N d(x_j, Tx_j) \leq r, \quad \forall j \in \mathbb{N} \text{ and } r > 0.$$

Suppose that the following assertions hold:

- (i) T is an α -admissible mapping;
- (ii) there exist a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
- (iii) T is continuous.

Then there exist a point x^* in $\overline{B(x_0, r)}$ such that $Tx^* = x^*$.

Proof. The proof is same as in Theorem 4. $\square$

Remark 3. Theorem 4 and 5 generalize the main result of Hussain [3] and also extends results of [2] and [1] on closed ball in a complete metric space.

3. FIXED POINT THEOREMS FOR GF-CONTRACTION ON CLOSED BALL

Definition 9. Let T be a self mapping in a metric space (X, d) and let x_0 be an arbitrary point in X . Also suppose that $\alpha : X \times X \rightarrow -\infty \cup (0, +\infty)$; $\eta : X \times X \rightarrow \mathbb{R}_+$ are two functions. We say that T is called modified $\alpha - \eta - \psi$ -GF-contraction II on closed ball if for all $x, y \in \overline{B(x_0, r)} \subseteq X$; with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$; we have

$$(15) \quad G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq \leq F(\psi(M(x, y))),$$

where

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty) \right\}.$$

Moreover,

$$\sum_{j=0}^N d(x_j, Tx_j) \leq r, \quad \forall j \in \mathbb{N} \text{ and } r > 0,$$

$G \in \Delta_G$, $\psi \in \Psi$, and $F \in \Delta_F$.

Theorem 6. Let (X, d) be a complete metric space. Let $T : X \rightarrow X$ be an α - η - ψ -GF-contraction II mapping on closed ball satisfying the following assertions:

- (i) T is an α -admissible mapping with respect to η ;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
- (iii) T is α - η -continuous.

Then there exist a point x^* in $\overline{B(x_0, r)}$ such that $Tx^* = x^*$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For $x_0 \in X$, we construct a sequence $\{x_n\}_{n=1}^\infty$ such that $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0$. Continuing this way, we have $x_{n+1} = Tx_n = T^{n+1}x_0, \forall n \in \mathbb{N}$.

Since T is an α -admissible mapping with respect to η , then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. Continuing this process, we have

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}.$$

If there exists an $n \in \mathbb{N}$ such that $d(x_n, Tx_n) = 0$. We assume that $x_n \neq x_{n+1}$ with

$$d(Tx_{n-1}, Tx_n) = d(x_n, Tx_n) > 0, \quad \forall n \in \mathbb{N}.$$

First we show that $x_n \in \overline{B(x_0, r)}, \quad \forall n \in \mathbb{N}$,

$$d(x_0, x_1) = d(x_0, Tx_0) \leq r.$$

Thus, $x_1 \in \overline{B(x_0, r)}$. Suppose that $x_2, \dots, x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$. Since, T is an $\alpha - \eta - \psi$ -GF-contraction on closed ball, such that

$$(16) \quad \begin{aligned} &G(d(x_{j-1}, Tx_{j-1}), d(x_j, Tx_j), d(x_{j-1}, Tx_j), d(x_j, Tx_{j-1})) \\ &+ F(d(Tx_{j-1}, Tx_j)) \leq F(\psi(M(x_{j-1}, x_j))). \end{aligned}$$

This implies

$$(17) \quad \begin{aligned} &G(d(x_{j-1}, x_j), d(x_j, x_{j+1}), d(x_{j-1}, x_{j+1}), 0) \\ &+ F(d(x_j, x_{j+1})) \leq F(\psi(M(x_{j-1}, x_j))). \end{aligned}$$

Since, $d(x_{j-1}, x_j) \cdot d(x_j, x_{j+1}) \cdot d(x_{j-1}, x_{j+1}) \cdot 0 = 0$, then there exist a $\tau > 0$ such that

$$F(d(x_j, x_{j+1})) = F(d(Tx_{j-1}, Tx_j)) \leq F(\psi(M(x_{j-1}, x_j))) - \tau.$$

The rest of the proof follows from the proof of the Theorem 4. $\square$

Along the same lines we introduce the modified $\alpha - \eta - \psi$ -GF-contraction III on a closed ball.

Definition 10. Let T be a self mapping in a metric space (X, d) and let x_0 be an arbitrary point in X . Also suppose that $\alpha : X \times X \rightarrow -\infty \cup (0, +\infty)$; $\eta : X \times X \rightarrow \mathbb{R}_+$ are two functions. We say that T is called modified $\alpha - \eta - \psi$ -GF-contraction III on a closed ball if for all $x, y \in \overline{B(x_0, r)} \subseteq X$; with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$; we have

$$(18) \quad \begin{aligned} &G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \\ &+ F(d(Tx, Ty)) \leq F(\psi(M'(x, y))), \end{aligned}$$

where

$$\begin{aligned} M'(x, y) = \max \left\{ d(x, y), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, \right. \\ \left. d(T^2x, Tx), d(T^2x, y), d(Tx, y) + d(y, Ty), d(T^2x, Ty) + d(x, Tx) \right\}, \end{aligned}$$

Moreover,

$$\sum_{j=0}^N d(x_j, Tx_j) \leq r, \quad \forall j \in \mathbb{N} \text{ and } r > 0,$$

$G \in \Delta_G$, $\psi \in \Psi$, and $F \in \Delta_F$.

Now, we obtain the following generalization of Theorem 6.

Theorem 7. Let (X, d) be a complete metric space. Let $T : X \rightarrow X$ be an $\alpha - \eta - \psi$ -GF-contraction III mapping on closed ball satisfying the following assertions:

- (i) T is an α -admissible mapping with respect to η ;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
- (iii) T is $\alpha - \eta$ -continuous.

Then there exist a point x^* in $\overline{B(x_0, r)}$ such that $Tx^* = x^*$.

Proof. The proof is same as in Theorem 6. $\square$

Remark 4. Theorem 6 and 7 generalize Theorem 3.2 of [3].

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