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Some best proximity point results for multivalued mappings on partial metric spaces

Mustafa Aslantas*, Doaa Riyadh Abed Al-Zuhairi

Abstract. In this paper, we introduce two new concepts of Feng-Liu type multivalued contraction mapping and cyclic Feng-Liu type multivalued contraction mapping. Then, we obtain some new best proximity point results for such mappings on partial metric spaces by considering Feng-Liu's technique. Finally, we provide examples to show the effectiveness of our results.

1. Introduction

In 1922, the Banach contraction principle which is considered as the starting of the fixed point theory on metric spaces has been proved a fundamental theorem [10]. This famous principle is a very useful tool for the existence and uniqueness of the solution of problems in various fields such as differential equations, integral equations, partial differential equations. Because of its applicability, many authors have studied to generalize this principle [4, 8, 16, 18, 24, 26, 27]. One of the interesting and famous generalizations was proved by Nadler [22] by taking into account multivalued mappings on metric spaces as follows.

Theorem 1 ([22]). *Let (Υ, ρ) be a complete metric space and $\varphi : \Upsilon \to CB(\Upsilon)$ be a multivalued mapping. If there exists k in $[0, 1)$ such that*

$$H_\rho(\varphi\varsigma, \varphi\xi) \le k\rho(\varsigma, \xi),$$

for all $\varsigma, \xi \in \Upsilon$, where $CB(\Upsilon)$ is the family of all nonempty bounded and closed subsets of Υ and $H_\rho : CB(\Upsilon) \times CB(\Upsilon)$ defined as

$$H_\rho(\wp, \Re) = \max\left\{\sup_{\varsigma\in\wp} \rho(\varsigma, \Re), \sup_{\xi\in\Re} \rho(\wp, \xi)\right\}$$

is a Pompei-Hausdorff metric, then φ has a fixed point in Υ.

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Later, many interesting fixed point theorems for multivalued mappings have been obtained [21, 25, 30]. In this sense, Nadler's result has been extended by Feng and Liu [15] by taking the image of φ for $C(\Upsilon)$ valued instead of $CB(\Upsilon)$ where $C(\Upsilon)$ is the family of all nonempty closed subsets of Υ. Therefore, they neglected the boundedness of $\varphi\varsigma$ for all $\varsigma \in \Upsilon$. Hence, they did not need to use the Pompeiu-Hausdorff metric in their result as follows.

Theorem 2. *Let $\varphi : \Upsilon \to C(\Upsilon)$ be a multivalued mapping and (Υ, ρ) be a complete metric space. Suppose for all $\varsigma \in \Upsilon$ there exists $\xi \in I_\beta^\varsigma$ such that*

$$\rho(\xi, \varphi\xi) \leq \gamma\rho(\varsigma, \xi),$$

where

$$I_\beta^\varsigma = \{\xi \in \varphi\varsigma : \beta\rho(\varsigma, \xi) \leq \rho(\varsigma, \varphi\varsigma)\}.$$

If the function $g(\varsigma) = \rho(\varsigma, \varphi\varsigma)$ is lower semicontinuous on Υ and $0 < \gamma < \beta < 1$, then φ has a fixed point in Υ.

On the other hand, Kirk et al. [19] proved another generalization of the Banach contraction principle. They obtained the following nice result by introducing a new notion of cyclic mapping.

Theorem 3. *Let (Υ, ρ) be a complete metric space, $\emptyset \neq \wp, \Re \subseteq \Upsilon$ where $\wp$ and $\Re$ are closed and $\varphi : \wp \cup \Re \to \wp \cup \Re$ be a mapping. Assume that φ is a cyclic mapping, that is, $\varphi(\wp) \subseteq \Re$ and $\varphi(\Re) \subseteq \wp$. If there exists k in $[0, 1)$ such that*

$$\rho(\varphi\varsigma, \varphi\xi) \leq k\rho(\varsigma, \xi), \tag{1}$$

for all $\varsigma \in \wp$ and $\xi \in \Re$, then φ has a fixed point in $\wp \cap \Re$.

Note that, unlike Banach contraction principle, the mapping φ is not necessary to be continuous in Theorem 3. Because of its applicability, there are many studies on this topic in the literature [9, 23]. In this sense, Eldred and Veeremani [14] introduced a concept of cyclic contraction mapping by considering $\wp \cap \Re = \emptyset$.

Definition 1 ([14]). Let (Υ, ρ) be a metric space and $\emptyset \neq \wp, \Re \subseteq \Upsilon$. A mapping $\varphi : \wp \cup \Re \to \wp \cup \Re$ is called a cyclic contraction if it satisfies $\varphi(\wp) \subseteq \Re$ and $\varphi(\Re) \subseteq \wp$ and the following condition:

$$\rho(\varphi\varsigma, \varphi\xi) \leq k\rho(\varsigma, \xi) + (1 - k)\rho(\wp, \Re), \tag{2}$$

for all $\varsigma \in \wp$ and $\xi \in \Re$, where $k \in [0, 1)$.

Hence, they gave a generalization of inequality (1). Indeed, if $\wp \cap \Re = \emptyset$, φ cannot have a fixed point. In this case, it is sensible to find the existence of a point ς such that $\rho(\varsigma, \varphi\varsigma) = \rho(\wp, \Re)$ which is called the best proximity point of φ [12]. Note that, the best proximity point becomes a fixed point in special case $\wp = \Re = \Upsilon$. Hence, best proximity point results are a natural generalization of fixed point results, For this reason, many authors have been studied to obtain existence of best proximity point [1, 3, 6, 7, 11, 28, 29]. Taking into account this situation, the inequality (1) was generalized different way from the results in the literature as in inequality (2). Hence, they obtained the existence of best proximity point of φ as follows:

Theorem 4. *Let (Υ,ρ) be a metric space, $\emptyset \neq \wp,\Re \subseteq \Upsilon$ and $\varphi : \wp \cup \Re \to \wp \cup \Re$ be a cyclic contraction mapping. Let $\varsigma_0 \in \wp$ and define $\varsigma_{n+1} = \varphi\varsigma_n$, for all $n \geq 1$. If $\{\varsigma_{2n-1}\}$ has a convergent subsequence in $\wp$, then there exists $\varsigma \in \wp$ such that $\rho(\varsigma,\varphi\varsigma) = \rho(\wp,\Re)$.*

On the other hand, to study of denotational semantics of dataflow networks, Matthews [20] obtained another generalization of mentioned principle by introducing a concept so called partial metric. Then, many authors obtained a lot of fixed point results in the settings of partial metric spaces [5, 13].

Now, we recall definition of the partial metric space and its topological properties (see for details [2, 17, 20]).

Definition 2 ([20])**.** Let Υ be a nonempty set, and $\sigma : \Upsilon \times \Upsilon \to [0,+\infty)$. If the following conditions hold:

p1) $\sigma(\varsigma,\varsigma) = \sigma(\varsigma,\xi) = \sigma(\xi,\xi)$ if and only if $\varsigma = \xi$,
p2) $\sigma(\varsigma,\varsigma) \leq \sigma(\varsigma,\xi)$,
p3) $\sigma(\varsigma,\xi) = \sigma(\xi,\varsigma)$,
p4) $\sigma(\varsigma,\vartheta) \leq \sigma(\varsigma,\xi) + \sigma(\xi,\vartheta) - \sigma(\xi,\xi)$,

for all $\varsigma,\xi,\vartheta \in \Upsilon$, then σ is said to be a partial metric. The pair (Υ,σ) is called partial metric space.

It is clear that every metric space is a partial metric space, but the converse may not be true. Indeed, let $\Upsilon = [0,+\infty) \times [0,+\infty)$ and a function $\sigma : \Upsilon \times \Upsilon \to [0,+\infty)$ defined as $\sigma(\varsigma,\xi) = \max\{\varsigma_1,\xi_1\} + |\varsigma_2 - \xi_2|$ for all $\varsigma = (\varsigma_1,\varsigma_2), \xi = (\xi_1,\xi_2) \in \Upsilon$. Then (Υ,σ) is not a metric space, but it is a partial metric space.

Let (Υ,σ) be a partial metric space. Then, σ generates T_0 topology τ_σ on Υ which has as a base the family open σ-balls

$$\{B_\sigma(\varsigma,\varepsilon) : \varsigma \in \Upsilon, \varepsilon > 0\},$$

where

$$B_\sigma(\varsigma,\varepsilon) = \{\xi \in \Upsilon : \sigma(\varsigma,\xi) < \sigma(\varsigma,\varsigma) + \varepsilon\},$$

for all $\varsigma \in \Upsilon$ and $\varepsilon > 0$.

Let $\{\varsigma_n\}$ be a sequence in Υ and $\varsigma \in \Upsilon$. It is easy to see that the sequence$\{\varsigma_n\}$ converges to ς with respect to τ_σ if and only if

$$\lim_{n\to+\infty} \sigma(\varsigma_n,\varsigma) = \sigma(\varsigma,\varsigma).$$

If $\lim_{n,m\to\infty} \sigma(\varsigma_n, \varsigma_m)$ exists and is finite, then $\{\varsigma_n\}$ is said to be Cauchy sequence. If every Cauchy sequence$\{\varsigma_n\}$ converges to a point ς in Υ such that

$$\lim_{n,m\to+\infty} \sigma(\varsigma_n, \varsigma_m) = \sigma(\varsigma, \varsigma),$$

then (Υ, σ) is said to be a complete partial metric space.

If (Υ, σ) is a partial metric space, then the function $\rho_\sigma : \Upsilon \times \Upsilon \to [0, +\infty)$ defined by

$$\rho_\sigma(\xi, \varsigma) = 2\sigma(\xi, \varsigma) - \sigma(\xi, \xi) - \sigma(\varsigma, \varsigma)$$

is a metric on Υ.

Now, we give the relations between partial metric space (Υ, σ) and corresponding metric space (Υ, ρ_σ) which are important for our main results.

Lemma 1. *Let (Υ, σ) be a partial metric space.*

(i) *$\{\varsigma_n\}$ is a Cauchy sequence in $(\Upsilon, \sigma) \Leftrightarrow \{\varsigma_n\}$ is a Cauchy sequence in (Υ, ρ_σ).*

(ii) *(Υ, σ) is a complete partial metric space $\Leftrightarrow (\Upsilon, \rho_\sigma)$ is a complete metric space.*

(iii) *Given a sequence $\{\varsigma_n\}$ in Υ and $\varsigma \in \Upsilon$. Then, we have*

$$\lim_{n\to+\infty} \rho_\sigma(\varsigma_n, \varsigma) = 0 \textit{if and only if} \sigma(\varsigma, \varsigma) = \lim_{n\to+\infty} \sigma(\varsigma_n, \varsigma) = \lim_{n,m\to+\infty} \sigma(\varsigma_n, \varsigma_m).$$

From Lemma 1 (iii), it can be seen that

$$\lim_{n\to+\infty} \rho_\sigma(\varsigma_n, \varsigma) = 0 = \lim_{n\to+\infty} \rho_\sigma(\xi_n, \xi) \text{implies} \lim_{n\to+\infty} \sigma(\varsigma_n, \xi_n) = \sigma(\varsigma, \xi).$$

The following definition is important for our results.

Definition 3. Let (Υ, σ) be a partial metric space and $g : \Upsilon \to \mathbb{R}$ be a function. If

$$g(\varsigma_0) \leq \lim_{n\to+\infty} \inf g(\varsigma_n)$$

for any sequence $\{\varsigma_n\}$ in Υ satisfying $\varsigma_n \to \varsigma_0$ as $n \to +\infty$, then the function g is called lower semicontinuous at a point $\varsigma_0 \in \Upsilon$

Now, we recall some definitions and notions related with best proximity point in the setting of partial metric. Let (Υ, σ) be a partial metric space and $\emptyset \neq \wp, \Re \subseteq \Upsilon$. Consider following subsets of $\wp$ and $\Re$, respectively:

$$\begin{aligned} \wp_0 &= \{\varsigma \in \wp : \sigma(\varsigma, \xi) = \sigma(\wp, \Re), \text{ for some } \xi \in \Re\}, \\ \Re_0 &= \{\xi \in \Re : \sigma(\varsigma, \xi) = \sigma(\wp, \Re), \text{ for some } \varsigma \in \wp\}. \end{aligned}$$

Definition 4. Let (Υ, ρ) be a partial metric space and $\emptyset \neq \wp, \Re \subseteq \Upsilon$ with $\wp_0 \neq \emptyset$. Then $(\wp, \Re)$ is said to have P-property if

$$\left.\begin{array}{l} \sigma(\varsigma_1, \xi_1) = \sigma(\wp, \Re) \\ \sigma(\varsigma_2, \xi_2) = \sigma(\wp, \Re) \end{array}\right\} \Rightarrow \sigma(\varsigma_1, \varsigma_2) = \sigma(\xi_1, \xi_2),$$

for all $\varsigma_1, \varsigma_2 \in \wp_0$ and $\xi_1, \xi_2 \in \Re_0$.

In this paper, introducing two new concepts of Feng-Liu type multivalued contraction mapping and cyclic Feng-Liu type multivalued contraction mapping, we obtain some new best proximity point results for multivalued mappings on partial metric spaces by considering Feng-Liu's technique. Finally, we provide examples to show the effectiveness of our results.

2. Main Result

We begin this section by introducing the definitions of best lower semicontinuity and Feng-Liu type multivalued contraction mapping.

Definition 5. Let (Υ, σ) be a partial metric space and $\emptyset \neq \wp, \Re \subseteq \Upsilon$ with $\wp_0 \neq \emptyset$. If

$$g(\varsigma_0, \xi_0) \leq \lim_{n\to+\infty} \inf g(\varsigma_n, \xi_n)$$

for every sequences $\{\varsigma_n\}$ in $\wp_0$ and $\{\xi_n\}$ in $\Re_0$ satisfying

$$\varsigma_n \to \varsigma_0 \in \wp \text{ and } \xi_n \to \xi_0 \in \Re \text{ as } n \to +\infty,$$

then a function $g : \wp \times \Re \to \mathbb{R}$ is called best lower semicontinuous at a point $(\varsigma_0, \xi_0) \in \wp \times \Re$.

Definition 6. Let (Υ, σ) be a partial metric space, $\emptyset \neq \wp, \Re \subseteq \Upsilon$ and $\varphi : \wp \to C(\Re)$ be a multivalued mapping. The mapping φ is said to be Feng-Liu type multivalued contraction mapping if for all $\varsigma \in \wp_0$ and $\xi \in \varphi\varsigma$ there exists $\vartheta \in \wp_0$ satisfying

$$\sigma(\xi, \vartheta) = \sigma(\wp, \Re)$$

and

$$\sigma(\xi, \varphi\vartheta) \leq \gamma\sigma(\varsigma, \vartheta),$$

for some $\gamma \in [0, 1)$.

Theorem 5. *Let (Υ, σ) be a complete partial metric space, $\emptyset \neq \wp, \Re \subseteq \Upsilon$ where $\wp, \Re$ are closed and $\varphi : \wp \to C(\Re)$ be a Feng-Liu type multivalued contraction mapping. Assume that the following conditions are satisfied:*

(i) $\wp_0 \neq \emptyset$ *and* $\varphi(\wp_0) \subseteq \Re_0$;
(ii) *the pair* $(\wp, \Re)$ *has the* P*-Property;*
(iii) *function* $g(\varsigma, \xi) = \sigma(\xi, \varphi\varsigma)$ *is best lower semicontinuous on* $\wp \times \Re$.

Then, φ *has a best proximity point* ς^* *in* $\wp$. *Moreover,* $\sigma(\varsigma^*, \varsigma^*) = 0$.

Proof. Let $\varsigma_0 \in \wp_0$ be an arbitrary point. Choose $\xi_0 \in \varphi\varsigma_0$. Then, since φ is a Feng-Liu type multivalued contraction mapping, there exists $\varsigma_1 \in \wp_0$ such that

$$\sigma(\xi_0, \varsigma_1) = \rho(\wp, \Re)$$

and

$$\sigma(\xi_0, \varphi\varsigma_1) \leq \gamma\sigma(\varsigma_0, \varsigma_1).$$

Let $\beta \in (\gamma, 1)$ be a constant. Then, from definition of infimum, there exists $\xi_1 \in \varphi\varsigma_1$ such that

$$\beta\sigma(\xi_0, \xi_1) \leq \sigma(\xi_0, \varphi\varsigma_1).$$

Again, since φ is a Feng-Liu type multivalued contraction mapping, there exists $\varsigma_2 \in \wp_0$ such that

$$\sigma(\xi_1, \varsigma_2) = \rho(\wp, \Re)$$

and

$$\sigma(\xi_1, \varphi\varsigma_2) \leq \gamma\sigma(\varsigma_1, \varsigma_2).$$

Also there exists $\xi_2 \in \varphi\varsigma_2$ such that

$$\beta\sigma(\xi_1, \xi_2) \leq \sigma(\xi_1, \varphi\varsigma_2).$$

Repeating this process, we can construct sequences $\{\varsigma_n\}$ and $\{\xi_n\}$ in $\wp$ and $\Re$ with $\xi_n \in \varphi\varsigma_n$ and

$$\sigma(\xi_n, \varsigma_{n+1}) = \sigma(\wp, \Re), \tag{3}$$

$$\sigma(\xi_n, \varphi\varsigma_{n+1}) \leq \gamma\sigma(\varsigma_n, \varsigma_{n+1}), \tag{4}$$

$$\beta\sigma(\xi_n, \xi_{n+1}) \leq \sigma(\xi_n, \varphi\varsigma_{n+1}),$$

for all $n \geq 1$. On the other hand, since the pair $(\wp, \Re)$ has the P-property, then we have

$$\sigma(\varsigma_n, \varsigma_{n+1}) = \sigma(\xi_{n-1}, \xi_n), \tag{5}$$

for all $n \geq 1$. Therefore, we get

$$\begin{aligned} \sigma(\varsigma_n, \varsigma_{n+1}) &= \sigma(\xi_{n-1}, \xi_n) \\ &\leq \frac{1}{\beta}\sigma(\xi_{n-1}, \varphi\varsigma_n) \\ &\leq \frac{\gamma}{\beta}\sigma(\varsigma_{n-1}, \varsigma_n), \end{aligned}$$

for all $n \geq 1$. Hence, we obtain

$$\begin{aligned} \sigma(\varsigma_n, \varsigma_{n+1}) &\leq \frac{\gamma}{\beta}\sigma(\varsigma_{n-1}, \varsigma_n) \\ &\leq \left(\frac{\gamma}{\beta}\right)^2 \sigma(\varsigma_{n-2}, \varsigma_{n-1}) \\ &\;\;\vdots \\ &\leq \left(\frac{\gamma}{\beta}\right)^n \sigma(\varsigma_0, \varsigma_1) \end{aligned}$$

and so

$$\lim_{n\to+\infty} \sigma(\varsigma_n, \varsigma_{n+1}) = 0. \tag{6}$$

From (4) and (6), we also have

$$\lim_{n\to+\infty}\sigma(\xi_n,\varphi\varsigma_{n+1})=0. \tag{7}$$

Then, for all $m>n>n_0$, we have

$$\begin{aligned}\sigma(\varsigma_n,\varsigma_m) &\le \sigma(\varsigma_n,\varsigma_{n+1})+\sigma(\varsigma_{n+1},\varsigma_{n+2})+\cdots+\sigma(\varsigma_{m-1},\varsigma_m)\\ &\le \left(\frac{\gamma}{\beta}\right)^n\sigma(\varsigma_0,\varsigma_1)+\left(\frac{\gamma}{\beta}\right)^{n+1}\sigma(\varsigma_0,\varsigma_1)+\cdots+\left(\frac{\gamma}{\beta}\right)^{m-1}\sigma(\varsigma_0,\varsigma_1)\\ &= \left(\frac{\gamma}{\beta}\right)^n\sigma(\varsigma_0,\varsigma_1)\left(1+\frac{\gamma}{\beta}+\cdots+\left(\frac{\gamma}{\beta}\right)^{m-n-1}\right)\\ &= \left(\frac{\gamma}{\beta}\right)^n\sigma(\varsigma_0,\varsigma_1)\frac{1-\left(\frac{\gamma}{\beta}\right)^{m-n}}{1-\frac{\gamma}{\beta}}\\ &\le \frac{\left(\frac{\gamma}{\beta}\right)^n}{1-\frac{\gamma}{\beta}}\sigma(\varsigma_0,\varsigma_1).\end{aligned}$$

Hence, $\lim_{n,m\to+\infty}\sigma(\varsigma_n,\varsigma_m)=0$, and so $\{\varsigma_n\}$ is a Cauchy sequence in $\wp$. Since $\wp$ is closed subset of the complete partial metric space (Υ,σ), there exists $\varsigma^*\in\wp$ such that

$$\lim_{n,m\to+\infty}\sigma(\varsigma_n,\varsigma_m)=\lim_{n\to+\infty}\sigma(\varsigma_n,\varsigma^*)=\sigma(\varsigma^*,\varsigma^*)=0. \tag{8}$$

From (5), we also get $\{\xi_n\}$ is a Cauchy sequence in $\Re$. Similarly, there exists $\xi^*\in\Re$ such that

$$\lim_{n,m\to+\infty}\sigma(\xi_n,\xi_m)=\lim_{n\to+\infty}\sigma(\xi_n,\xi^*)=\sigma(\xi^*,\xi^*)=0.$$

Letting $n\to+\infty$ in (3), we have

$$\sigma(\varsigma^*,\xi^*)=\rho(\wp,\Re). \tag{9}$$

Finally, since $g(\varsigma,\xi)=\sigma(\xi,\varphi\varsigma)$ is best lower semicontinuous on $\wp\times\Re$ and for the sequences $\{\varsigma_{n+1}\}$ in $\wp_0$ and $\{\xi_n\}$ in $\Re_0$, we have $\varsigma_{n+1}\to\varsigma^*$ and $\xi_n\to\xi^*$ as $n\to+\infty$, from (7) we get

$$\sigma(\xi^*,\varphi\varsigma^*)=g(\varsigma^*,\xi^*)\le\liminf g(\varsigma_{n+1},\xi_n)=\liminf\sigma(\xi_n,\varphi\varsigma_{n+1})=0,$$

hence $\xi^*\in\overline{\varphi\varsigma^*}=\varphi\varsigma^*$. Therefore, from (9) we have

$$\sigma(\wp,\Re)\le\sigma(\varsigma^*,\varphi\varsigma^*)\le\sigma(\varsigma^*,\xi^*)=\sigma(\wp,\Re),$$

that is,

$$\sigma(\varsigma^*,\varphi\varsigma^*)=\rho(\wp,\Re).$$

Hence, φ has a best proximity point ς^* in $\wp$. Moreover, from (8) we have $\sigma(\varsigma^*,\varsigma^*)=0$. □

Example 1. Let $\Upsilon = [0, +\infty) \times [0, +\infty)$ and $\sigma : \Upsilon \times \Upsilon \to [0, +\infty)$ be a function defined by

$$\sigma(\varsigma, \xi) = \max\{\varsigma_1, \xi_1\} + |\varsigma_2 - \xi_2|,$$

for all $\varsigma = (\varsigma_1, \varsigma_2), \xi = (\xi_1, \xi_2) \in \Upsilon$. Then, (Υ, σ) is a complete partial metric space. Let $\wp = \{(u, 1) : u \in [0, +\infty)\}$ and $\Re = \{(v, 0) : v \in [0, +\infty)\}$ be subsets of Υ. It can be easily seen that $\sigma(\wp, \Re) = 1$ and $\wp,\Re$ are closed subsets. Also, we have $\wp_0 = \{(0, 1)\}$, $\Re_0 = \{(0, 0)\}$and the pair $(\wp, \Re)$ has the P-Property. Now, we define a mapping $\varphi : \wp \to C(\Re)$ as

$$\varphi\varsigma = \begin{cases} \{(0,0)\}, & \varsigma = (0,1), \\ \{(v,0) : v \geq u\}, & \varsigma = (u,1), \end{cases}$$

for all $\varsigma = (u, 1) \in \wp$. Hence, we have $\varphi(\wp_0) \subseteq \Re_0$. It can be easily seen that φ is a Feng-Liu type multivalued contraction mapping for all $\gamma \in [0, 1)$ and the function $g(\varsigma, \xi) = \sigma(\xi, \varphi\varsigma)$ is best lower semicontinuous on $\wp \times \Re$. Then, all hypotheses of Theorem 5 hold. Therefore, φ has a best proximity point ς^* in $\wp$. Moreover, $\sigma(\varsigma^*, \varsigma^*) = \sigma((0, 1), (0, 1)) = 0$.

If we take $\wp = \Re = \Upsilon$ in Theorem 5, we can present the following result.

Corollary 1. *Let (Υ, σ) be a complete partial metric space and $\varphi : \Upsilon \to C(\Upsilon)$ be a multivalued mapping. If there exists $\gamma \in (0, 1)$ such that*

$$\sigma(\xi, \varphi\xi) \leq \gamma\sigma(\varsigma, \xi),$$

for all $\varsigma \in \Upsilon$ and $\xi \in \varphi\varsigma$ and $g(\varsigma, \xi) = \sigma(\xi, \varphi\varsigma)$ is lower semicontinuous on $\Upsilon \times \Upsilon$, then φ has a fixed point in Υ.

Now, we introduce the definition of cyclic Feng-Liu type multivalued contraction mapping.

Definition 7. Let (Υ, σ) be a partial metric space, $\emptyset \neq \wp, \Re \subseteq \Upsilon$ and $\varphi : \wp \cup \Re \to P(\wp \cup \Re)$ be a multivalued mapping. Then φ is said to be cyclic Feng-Liu type multivalued contraction mapping if there exist $\beta, \gamma \in (0, 1)$ with $\gamma < \beta$ satisfying for all $\varsigma \in \wp$ with $\sigma(\varsigma, \varphi\varsigma) > \sigma(\wp, \Re)$, there exists $\xi \in \varphi^{\varsigma}_{\beta} = \{\xi \in \varphi\varsigma : \beta\sigma(\varsigma, \xi) \leq \sigma(\varsigma, \varphi\varsigma)\}$ such that

$$\sigma(\xi, \varphi\xi) \leq \gamma\sigma(\varsigma, \xi) + (\beta - \gamma)\sigma(\wp, \Re).$$

Theorem 6. *Let (Υ, σ) be a partial metric space and $\emptyset \neq \wp, \Re \subseteq \Upsilon$. Let $\varphi : \wp \cup \Re \to P(\wp \cup \Re)$ be a cyclic Feng-Liu type multivalued contraction mapping, $\varsigma_0 \in \wp$ and define a sequence $\{\varsigma_n\}$ by $\varsigma_{n+1} = \varphi\varsigma_n$ for all $n \geq 1$. If $\{\varsigma_{2n}\}$ has a convergent subsequence in $\wp$ and $g(\varsigma) = \rho(\varsigma, \varphi\varsigma)$ is lower semicontinuous on $\wp$, then φ has a best proximity point in $\wp$.*

Proof. Let $\varsigma_0 \in \wp$ be an arbitrary point. If $\sigma(\varsigma_0, \varphi\varsigma_0) = \sigma(\wp, \Re)$, then the proof is finished. Then, we assume $\sigma(\varsigma_0, \varphi\varsigma_0) > \sigma(\wp, \Re)$. Since φ is cyclic Feng-Liu type multivalued contraction mapping, there exists $\varsigma_1 \in \varphi^{\varsigma_0}_{\beta}$ such that

$$\sigma(\varsigma_1, \varphi\varsigma_1) \leq \gamma\sigma(\varsigma_0, \varsigma_1) + (\beta - \gamma)\sigma(\wp, \Re).$$

If $\sigma(\varsigma_1, \varphi\varsigma_1) = \sigma(\wp, \Re)$, then the proof is finished. Hence, we assume

$$\sigma(\varsigma_1, S\varsigma_1) > \sigma(\wp, \Re).$$

Since φ is cyclic Feng-Liu type multivalued contraction mapping, there exists $\varsigma_2 \in \varphi_\beta^{\varsigma_1}$ such that

$$\sigma(\varsigma_2, \varphi\varsigma_2) \leq \gamma\sigma(\varsigma_1, \varsigma_2) + (\beta - \gamma)\sigma(\wp, \Re).$$

Repeating this process, we can construct a sequence $\{\varsigma_n\}$ in $\wp \cup \Re$ (we can suppose $\sigma(\varsigma_{2n+1}, \varphi\varsigma_{2n+1}) > \sigma(\wp, \Re)$ for all $n \geq 1$, otherwise the proof is finished)

$$\sigma(\varsigma_{2n+1}, \varphi\varsigma_{2n+1}) \leq \gamma\sigma(\varsigma_{2n}, \varsigma_{2n+1}) + (\beta - \gamma)\sigma(\wp, \Re), \tag{10}$$

for all $n \geq 1$. Since $\varsigma_{2n+1} \in \varphi_\beta^{\varsigma_{2n}}$, we have

$$\beta\sigma(\varsigma_{2n}, \varsigma_{2n+1}) \leq \sigma(\varsigma_{2n}, \varphi\varsigma_{2n}), \tag{11}$$

for all $n \geq 1$. Hence, from (10) and (11) we get

$$\beta\sigma(\varsigma_{2n}, \varsigma_{2n+1}) \leq \gamma\sigma(\varsigma_{2n-1}, \varsigma_{2n}) + (\beta - \gamma)\sigma(\wp, \Re),$$

for all $n \geq 1$. Therefore, we have

$$\begin{aligned}
\sigma(\wp, \Re) &\leq \sigma(\varsigma_{2n}, \varsigma_{2n+1}) \\
&\leq \frac{\gamma}{\beta}\sigma(\varsigma_{2n-1}, \varsigma_{2n}) + \left(1 - \frac{\gamma}{\beta}\right)\sigma(\wp, \Re) \\
&\leq \frac{\gamma}{\beta}\left\{\frac{\gamma}{\beta}\sigma(\varsigma_{2n-2}, \varsigma_{2n-1}) + \left(1 - \frac{\gamma}{\beta}\right)\sigma(\wp, \Re)\right\} + \left(1 - \frac{\gamma}{\beta}\right)\sigma(\wp, \Re) \\
&= \left(\frac{\gamma}{\beta}\right)^2\sigma(\varsigma_{2n-2}, \varsigma_{2n-1}) + \left(1 + \frac{\gamma}{\beta}\right)\left(1 - \frac{\gamma}{\beta}\right)\sigma(\wp, \Re) \\
&\leq \left(\frac{\gamma}{\beta}\right)^2\left\{\frac{\gamma}{\beta}\sigma(\varsigma_{2n-3}, \varsigma_{2n-2}) + \left(1 - \frac{\gamma}{\beta}\right)\sigma(\wp, \Re)\right\} \\
&\quad + \left(1 + \frac{\gamma}{\beta}\right)\left(1 - \frac{\gamma}{\beta}\right)\sigma(\wp, \Re) \\
&= \left(\frac{\gamma}{\beta}\right)^3\sigma(\varsigma_{2n-3}, \varsigma_{2n-2}) + \left(1 + \frac{\gamma}{\beta} + \left(\frac{\gamma}{\beta}\right)^2\right)\left(1 - \frac{\gamma}{\beta}\right)\sigma(\wp, \Re) \\
&\quad \vdots \\
&\leq \left(\frac{\gamma}{\beta}\right)^{2n}\sigma(\varsigma_0, \varsigma_1) + \left(1 + \frac{\gamma}{\beta} + \cdots + \left(\frac{\gamma}{\beta}\right)^{2n-1}\right)\left(1 - \frac{\gamma}{\beta}\right)\sigma(\wp, \Re) \\
&= \left(\frac{\gamma}{\beta}\right)^{2n}\sigma(\varsigma_0, \varsigma_1) + \left(1 - \left(\frac{\gamma}{\beta}\right)^{2n}\right)\sigma(\wp, \Re),
\end{aligned}$$

for all $n \geq 1$. Since $\frac{\gamma}{\beta} \in (0,1)$, we have

$$\lim_{n\to+\infty} \sigma(\varsigma_{2n}, \varsigma_{2n+1}) = \sigma(\wp, \Re). \tag{12}$$

Now, since $\{\varsigma_{2n}\}$ has a convergent subsequence, then there exists a subsequence $\{\varsigma_{2n_k}\}$ of $\{\varsigma_{2n}\}$ such that $\varsigma_{2n_k} \to \varsigma^* \in \wp$. Because of the fact that g is a lower semicontinuous function, from (12), we have

$$\begin{aligned}
\sigma(\wp, \Re) &\leq \sigma(\varsigma^*, \varphi\varsigma^*) \\
&= g(\varsigma^*) \\
&\leq \lim_{k\to+\infty} \inf g(\varsigma_{2n_k}) \\
&= \lim_{k\to+\infty} \inf \sigma(\varsigma_{2n_k}, \varphi\varsigma_{2n_k}) \\
&\leq \lim_{k\to+\infty} \inf \sigma(\varsigma_{2n_k}, \varsigma_{2n_k+1}) \\
&= \sigma(\wp, \Re).
\end{aligned}$$

Therefore, φ has a best proximity point ς^* in $\wp$. □

Remark 1. Note that, if $g(\varsigma) = \rho(\varsigma, \varphi\varsigma)$ is lower semicontinuous on $\Re$ and $\{\varsigma_{2n+1}\}$ has a convergent subsequence in Theorem 6, then φ has a best proximity point in $\Re$.

Example 2. Let $\Upsilon = [0, +\infty)$ and $\sigma : \Upsilon \times \Upsilon \to [0, +\infty)$ be a function defined by

$$\sigma(\varsigma, \xi) = \max\{\varsigma, \xi\},$$

for $\varsigma, \xi \in \Upsilon$. Then, it can be seen that (Υ, σ) is a partial metric space. Let $\wp = \left\{\frac{1}{3^{2n}} : n \geq 1\right\} \cup \{0\}$ and $\Re = \left\{\frac{1}{3^{2n+1}} : n \geq 1\right\}$ be subsets of Υ. Hence, we have $\sigma(\wp, \Re) = 0$. Now, we define a mapping $\varphi : \wp \cup \Re \to P(\wp \cup \Re)$ as

$$\varphi\varsigma = \begin{cases} \left\{\frac{1}{3^{2n+1}}, \frac{1}{3^{2n+3}}, \cdots\right\}, & \varsigma = \frac{1}{3^{2n}}, \\ \Re, & \varsigma = 0, \\ \left\{\frac{1}{3^{2n+2}}, \frac{1}{3^{2n+4}}, \cdots\right\}, & \varsigma = \frac{1}{3^{2n+1}}, \end{cases}$$

for all $n \geq 1$. We shall show that φ is a cyclic Feng-Liu type multivalued contraction mapping with $\gamma = \frac{1}{3}$ and $\beta \in (\gamma, 1)$. Indeed, let $\varsigma \in \wp \cup \Re$ with $\sigma(\varsigma, \varphi\varsigma) > \sigma(\wp, \Re)$. We consider the following cases:
Case 1: Let $\varsigma = \frac{1}{3^{2n}}$ for all $n \geq 1$. Then, for $\xi = \frac{1}{3^{2n+1}}$, we have

$$\beta\sigma(\varsigma, \xi) = \frac{\beta}{3^{2n}} \leq \frac{1}{3^{2n}} = \max\left\{\frac{1}{3^{2n}}, \varphi\frac{1}{3^{2n}}\right\} = \sigma(\varsigma, \varphi\varsigma)$$

and

$$\sigma\left(\frac{1}{3^{2n+1}}, \varphi\frac{1}{3^{2n+1}}\right) = \frac{1}{3^{2n+1}}$$

$$\leq \frac{1}{3}\max\left\{\frac{1}{3^{2n}}, \frac{1}{3^{2n+1}}\right\}$$
$$= \gamma\sigma(\varsigma,\xi) + (\beta-\gamma)\sigma(\wp,\Re).$$

Case 2: Let $\varsigma = \frac{1}{3^{2n+1}}$ for all $n \geq 1$. Then, for $\xi = \frac{1}{3^{2n+2}}$, we have

$$\beta\sigma(\varsigma,\xi) = \frac{\beta}{3^{2n+1}} \leq \frac{1}{3^{2n+1}} = \max\left\{\frac{1}{3^{2n+1}}, \varphi\frac{1}{3^{2n+1}}\right\} = \sigma(\varsigma,\varphi\varsigma)$$

and

$$\begin{aligned}\sigma\left(\frac{1}{3^{2n+2}}, \varphi\frac{1}{3^{2n+2}}\right) &= \frac{1}{3^{2n+2}} \\ &\leq \frac{1}{3}\max\left\{\frac{1}{3^{2n+1}}, \frac{1}{3^{2n+2}}\right\} \\ &= \gamma\sigma(\varsigma,\xi) + (\beta-\gamma)\sigma(\wp,\Re).\end{aligned}$$

Hence, φ is a cyclic Feng-Liu type multivalued contraction mapping with $\gamma = \frac{1}{3}$ and $\beta \in (\gamma, 1)$. Also, every sequence in $\wp$ has a convergent subsequence. Further, it is clear that

$$g(\varsigma) = \begin{cases} \frac{1}{3^{2n}}, & \varsigma = \frac{1}{3^{2n}}, \\ 0, & \varsigma = 0 \end{cases}$$

is a lower semicontinuous function on $\wp$. Therefore, all condition of Theorem 6 hold, and so φ best proximity point ς^* in $\wp$.

If we take $\wp = \Re = \Upsilon$ and $\sigma(\wp,\Re) = 0$ in Theorem 6, we obtain the following result.

Corollary 2. *Let (Υ,σ) be a partial metric space, $\varphi : \Upsilon \to P(\Upsilon)$ be a multivalued mapping and define a sequence $\{\varsigma_n\}$ by $\varsigma_{n+1} = \varphi\varsigma_n$ for all $n \geq 1$ with any initial point $\varsigma_0 \in \Upsilon$. Assume that $\{\varsigma_{2n}\}$ has a convergent subsequence in Υ and $g(\varsigma) = \rho(\varsigma,\varphi\varsigma)$ is lower semicontinuous on Υ. If for all $\varsigma \in X$ there exists $\xi \in I_\beta^\varsigma$ satisfying*

$$\sigma(\xi,\varphi\xi) \leq \gamma\sigma(\varsigma,\xi),$$

where $\beta,\gamma \in (0,1)$ with $\gamma < \beta$, then φ has a fixed point in Υ.

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MUSTAFA ASLANTAS
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ÇANKIRI KARATEKIN UNIVERSITY
18100 ÇANKIRI
TURKEY
E-mail address: maslantas@karatekin.edu.tr

DOAA RIYADH ABED AL-ZUHAIRI
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ÇANKIRI KARATEKIN UNIVERSITY
18100 ÇANKIRI
TURKEY
E-mail address: doaariyadh68@gmail.com