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Applications of Borel distribution series on holomorphic and bi-univalent functions

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Abstract. In present manuscript, we introduce and study two families $\mathcal{B}_\Sigma(\lambda, \delta; \alpha)$ and $\mathcal{B}^*_\Sigma(\lambda, \delta; \beta)$ of holomorphic and bi-univalent functions which involve the Borel distribution series. We establish upper bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in each of these families. We also point out special cases and consequences of our results.

1. Introduction

We indicate by $\mathcal{A}$ the family of functions which are holomorphic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and have the following normalized type:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

We also indicate by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. According to the Koebe one-quarter theorem [8], every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}\big(f(z)\big) = z, \quad (z \in \mathbb{U}),$$

and

$$f\big(f^{-1}(w)\big) = w, \; quad \, |w| < r_0(f); \; r_0(f) \geqq \tfrac{1}{4}),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ stand for the class of normalized bi-univalent functions

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in $\mathbb{U}$ given by (1). For a brief historical account and for several interesting examples of functions in the class Σ, see the pioneering work on this subject by Srivastava *et al.* [18], which actually revived the study of bi-univalent functions in recent years. From the work of Srivastava *et al.* [18], we choose to recall here the following examples of functions in the class Σ :

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

We notice that the class Σ is not empty. However, the Koebe function is not a member of Σ.

In a considerably large number of sequels to the aforementioned work of Srivastava *et al.* [18], several different subclasses of the bi-univalent function class Σ were introduced and studied analogously by the many authors (see, for example, [1–7, 9–11, 13, 14, 16, 17, 19–28, 30, 31]), but only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin expansion (1) were obtained in many of these recent papers. The problem to find the general coefficient bounds on the Taylor-Maclaurin coefficients

$$|a_n|, \quad (n \in \mathbb{N} \setminus \{1,2\};\ \mathbb{N} := \{1,2,3,\dots\})$$

for functions $f \in \Sigma$ is still not completely addressed for many of the subclasses of the bi-univalent function class Σ (see, for example, [14, 19, 21]).

Recently, Srivastava [12] in his survey-cum-expository review article, explored the mathematical application of q-calculus, fractional q- calculus and fractional q-differential operators in Geometric Function Theory.

A discrete random variable x is said to have a Borel distribution, if it takes the values $1, 2, 3, \dots,$ with the probabilities

$$\frac{e^{-\delta}}{1!}, \quad \frac{2\delta e^{-2\delta}}{2!}, \quad \frac{9\delta^2 e^{-3\delta}}{3!}, \dots,$$

respectively, where δ are called the parameters. Hence

$$\mathrm{P}rob(x = r) = \frac{(\delta r)^{r-1}\, e^{-\delta r}}{r!}, \quad (r = 1, 2, 3, \dots).$$

Wanas and Khuttar [29] introduced the following power series whose coefficients are probabilities of the Borel distribution:

$$\mathcal{M}(\delta, z) = z + \sum_{k=2}^{\infty} \frac{(\delta(k-1))^{k-2}\, e^{-\delta(k-1)}}{(k-1)!} z^k, \quad (z \in \mathbb{U};\ 0 < \delta \leqq 1).$$

We note by the familiar Ratio Test that the radius of convergence of the above series is infinity.

Now, we considered the linear operator $\mathcal{B}_\delta : \mathcal{A} \longrightarrow \mathcal{A}$ which is defined as follows:

$$\mathcal{B}_\delta f(z) = \mathcal{M}(\delta, z) * f(z) = z + \sum_{k=2}^{\infty} \frac{(\delta(k-1))^{k-2}\, e^{-\delta(k-1)}}{(k-1)!} a_k z^k, \quad z \in \mathbb{U},$$

where $(*)$ indicate the Hadamard product (or convolution) of two series.

Very recently, Srivastava and El-Deeb [15] have introduced some applications of the Borel distribution.

We now recall the following lemma that will be used to prove our main results.

Lemma 1 (see [8]). *If $h \in \mathcal{P}$, then*

$$|c_k| \leqq 2, \quad (\forall k \in \mathbb{N}),$$

where $\mathcal{P}$ is the family of all functions h, holomorphic in $\mathbb{U}$, for which

$$\Re\big(h(z)\big) > 0, \quad (z \in \mathbb{U}),$$

with

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad (z \in \mathbb{U}).$$

2. Coefficient estimates for the bi-univalent function class $\mathcal{B}_\Sigma(\lambda, \delta; \alpha)$

In this section, we first define the bi-univalent function class $\mathcal{B}_\Sigma(\lambda, \delta; \alpha)$.

Definition 1. A function $f \in \Sigma$, given by (1), in said to be the bi-univalent function class $\mathcal{B}_\Sigma(\lambda, \delta; \alpha)$ if it satisfies the following conditions:

$$\left|\arg\left(1 + \frac{z\,(\mathcal{B}_\delta f(z))'}{\mathcal{B}_\delta f(z)} + \frac{z\,(\mathcal{B}_\delta f(z))''}{(\mathcal{B}_\delta f(z))'} - \frac{\lambda z^2\,(\mathcal{B}_\delta f(z))'' + z\,(\mathcal{B}_\delta f(z))'}{\lambda z\,(\mathcal{B}_\delta f(z))' + (1-\lambda)\mathcal{B}_\delta f(z)}\right)\right| < \frac{\alpha\pi}{2} \tag{3}$$

and

$$\left|\arg\left(1 + \frac{w\,(\mathcal{B}_\delta g(w))'}{\mathcal{B}_\delta g(w)} + \frac{w\,(\mathcal{B}_\delta g(w))''}{(\mathcal{B}_\delta g(w))'} - \frac{\lambda w^2\,(\mathcal{B}_\delta g(w))'' + w\,(\mathcal{B}_\delta g(w))'}{\lambda w\,(\mathcal{B}_\delta g(w))' + (1-\lambda)\mathcal{B}_\delta g(w)}\right)\right| < \frac{\alpha\pi}{2}, \tag{4}$$

where

$$z, w \in \mathbb{U}, \;\; 0 < \alpha \leqq 1, \;\; 0 \leqq \lambda \leqq 1 \quad \text{and} \quad 0 < \delta \leqq 1,$$

and $g = f^{-1}$ is given by (2).

In particular, if we choose $\lambda = 1$ in Definition 1, the family $\mathcal{B}_\Sigma(\lambda, \delta; \alpha)$ reduces to the family $S_\Sigma(\delta; \alpha)$ of bi-starlike functions which satisfying the following conditions

$$\left|\arg\left(\frac{z\,(\mathcal{B}_\delta f(z))'}{\mathcal{B}_\delta f(z)}\right)\right| < \frac{\alpha\pi}{2}$$

and

$$\left|\arg\left(\frac{w\,(\mathcal{B}_\delta g(w))'}{\mathcal{B}_\delta g(w)}\right)\right| < \frac{\alpha\pi}{2}.$$

If we choose $\lambda = 0$ in Definition 1, the family $\mathcal{B}_\Sigma(\lambda, \delta; \alpha)$ reduces to the family $\mathcal{K}_\Sigma(\delta; \alpha)$ of bi-convex functions which satisfying the following conditions:

$$\left|\arg\left(1 + \frac{z\,(\mathcal{B}_\delta f(z))''}{(\mathcal{B}_\delta f(z))'}\right)\right| < \frac{\alpha\pi}{2}$$

and

$$\left|\arg\left(1+\frac{w\left(\mathcal{B}_\delta g(w)\right)''}{\left(\mathcal{B}_\delta g(w)\right)'}\right)\right|<\frac{\alpha\pi}{2}.$$

Our first main result is asserted by Theorem 1 below.

Theorem 1. *Let the function* $f\in\mathcal{B}_\Sigma(\lambda,\delta;\alpha)$ $(0<\alpha\leqq 1; 0\leqq\lambda\leqq 1; 0<\delta\leqq 1)$ *be given by* (1). *Then*

$$|a_2|\leqq\frac{2\alpha}{\sqrt{\left|2\alpha e^{-2\delta}\left[(\lambda+1)^2+2\delta(3-2\lambda)-5\right]+(1-\alpha)\left(2-\lambda\right)^2e^{-2\delta}\right|}}$$

and

$$|a_3|\leqq\frac{4\alpha^2e^{2\delta}}{(2-\lambda)^2}+\frac{\alpha e^{2\delta}}{\delta(3-2\lambda)}.$$

Proof. In light of the conditions (3) and (4), we have

$$1+\frac{z\left(\mathcal{B}_\delta f(z)\right)'}{\mathcal{B}_\delta f(z)}+\frac{z\left(\mathcal{B}_\delta f(z)\right)''}{\left(\mathcal{B}_\delta f(z)\right)'}-\frac{\lambda z^2\left(\mathcal{B}_\delta f(z)\right)''+z\left(\mathcal{B}_\delta f(z)\right)'}{\lambda z\left(\mathcal{B}_\delta f(z)\right)'+(1-\lambda)\mathcal{B}_\delta f(z)}=[p(z)]^\alpha \tag{5}$$

and

$$1+\frac{w\left(\mathcal{B}_\delta g(w)\right)'}{\mathcal{B}_\delta g(w)}+\frac{w\left(\mathcal{B}_\delta g(w)\right)''}{\left(\mathcal{B}_\delta g(w)\right)'}-\frac{\lambda w^2\left(\mathcal{B}_\delta g(w)\right)''+w\left(\mathcal{B}_\delta g(w)\right)'}{\lambda w\left(\mathcal{B}_\delta g(w)\right)'+(1-\lambda)\mathcal{B}_\delta g(w)}=[q(w)]^\alpha, \tag{6}$$

where $g=f^{-1}$ and the functions $p,q\in\mathcal{P}$ have the following series representations:

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots \tag{7}$$

and

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\cdots. \tag{8}$$

By comparing the corresponding coefficients of (5) and (6), we find that

$$(2-\lambda)e^{-\delta}a_2=\alpha p_1, \tag{9}$$

$$2\delta(3-2\lambda)e^{-2\delta}a_3-\left(5-(\lambda+1)^2\right)e^{-2\delta}a_2^2=\alpha p_2+\frac{\alpha(\alpha-1)}{2}\,p_1^2, \tag{10}$$

$$-(2-\lambda)e^{-\delta}a_2=\alpha q_1 \tag{11}$$

and

$$2\delta(3-2\lambda)e^{-2\delta}\left(2a_2^2-a_3\right)-\left(5-(\lambda+1)^2\right)e^{-2\delta}a_2^2=\alpha q_2+\frac{\alpha(\alpha-1)}{2}\,q_1^2. \tag{12}$$

Thus, by using (9) and (11), we conclude that

$$p_1=-q_1 \tag{13}$$

and

$$2\left(2-\lambda\right)^2e^{-2\delta}a_2^2=\alpha^2(p_1^2+q_1^2). \tag{14}$$

If we add (10) to (12), we obtain

$$(15)\quad 2e^{-2\delta}\left[(\lambda+1)^2+2\delta(3-2\lambda)-5\right]a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2+q_1^2\right).$$

Substituting the value of $p_1^2+q_1^2$ from (14) into the right-hand side of (15), and after some computations, we deduce that

$$(16)\qquad a_2^2=\frac{\alpha^2(p_2+q_2)}{2\alpha e^{-2\delta}\left[(\lambda+1)^2+2\delta(3-2\lambda)-5\right]+(1-\alpha)(2-\lambda)^2e^{-2\delta}}.$$

By taking the moduli of both sides of (16) and applying the Lemma 1 for the coefficients p_2 and q_2, we have

$$|a_2|\leqq\frac{2\alpha}{\sqrt{\left|2\alpha e^{-2\delta}\left[(\lambda+1)^2+2\delta(3-2\lambda)-5\right]+(1-\alpha)(2-\lambda)^2e^{-2\delta}\right|}}.$$

Next, in order to determinate the bound on $|a_3|$, by subtracting (12) from (10), we get

$$(17)\qquad 4\delta(3-2\lambda)e^{-2\delta}\left(a_3-a_2^2\right)=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2-q_1^2\right).$$

Now, upon substituting the value of a_2^2 from (14) into (17) and using (13), we deduce that

$$(18)\qquad a_3=\frac{\alpha^2\left(p_1^2+q_1^2\right)}{2(2-\lambda)^2e^{-2\delta}}+\frac{\alpha(p_2-q_2)}{4\delta(3-2\lambda)e^{-2\delta}}.$$

Finally, by taking the moduli on both sides of (18) and applying the Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, it follows that

$$|a_3|\leqq\frac{4\alpha^2e^{2\delta}}{(2-\lambda)^2}+\frac{\alpha e^{2\delta}}{\delta(3-2\lambda)}.$$

This completes the proof of Theorem 1. □

Putting $\lambda=1$ in Theorem 1, we state:

Corollary 1. *For* $0<\alpha\leqq1$ *and* $0<\delta\leqq1$, *let the function* $f\in S_\Sigma(\delta;\alpha)$ *be given by* (1). *Then*

$$|a_2|\leqq\frac{2\alpha}{\sqrt{\left|2\alpha(2\delta-1)e^{-2\delta}+(1-\alpha)e^{-2\delta}\right|}}$$

and

$$|a_3|\leqq4\alpha^2e^{2\delta}+\frac{1}{\delta}\alpha e^{2\delta}.$$

Putting $\lambda=0$ in Theorem 1, we state:

Corollary 2. *For $0 < \alpha \leqq 1$ and $0 < \delta \leqq 1$, let the function $f \in \mathcal{K}_\Sigma(\delta;\alpha)$ be given by* (1)*. Then*

$$|a_2| \leqq \frac{\alpha}{\sqrt{|\alpha e^{-2\delta}(3\delta-2)+(1-\alpha)e^{-2\delta}|}}$$

and

$$|a_3| \leqq \alpha^2 e^{2\delta} + \frac{1}{3\delta}\alpha e^{2\delta}.$$

3. COEFFICIENT ESTIMATES FOR THE BI-UNIVALENT FUNCTION CLASS $\mathcal{B}^*_\Sigma(\lambda,\delta;\beta)$

In this section, we first define the bi-univalent function class $\mathcal{B}^*_\Sigma(\lambda,\delta;\beta)$.

Definition 2. A function $f \in \Sigma$, given by (1), is said to be in the bi-univalent function class $\mathcal{B}^*_\Sigma(\lambda,\delta;\beta)$ if it satisfies the following conditions:

$$\Re\left\{1+\frac{z(\mathcal{B}_\delta f(z))'}{\mathcal{B}_\delta f(z)}+\frac{z(\mathcal{B}_\delta f(z))''}{(\mathcal{B}_\delta f(z))'}-\frac{\lambda z^2(\mathcal{B}_\delta f(z))''+z(\mathcal{B}_\delta f(z))'}{\lambda z(\mathcal{B}_\delta f(z))'+(1-\lambda)\mathcal{B}_\delta f(z)}\right\} > \beta \tag{19}$$

and

$$\Re\left\{1+\frac{w(\mathcal{B}_\delta g(w))'}{\mathcal{B}_\delta g(w)}+\frac{w(\mathcal{B}_\delta g(w))''}{(\mathcal{B}_\delta g(w))'}-\frac{\lambda w^2(\mathcal{B}_\delta g(w))''+w(\mathcal{B}_\delta g(w))'}{\lambda w(\mathcal{B}_\delta g(w))'+(1-\lambda)\mathcal{B}_\delta g(w)}\right\} > \beta, \tag{20}$$

where

$$z,w \in \mathbb{U},\ \ 0 \leqq \beta < 1,\ \ 0 \leqq \lambda \leqq 1 \quad \text{and} \quad 0 < \delta \leqq 1,$$

and $g = f^{-1}$ is given by (2).

In particular, if we choose $\lambda = 1$ in Definition 2, the family $\mathcal{B}^*_\Sigma(\lambda,\delta;\beta)$ reduces to the family $S^*_\Sigma(\delta;\beta)$ of bi-starlike functions which satisfying the following conditions

$$\Re\left\{\frac{z(\mathcal{B}_\delta f(z))'}{\mathcal{B}_\delta f(z)}\right\} > \beta$$

and

$$\Re\left\{\frac{w(\mathcal{B}_\delta g(w))'}{\mathcal{B}_\delta g(w)}\right\} > \beta.$$

Also, if we choose $\lambda = 0$ in Definition 2, the family $\mathcal{B}^*_\Sigma(\lambda,\delta;\beta)$ reduces to the family $\mathcal{K}^*_\Sigma(\delta;\beta)$ of bi-convex functions which satisfying the following conditions

$$\Re\left\{1+\frac{z(\mathcal{B}_\delta f(z))''}{(\mathcal{B}_\delta f(z))'}\right\} > \beta$$

and

$$\Re\left\{1+\frac{w(\mathcal{B}_\delta g(w))''}{(\mathcal{B}_\delta g(w))'}\right\} > \beta.$$

Our second main result is asserted by Theorem 2 below.

Theorem 2. *Let the function* $f \in \mathcal{B}^*_\Sigma(\lambda, \delta; \beta)$ $(0 \leqq \beta < 1; 0 \leqq \lambda \leqq 1; 0 < \delta \leqq 1)$ *be given by* (1). *Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\beta)}{\left| e^{-2\delta}\left[(\lambda+1)^2 + 2\delta(3-2\lambda) - 5\right]\right|}}$$

and

$$|a_3| \leqq \frac{4\left(1-\beta\right)^2 e^{2\delta}}{(2-\lambda)^2} + \frac{(1-\beta)e^{2\delta}}{\delta(3-2\lambda)}.$$

Proof. In view of the conditions (19) and (20), there exist the functions $p, q \in \mathcal{P}$ such that

$$\begin{aligned} &1 + \frac{z\left(\mathcal{B}_\delta f(z)\right)'}{\mathcal{B}_\delta f(z)} + \frac{z\left(\mathcal{B}_\delta f(z)\right)''}{\left(\mathcal{B}_\delta f(z)\right)'} \\ &\quad - \frac{\lambda z^2\left(\mathcal{B}_\delta f(z)\right)'' + z\left(\mathcal{B}_\delta f(z)\right)'}{\lambda z\left(\mathcal{B}_\delta f(z)\right)' + (1-\lambda)\mathcal{B}_\delta f(z)} = \beta + (1-\beta)p(z) \end{aligned} \tag{21}$$

and

$$\begin{aligned} &1 + \frac{w\left(\mathcal{B}_\delta g(w)\right)'}{\mathcal{B}_\delta g(w)} + \frac{w\left(\mathcal{B}_\delta g(w)\right)''}{\left(\mathcal{B}_\delta g(w)\right)'} \\ &\quad - \frac{\lambda w^2\left(\mathcal{B}_\delta g(w)\right)'' + w\left(\mathcal{B}_\delta g(w)\right)'}{\lambda w\left(\mathcal{B}_\delta g(w)\right)' + (1-\lambda)\mathcal{B}_\delta g(w)} = \beta + (1-\beta)q(w), \end{aligned} \tag{22}$$

where $g = f^{-1}$ and the functions $p, q \in \mathcal{P}$ have the series expansions given by (7) and (8), respectively. Thus, by comparing the corresponding coefficients in (21) and (22), we get

$$(2-\lambda)e^{-\delta}a_2 = (1-\beta)p_1, \tag{23}$$

$$2\delta(3-2\lambda)e^{-2\delta}a_3 - \left(5-(\lambda+1)^2\right)e^{-2\delta}a_2^2 = (1-\beta)p_2, \tag{24}$$

$$-(2-\lambda)e^{-\delta}a_2 = (1-\beta)q_1 \tag{25}$$

and

$$2\delta(3-2\lambda)e^{-2\delta}\left(2a_2^2 - a_3\right) - \left(5-(\lambda+1)^2\right)e^{-2\delta}a_2^2 = (1-\beta)q_2. \tag{26}$$

We now find from (23)) and (25)) that

$$p_1 = -q_1$$

and

$$2\left(2-\lambda\right)^2 e^{-2\delta}a_2^2 = (1-\beta)^2\left(p_1^2 + q_1^2\right). \tag{27}$$

By adding (24) and (26), we obtain

$$2e^{-2\delta}\left[(\lambda+1)^2 + 2\delta(3-2\lambda) - 5\right]a_2^2 = (1-\beta)(p_2+q_2).$$

Consequently, we have

$$a_2^2 = \frac{(1-\beta)(p_2+q_2)}{2e^{-2\delta}\left[(\lambda+1)^2 + 2\delta(3-2\lambda) - 5\right]}.$$

Next, by applying the Lemma 1 for the coefficients p_2 and q_2, we deduce that

$$|a_2| \leqq \sqrt{\frac{2(1-\beta)}{\left|e^{-2\delta}\left[(\lambda+1)^2 + 2\delta(3-2\lambda) - 5\right]\right|}}.$$

In order to determine the bound on $|a_3|$, by subtracting (26) from (24), we get

$$4\delta(3-2\lambda)e^{-2\delta}\left(a_3 - a_2^2\right) = (1-\beta)\left(p_2 - q_2\right)$$

or, equivalently,

$$a_3 = a_2^2 + \frac{(1-\beta)\left(p_2-q_2\right)}{4\delta(3-2\lambda)e^{-2\delta}}. \tag{28}$$

Substituting the value of a_2^2 from (27) into (28), it follows that

$$a_3 = \frac{(1-\beta)^2\left(p_1^2+q_1^2\right)}{2\left(2-\lambda\right)^2 e^{-2\delta}} + \frac{(1-\beta)\left(p_2-q_2\right)}{4\delta(3-2\lambda)e^{-2\delta}}.$$

Finally, by applying the Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \leqq \frac{4\left(1-\beta\right)^2 e^{2\delta}}{(2-\lambda)^2} + \frac{(1-\beta)e^{2\delta}}{\delta(3-2\lambda)}.$$

We have thus completed the proof of Theorem 2. □

Putting $\lambda = 1$ in Theorem 2, we state:

Corollary 3. *For $0 \leqq \beta < 1$ and $0 < \delta \leqq 1$, let $f \in S_{\Sigma}^{*}(\delta;\beta)$ be given by (1). Then*

$$|a_2| \leqq \sqrt{\frac{2(1-\beta)}{\left|(2\delta-1)\,e^{-2\delta}\right|}}$$

and

$$|a_3| \leqq 4\left(1-\beta\right)^2 e^{2\delta} + \frac{1}{\delta}(1-\beta)e^{2\delta}.$$

Putting $\lambda = 0$ in Theorem 2, we state:

Corollary 4. *For $0 \leqq \beta < 1$ and $0 < \delta \leqq 1$, let $f \in \mathcal{K}_{\Sigma}^{*}(\delta;\beta)$ be given by (1). Then*

$$|a_2| \leqq \sqrt{\frac{1-\beta}{\left|(3\delta-2)\,e^{-2\delta}\right|}}$$

and

$$|a_3| \leqq \left(1-\beta\right)^2 e^{2\delta} + \frac{1}{3\delta}(1-\beta)e^{2\delta}.$$

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References

[1] C. Abirami, N. Magesh, J. Yamini, *Initial bounds for certain classes of bi-univalent functions defined by Horadam polynomials*, Abstract and Applied Analysis, (2020), Article ID: 7391058, 1–8.

[2] E. A. Adegani, S. Bulut, A. A. Zireh, *Coefficient estimates for a subclass of analytic bi-univalent functions*, Bulletin of the Korean Mathematical Society, 55 (2) (2018), 405–413.

[3] A. G. Alamoush, *Certain subclasses of bi-univalent functions involving the Poisson distribution associated with Horadam polynomials*, Malaya Journal of Matematik, 7 (2019), 618–624.

[4] A. G. Alamoush, *Coefficient estimates for a new subclasses of lambda-pseudo bi-univalent functions with respect to symmetrical points associated with the Horadam Polynomials*, Turkish Journal of Mathematics, 3 (2019), 2865–2875.

[5] A. G. Alamoush, *On a subclass of bi-univalent functions associated to Horadam polynomials*, International Journal of Open Problems in Complex Analysis, 12 (2020), 58–66.

[6] A. G. Alamoush, *On subclass of analytic bi-close-to-convex functions*, International Journal of Open Problems in Complex Analysis, 13 (2021), 10–18.

[7] S. Bulut, A. K. Wanas, *Coefficient estimates for families of bi-univalent functions defined by Ruscheweyh derivative operator*, Mathematica Moravica, 25 (1) (2021), 71–80.

[8] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[9] H. Ö. Güney, G. Murugusundaramoorthy, J. Sokół, *Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers*, Acta Universitatis Sapientiae Mathematica, 10 (2018), 70–84.

[10] N. Magesh, S. Bulut, *Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions*, Afrika Matematika, 29 (1-2) (2018), 203–209.

[11] B. Şeker, *On a new subclass of bi-univalent functions defined by using Salagean operator*, Turkish Journal of Mathematics, 42 (2018), 2891–2896.

[12] H. M. Srivastava, *Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis*, Iranian Journal of Science and Technology, Transactions A: Science, 44 (2020), 327–344.

[13] H. M. Srivastava, Ş. Altınkaya, S. Yalçin, *Certain subclasses of bi-univalent functions associated with the Horadam polynomials*, Iranian Journal of Science and Technology, Transactions A: Science, 43 (2019), 1873–1879.

[14] H. M. Srivastava, S. S. Eker, S. G. Hamidi, J. M. Jahangiri, *Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator*, Bulletin of the Iranian Mathematical Society, 44 (1) (2018), 149–157.

[15] H. M. Srivastava, S. M. El-Deeb, *Fuzzy differential subordinations based upon the Mittag-Leffler type Borel distribution*, Symmetry, 13 (2021), Article ID: 1023, 1–15.

[16] H. M. Srivastava, S. Gaboury, F. Ghanim, *Coefficient estimates for some general subclasses of analytic and bi-univalent functions*, Afrika Matematika, 28 (2017), 693–706.

[17] H. M. Srivastava, S. Gaboury, F. Ghanim, *Coefficient estimates for a general subclass of analytic and bi-univalent functions of the Ma-Minda type*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas (RACSAM), 112 (2018), 1157–1168.

[18] H. M. Srivastava, A. K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Applied Mathematics Letters, 23 (2010), 1188–1192.

[19] H. M. Srivastava, A. Motamednezhad, E. A. Adegani, *Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator*, Mathematics, 8 (2020), Article ID: 172, 1–12.

[20] H. M. Srivastava, A. Motamednezhad, S. Salehian, *Coefficients of a comprehensive subclass of meromorphic bi-univalent functions associated with the Faber polynomial expansion*, Axioms, 10 (2021), Article ID: 27, 1–13.

[21] H. M. Srivastava, F. M. Sakar, H. Ö. Güney, *Some general coefficient estimates for a new class of analytic and bi-univalent functions defined by a linear combination*, Filomat, 32 (2018), 1313–1322.

[22] H. M. Srivastava, A. K. Wanas, *Initial Maclaurin coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions defined by a linear combination*, Kyungpook Mathematical Journal, 59 (3) (2019), 493–503.

[23] H. M. Srivastava, A. K. Wanas, G. Murugusundaramoorthy, *Certain family of bi-univalent functions associated with Pascal distribution series based on Horadam polynomials*, Surveys in Mathematics and its Applications, 16 (2021), 193–205.

[24] S. R. Swamy, A. K. Wanas, Y. Sailaja, *Some special families of holomorphic and Sălăgean type bi-univalent functions associated with (m,n)-Lucas polynomials*, Communications in Mathematics and Applications, 11 (4) (2020), 563–574.

[25] A. K. Wanas, *Applications of (M,N)-Lucas polynomials for holomorphic and bi-univalent functions*, Filomat, 34 (10) (2020), 3361–3368.

[26] A. K. Wanas, *Coefficient estimates for Bazilevič functions of bi-prestarlike functions*, Miskolc Mathematical Notes, 21 (2) (2020), 1031–1040.

[27] A. K. Wanas, *Applications of Chebyshev polynomials on λ-pseudo bi-starlike and λ-pseudo bi-convex functions with respect to symmetrical points*, TWMS Journal of Applied and Engineering Mathematics, 10 (3) (2020), 568–573.

[28] A. K. Wanas, A. L. Alina, *Applications of Horadam polynomials on Bazilevič bi-univalent function satisfying subordinate conditions*, Journal of Physics: Conference Series, 1294 (2019), 1–6.

[29] A. K. Wanas, J. A. Khuttar, *Applications of Borel distribution series on analytic functions*, Earthline Journal of Mathematical Sciences, 4 (2020), 71–82.

[30] A. K. Wanas, A. H. Majeed, *On subclasses of analytic and m-fold symmetric bi-univalent functions*, Iranian Journal of Mathematical Sciences and Informatics, 15 (2) (2020), 51–60.

[31] A. K. Wanas, H. Tang, *Initial Coefficient estimates for a classes of m-fold symmetric bi-univalent functions involving Mittag-Leffler function*, Mathematica Moravica, 24 (2) (2020), 51–61.

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