Lower bounds for blow up time of the 
p-Laplacian equation with damping term

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Abstract. In this work deals with the p-Laplacian wave equation with damping terms in a bounded domain. Under suitable conditions, we obtain a lower bounds for the blow up time. Our result extends the recent results obtained by Baghaei (2017) and Zhou (2015), for $p > 2$.

1. Introduction

In this work, we study the following p-Laplacian equation with strong and weak damping terms

\[ \begin{cases} u_{tt} - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) - a\Delta u_t + bu_t = |u|^{q-2} u, & x \in \Omega, \ t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial \Omega, \ t > 0, \end{cases} \]

where $\Omega \subset \mathbb{R}^n$ $(n = 2, 3, \ldots)$ is a bounded domain with a smooth boundary $\partial \Omega$; and $u_0(x) \in W_0^{1,p}(\Omega)$, $u_1(x) \in L^2(\Omega)$, $p > 2$. $a \geq 0$, $b > -a\rho_1$ with $\rho_1 > 0$ is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions and

\[ \begin{cases} 2 < q < \infty & \text{if } n = 2, \\ 2 < q \leq \begin{cases} \frac{2n}{n-2}, & \text{for } a > 0 \\ \frac{2n-2}{n-2}, & \text{for } a = 0 \end{cases} & \text{if } n \geq 3. \end{cases} \]

When $p = 2$, (1) is reduced to the following wave equation

\[ u_{tt} - \Delta u - a\Delta u_t + bu_t = |u|^{q-2} u. \]

In 2006, Gazzalo and Squassina [2] studied problem (3). They proved the local existence, global existence and blow up of solutions. Later, some authors studied the lower bounds for the blow up time under some conditions,
see [1,7,9]. Also, in recent years, some authors investigate the lower bounds for blow up time for hyperbolic type equations, see [3,4,6,8].

Inspired by the above papers, in this paper we consider the lower bound for the blow up time of solutions (1). Our result improves the recent results obtained by Baghaei [1] and Zhou [9], for \( p > 2 \).

Now, for the problem (1), we define the functionals, the potential well depth \( d \) and the unstable set \( U \), are given as

\[
I (u) = \| \nabla u \|^p_p - \| u \|^q_q, \\
J (u) = \frac{1}{p} \| \nabla u \|^p_p - \frac{1}{q} \| u \|^q_q, \\
E (t) := E (u, u_t) = J (u) + \frac{1}{2} \| u_t \|^2, \\
d = \inf_{u \in H^1_0(\Omega) / \{0\}} \max_{\lambda \geq 0} J (\lambda u), \\
U = \left\{ u \in W^{1,p}_0 : J (u) \leq d \text{ and } I (u) < 0 \right\}
\]

with above assumptions, the results on the existence of local solutions and the nonexistence of the solutions of the problem (1) can be reformulated as follows:

(i) Assume that \( q \) satisfy (2), then there exist \( T > 0 \) and a unique solution \( u \) problem (1) satisfying

\[
u \in C \left( [0,T), W^{1,p}_0(\Omega) \right) \cap C^1 \left( [0,T), L^2(\Omega) \right) \cap C^2 \left( [0,T), H^{-1}(\Omega) \right), \\
u_t \in L^2 \left( (0,T), H^1_0(\Omega) \right).
\]

(ii) The solution \( u \) of problem (1) blows up at a finite time \( T \) if and only if there exists \( \bar{t} \in [0,T) \) such that \( u (\bar{t}) \in U \) and \( E \left( u (\bar{t}), u_t (\bar{t}) \right) \leq d \).

\[
E (t) = \frac{1}{2} \| u_t \|^2 + \frac{1}{p} \| \nabla u \|^p_p - \frac{1}{q} \| u \|^q_q, \\
E (0) = \frac{1}{2} \| u_1 \|^2 + \frac{1}{p} \| \nabla u_0 \|^p_p - \frac{1}{q} \| u_0 \|^q_q.
\]

Lemma 1 ([5]). Let \( \Omega \subset R^n \) \((n \geq 2)\). Let \( u \) be a non-negative piecewise \( C^1 \) function defined in \( \Omega \) with \( u (x) = 0, x \in \partial \Omega \). So

\[
\int_\Omega |u|^{2s} \, ds \leq \delta \left( \int_\Omega |\nabla u|^2 \, dx \right)^s
\]

satisfies for

\[
\begin{align*}
s & > 1, \quad \text{if } n = 2, \\
1 & < s < \frac{n}{n-2}, \quad \text{if } n \geq 3,
\end{align*}
\]

with \( \delta = \left( \frac{n-1}{n^2} \right)^{2s} |\Omega|^{1-\frac{(n-2)}{n}s} \).
Theorem 1. Assume that \( q > \frac{p+2}{2} \) holds. Let \( u(x,t) \) be the solution of the problem (1), which blows up at a finite time \( T^* \). Then

\[
\int_{\psi(0)}^{\infty} qM + \tau + \varepsilon qp \frac{2^{(q-1)} - 2}{p} M^{\frac{p-2q+2}{p-2}} \tau^{\frac{2-(q-1)}{p-2}} \leq T^*,
\]

where

\[
(7) \begin{cases}
\psi(0) = \|u_0\|_q^q, \\
\varepsilon = \left( \frac{n-1}{n^2} \right)^{2(q-1)} |\Omega|^{rac{1}{n}} - \left( \frac{n-2}{n} \right)^{q-1} \Omega, \\
M = \frac{1}{2} \|u_1\|^2 + \frac{1}{p} \|\nabla u_0\|^p - \frac{1}{q} \|u_0\|^q.
\end{cases}
\]

Proof. Multiplying the equation of (1) by \( u_t \), then integrating the result over \( \Omega \), we have

\[
\int_{\Omega} u_t \left[ u_{tt} - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) - a\Delta u + bu \right] \, dx = \int_{\Omega} u_t |u|^{q-2} u \, dx,
\]

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_t|^2 \, dx + \frac{1}{p} \frac{d}{dt} \int_{\Omega} |\nabla u|^p \, dx + a \int_{\Omega} |\nabla u_t|^2 \, dx + b \int_{\Omega} |u_t|^2 \, dx = \frac{1}{q} \frac{d}{dt} \int_{\Omega} |u|^q \, dx.
\]

Since

\[
\frac{d}{dt} \left( \frac{1}{2} \|u_t\|^2 + \frac{1}{p} \|\nabla u\|^p - \frac{1}{q} \|u\|^q \right) = -a \|\nabla u_t\|^2 - b \|u_t\|^2,
\]

the above calculations imply \( E'(t) \leq 0 \), with

\[
E(t) := \frac{1}{2} \|u_t\|^2 + \frac{1}{p} \|\nabla u\|^p - \frac{1}{q} \|u\|^q.
\]

So,

\[
(8) \quad E(t) \leq E(0) = M,
\]

where \( M \) is defined in (7) and

\[
(9) \quad \|u_t\|^2 + \frac{2}{p} \|\nabla u\|^p = 2E(t) + \frac{2}{q} \|u\|^q \leq 2M + \frac{2}{q} \|u\|^q.
\]

Now define

\[
\psi(t) = \|u\|^q_q.
\]

then thanks to Cauchy’s and embedding \( (L^p(\Omega) \hookrightarrow L^2(\Omega), \ p > 2) \) inequalities and Lemma 1 with \( s = q - 1 \) and \( \delta = \varepsilon \), we obtain

\[
\psi'(t) = q \int_{\Omega} |u|^{q-2} uu_t \, dx \leq \frac{q}{2} \left( \|u_t\|^2 + \|u\|^2_{2(q-1)} \right) \leq \frac{q}{2} \left( \|u_t\|^2 + \varepsilon \|\nabla u\|^2_{2(q-1)} \right)
\]
\[
\frac{q}{2} \left( \|u_t\|^2 + \varepsilon \|\nabla u\|_p^{2(q-1)} \right) \\
= \frac{q}{2} \left( \|u_t\|^2 + \varepsilon \left( \|\nabla u\|_p^{p(2q-1) - 1} \right) \right).
\]

By the (9) and definition of \( \psi \), we get
\[
\psi'(t) \leq \frac{q}{2} \left[ 2M + \frac{2}{q} \psi(t) + \varepsilon \left( \frac{p}{2} \right)^{2(q-1)} \left( 2M + \frac{2}{q} \psi(t) \right)^{2(q-1)} \right] \\
\leq \frac{q}{2} \left[ 2M + \frac{2}{q} \psi(t) + \varepsilon \left( \frac{p}{2} \right)^{2(q-1)} \left( 2M \frac{2}{p} + \frac{2}{q} \psi(t) \right)^{2(q-1)} \right] \\
= \frac{q}{2} \left[ 2M + \frac{2}{q} \psi(t) + \varepsilon \left( \frac{p}{2} \right)^{2(q-1)} \left( 2M \frac{2}{p} + \frac{2}{q} \psi(t) \right)^{2(q-1)} \right] \\
= qM + \psi(t) + \varepsilon \left( \frac{p}{2} \right)^{2(q-1)} \left( 2M \frac{2}{p} + \frac{2}{q} \psi(t) \right)^{2(q-1)} \\
+ \varepsilon \left( \frac{p}{2} \right)^{2(q-1)} \left( \frac{2}{q} \psi(t) \right)^{2(q-1)} (t),
\]

where used \((a + b)^\omega \leq 2^\omega (a^\omega + b^\omega)\). Thus, the above inequality implies that
\[
\frac{d\psi(t)}{dt} \leq qM + \psi(t) + \varepsilon \left( \frac{p}{2} \right)^{2(q-1)} 2^{\frac{2(q-1)}{p}} 2^{-2} M^{\frac{2(q-1)}{p}} \\
+ \varepsilon \left( \frac{p}{2} \right)^{2(q-1)} q^{-2g+2} 2^{\frac{2(q-1)}{p}} 2^{-2} \psi \left( \frac{2}{q} \right) (t).
\]

(10)

Since \( \lim_{t \to T^*} \psi(t) = \infty \), we get from (10)
\[
\int_{\psi(0)}^{\infty} qM + \tau + \varepsilon \left( \frac{p}{2} \right)^{2(q-1)} 2^{\frac{2(q-1)}{p}} 2^{-2} M^{\frac{2(q-1)}{p}} + \varepsilon \left( \frac{p}{2} \right)^{2(q-1)} q^{-2g+2} 2^{\frac{2(q-1)}{p}} 2^{-2} \psi \left( \frac{2}{q} \right) (t) \leq T^*.
\]

This completes the proof of the main theorem. \( \square \)

2. Conclusion

In this work, we obtained the lower bounds for blow up time of the nonlinear \( p \)-Laplacian equation with damping terms in a bounded domain. This improves and extends many results in the literature.
REFERENCES


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