

Certain integral representations involving hypergeometric functions in two variables

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ABSTRACT. Various integral representations of hypergeometric functions have been introduced and investigated due to their important applications in divers fields. In this article, we define some new Euler-type integral representations for the Horn's functions of two variables G_1, G_2, G_3 and H_1 .

1. INTRODUCTION

Integral representations of hypergeometric functions have found applications in divers fields such as mathematics, physics, statistics, and engineering. Praveen et al. [21-29] studied several properties, various relations, different integral representations and many extensions of the hypergeometric functions. Hasanov et al. [12] studied some of the properties of the Horn type second-order double hypergeometric function H_2^* involving integral representations, differential equations, and generating functions. Choi et al. [9] introduced certain integral representations for Srivastava's triple hypergeometric functions H_A, H_B and H_C . Younis and Bin-Saad [18, 19] establish several integral representations and operational relations involving quadruple hypergeometric functions $X_i^{(4)}$ ($i = 38, 40, 45, 48, 50$). Younis and Nisar [20] introduce new integral representations of Euler-type for Exton's hypergeometric functions of four variables D_1, D_2, D_3, D_4 and D_5 . Recently, several interesting and useful integral representations of many special functions have been investigated (see, e.g., [2-8, 11, 14]).

For our purpose, we begin by recalling Horn's hypergeometric functions of two variables $G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6$ and H_7 defined by (see

2020 *Mathematics Subject Classification*. Primary: 33C20; Secondary: 33C65.

Key words and phrases. Beta function, Horn double functions, Appell functions, Eulerian integrals.

Full paper. Received 8 February 2021, revised 4 May 2021, accepted 3 September 2021, available online 2 November 2021.

[10, 13]):

$$\begin{aligned}
 G_1(a, b, c; x, y) &= \sum_{m, n=0}^{\infty} (a)_{m+n} (b)_{n-m} (c)_{m-n} \frac{x^m y^n}{m! n!}, \\
 (1) \quad G_2(a, b, c, d; x, y) &= \sum_{m, n=0}^{\infty} (a)_m (b)_n (c)_{n-m} (d)_{m-n} \frac{x^m y^n}{m! n!}, \\
 G_3(a, b; x, y) &= \sum_{m, n=0}^{\infty} (a)_{2n-m} (b)_{2m-n} \frac{x^m y^n}{m! n!}, \\
 H_1(a, b, c, d; x, y) &= \sum_{m, n=0}^{\infty} \frac{(a)_{m-n} (b)_{m+n} (c)_n}{(d)_m} \frac{x^m y^n}{m! n!}, \\
 H_2(a, b, c, d; e; x, y) &= \sum_{m, n=0}^{\infty} \frac{(a)_{m-n} (b)_m (c)_n (d)_n}{(e)_m} \frac{x^m y^n}{m! n!}, \\
 H_3(a, b; c; x, y) &= \sum_{m, n=0}^{\infty} \frac{(a)_{2m+n} (b)_n}{(c)_{m+n}} \frac{x^m y^n}{m! n!}, \\
 H_4(a, b; c, d; x, y) &= \sum_{m, n=0}^{\infty} \frac{(a)_{2m+n} (b)_n}{(c)_m (c)_n} \frac{x^m y^n}{m! n!}, \\
 H_5(a, b; c; x, y) &= \sum_{m, n=0}^{\infty} \frac{(a)_{2m+n} (b)_{n-m}}{(c)_n} \frac{x^m y^n}{m! n!}, \\
 H_6(a, b, c; x, y) &= \sum_{m, n=0}^{\infty} (a)_{2m-n} (b)_{n-m} (c)_n \frac{x^m y^n}{m! n!}, \\
 H_7(a, b, c; d; x, y) &= \sum_{m, n=0}^{\infty} \frac{(a)_{2m-n} (b)_n (c)_n}{(d)_m} \frac{x^m y^n}{m! n!},
 \end{aligned}$$

where $(a)_m$ is the well known Pochhammer symbol given by

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \begin{cases} 1, & (m=0), \\ a(a+1) \cdots (a+m-1), & (m \in \mathbb{N} := \{1, 2, \dots\}). \end{cases}$$

Appell [1, 17] defined four hypergeometric functions of two variables denoted these by F_1, F_2, F_3 and F_4 . We present here the definition of one of these functions

$$F_3(a, b, c, d; e; x, y) = \sum_{m, n=0}^{\infty} \frac{(a)_m (b)_n (c)_m (d)_n}{(e)_{m+n}} \frac{x^m y^n}{m! n!}.$$

The aim of this paper is to present several new integral representations of Euler-type involving Horn's functions G_1, G_2, G_3 and H_1 in terms of hypergeometric functions of two variables.

2. MAIN RESULTS

Theorem 2.1. *Each of the following integral representations holds true:*

$$\begin{aligned}
 G_1(a, b, c; x, y) &= \\
 &= \frac{\Gamma(a+a')(S-T)^a(R-T)^{a'}}{\Gamma(a)\Gamma(a')(S-R)^{a+a'-1}} \int_R^S \frac{(\alpha-R)^{a-1}(S-\alpha)^{a'-1}}{(\alpha-T)^{a+a'}} \\
 (2) \quad &\times G_2\left(a+a', 1-a', b, c; \frac{(S-T)(\alpha-R)x}{(S-R)(\alpha-T)}, \frac{-y}{(R-T)(S-\alpha)}\right) d\alpha, \\
 &\quad (\operatorname{Re}(a) > 0, \operatorname{Re}(a') > 0, T < R < S),
 \end{aligned}$$

$$\begin{aligned}
 G_1(a, b, c; x, y) &= \frac{4M_1^{a'}M_2^b\Gamma(a+a')\Gamma(b+b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')} \\
 (3) \quad &\cdot \int_0^\infty \int_0^\infty \frac{\cosh \alpha (\sinh^2 \alpha)^{a'-\frac{1}{2}}}{(1+M_1 \sinh^2 \alpha)^{a+a'}} \times \frac{\cosh \beta (\sinh^2 \beta)^{b-\frac{1}{2}}}{(1+M_2 \sinh^2 \beta)^{b+b'}} \\
 &\cdot H_2\left(c, 1-a', a+a', b+b'; b'; -\frac{x \operatorname{csch}^2 \alpha \operatorname{csch}^2 \beta}{M_1 M_2}, \right. \\
 &\quad \left. \frac{M_2 y}{(1+M_1 \sinh^2 \alpha)(1+M_2 \sinh^2 \beta)}\right) d\alpha d\beta, \\
 &\quad (\operatorname{Re}(a) > 0, \operatorname{Re}(a') > 0, \operatorname{Re}(b) > 0, \\
 &\quad \operatorname{Re}(b') > 0, M_1 > 0, M_2 > 0),
 \end{aligned}$$

$$\begin{aligned}
 G_1(a, b, c; x, y) &= \frac{\Gamma(a+b)\Gamma(c+c')}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(c')} \\
 (4) \quad &\cdot \int_0^1 \int_0^1 \alpha^{c-1} (1-\alpha)^{c'-1} \beta^{a-1} (1-\beta)^{b-1} \times \\
 &\quad \times H_7\left(1-c', \frac{a+b}{2}, \frac{a+b+1}{2}; 1-c-c'; \right. \\
 &\quad \left. \frac{\alpha\beta x}{(1-\alpha)^2(\beta-1)}, \frac{4(\alpha-1)\beta(1-\beta)y}{\alpha}\right) d\alpha d\beta, \\
 &\quad (\operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0, \operatorname{Re}(c) > 0, \operatorname{Re}(c') > 0),
 \end{aligned}$$

$$\begin{aligned}
 G_1(a, b, c; x, y) &= \frac{4\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \\
 (5) \quad &\cdot \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (\sin^2 \alpha)^{a+b-\frac{1}{2}} (\cos^2 \alpha)^{c-\frac{1}{2}} (\sin^2 \beta)^{b-\frac{1}{2}} (\cos^2 \beta)^{a-\frac{1}{2}} \\
 &\times \left[1 - x \cos^2 \alpha \cot^2 \beta - \frac{1}{4} y \sin^2 \alpha \tan^2 \alpha \sin^2 2\beta\right]^{-(a+b+c)} d\alpha, \\
 &\quad (\operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0, \operatorname{Re}(c) > 0).
 \end{aligned}$$

Proof. To prove the result in equality (2) asserted in Theorem 2.1, let \mathfrak{U} denote the right-hand side of the equality (2). Then from the definition of Horn's function G_2 in (1), we get

$$\begin{aligned}
 \mathfrak{U} &= \frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')} \sum_{m,n=0}^{\infty} \frac{(a+a')_m (b)_{n-m} (c)_{m-n}}{(a)_{-n}} \times \\
 (6) \quad &\times \frac{(S-T)^{a+m} (R-T)^{a'-n}}{(S-R)^{a+a'+m-1}} \cdot \int_R^S \frac{(\alpha-R)^{a+m-1} (S-\alpha)^{a'-n-1}}{(\alpha-T)^{a+a'+m}} \times \frac{x^m y^n}{m! n!} d\alpha.
 \end{aligned}$$

Applying the integral representation of the Beta function (see, e.g., [10, p. 10, (14)])

$$\begin{aligned}
 B(a, b) &= \frac{(S-T)^a (R-T)^b}{(S-R)^{a+b-1}} \int_R^S \frac{(\alpha-R)^{a-1} (S-\alpha)^{b-1}}{(\alpha-T)^{a+b}} d\alpha, \\
 &\quad (T < R < S, \operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0).
 \end{aligned}$$

in (6), we have

$$(7) \quad \mathfrak{U} = \frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')} \sum_{m,n=0}^{\infty} \frac{(a+a')_m (b)_{n-m} (c)_{m-n} B(a+m+n, a'-n)}{(a)_{-n}}.$$

Now, applying well known beta function (see, e.g., [15, 16])

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

in (7), we get the required result. On the same way, we find the results (3)–(5). □

A similar argument as in the proof of the Theorem 2.1. we can give the following theorems without proof.

Theorem 2.2. *The following integral representations holds:*

$$\begin{aligned}
 G_2(a, b, c, d; x, y) &= \\
 (8) \quad &= \frac{\Gamma(b+d)\Gamma(c+c')}{\Gamma(b)\Gamma(c)\Gamma(c')\Gamma(d)} \int_0^\infty \int_0^\infty \frac{\alpha^{c-1}}{(1+\alpha)^{c+c'}} \frac{\beta^{d-1}}{(1+\beta)^{b+d}} \times \\
 &\times F_3\left(a, \frac{c+c'}{2}, b+d, \frac{c+c'+1}{2}; c'; \frac{\beta x}{\alpha(1+\beta)}, \frac{4\alpha y}{(1+\alpha)^2 \beta}\right) d\alpha d\beta, \\
 &\quad (\operatorname{Re}(b) > 0, \operatorname{Re}(c) > 0, \operatorname{Re}(c') > 0, \operatorname{Re}(d) > 0),
 \end{aligned}$$

$$\begin{aligned}
 G_2(a, b, c, d; x, y) &= \frac{(1 + M_1)^a (1 + M_2)^b \Gamma(a + d) \Gamma(b + c)}{\Gamma(a) \Gamma(b) \Gamma(c) \Gamma(d)} \\
 &\cdot \int_0^1 \int_0^1 \frac{\alpha^{a-1} (1 - \alpha)^{d-1} \beta^{b-1} (1 - \beta)^{c-1}}{(1 + M_1 \alpha)^{a+d} (1 + M_2 \beta)^{b+c}} \\
 (9) \quad &\cdot G_3 \left(b + c, a + d; \frac{(1 + M_1) \alpha (1 - \alpha) (1 + M_2 \beta) x}{(1 + M_1 \alpha)^2 (1 - \alpha)}, \right. \\
 &\quad \left. \frac{(1 + M_2) \beta (1 - \beta) (1 + M_1 \alpha) y}{(1 + M_2 \beta)^2 (1 - \alpha)} \right) d\alpha d\beta, \\
 &(\operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0, \operatorname{Re}(c) > 0, \operatorname{Re}(d) > 0, M_1 > -1, M_2 > -1), \\
 (10)
 \end{aligned}$$

$$\begin{aligned}
 G_2(a, b, c, d; x, y) &= \frac{\Gamma(a + a') \Gamma(c + d)}{\Gamma(a) \Gamma(a') \Gamma(c) \Gamma(d)} \\
 &\cdot \int_0^\infty \int_0^\infty (e^{-\alpha})^a (1 - e^{-\alpha})^{a'-1} (e^{-\beta})^c (1 - e^{-\beta})^{d-1} \times \\
 &\times H_6 \left(1 - a', a + a', b; \frac{e^{-(\alpha-\beta)} (1 - e^{-\beta}) x}{(1 - e^{-\alpha})^2}, \frac{(e^{-\alpha} - 1) e^{-\beta} y}{(1 - e^{-\beta})} \right) d\alpha d\beta, \\
 &(\operatorname{Re}(a) > 0, \operatorname{Re}(a') > 0, \operatorname{Re}(c) > 0, \operatorname{Re}(d) > 0),
 \end{aligned}$$

Theorem 2.3. *The following integral representations hold:*

$$\begin{aligned}
 G_3(a, b; x, y) &= \frac{\Gamma(a + a') \Gamma(b + b')}{\Gamma(a) \Gamma(a') \Gamma(b) \Gamma(b') (S_1 - R_1)^{a+a'-1} (S_2 - R_2)^{b+b'-1}} \\
 &\cdot \int_{R_1}^{S_1} \int_{R_2}^{S_2} (\alpha - R_1)^{a-1} (S_1 - \alpha)^{a'-1} (\beta - R_2)^{b-1} (S_2 - \beta)^{b'-1} \\
 (11) \quad &\times G_2 \left(b + b', a + a', 1 - a', 1 - b'; \frac{(S_1 - \alpha) (\beta - R_2)^2 x}{(S_2 - R_2) (\alpha - R_1) (S_2 - \beta)}, \right) \\
 &\cdot \left(\frac{(\alpha - R_1)^2 (S_2 - \beta) y}{(S_1 - R_1) (S_1 - \alpha) (\beta - R_2)} \right) d\alpha d\beta, \\
 &(\operatorname{Re}(a) > 0, \operatorname{Re}(a') > 0, \operatorname{Re}(b) > 0, \operatorname{Re}(b') > 0, R_1 < S_1, R_2 < S_2),
 \end{aligned}$$

$$\begin{aligned}
 (12) \quad G_3(a, b; x, y) &= \frac{\Gamma(a + a')}{2^{a+a'-2} \Gamma(a) \Gamma(a')} \int_{-1}^1 \frac{[(1 + \alpha)^2]^{a'-\frac{1}{2}} [(1 - \alpha)^2]^{a-\frac{1}{2}}}{(1 + \alpha^2)^{a+a'}} \times \\
 &\times G_3 \left(1 - a', b; - \left(\frac{1 + \alpha}{1 - \alpha} \right)^2 x, \left(\frac{1 - \alpha}{1 + \alpha} \right)^4 y \right) d\alpha, \\
 &(\operatorname{Re}(a) > 0, \operatorname{Re}(a') > 0).
 \end{aligned}$$

$$\begin{aligned}
 G_3(a, b; x, y) &= \frac{4(1 + M_1)^a(1 + M_2)^b\Gamma(a + a')\Gamma(b + b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')} \\
 &\cdot \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{(\sin^2 \alpha)^{a-\frac{1}{2}} (\cos^2 \alpha)^{a'-\frac{1}{2}} (\sin^2 \beta)^{b-\frac{1}{2}} (\cos^2 \beta)^{b'-\frac{1}{2}}}{(1 + M_1 \sin^2 \alpha)^{a+a'} (1 + M_2 \sin^2 \beta)^{b+b'}} \\
 &\cdot H_2\left(a + a', 1 - a', \frac{b + b'}{2}, \frac{b + b' + 1}{2}; b; \right) \\
 (13) \quad &\cdot \left(-\frac{(1 + M_1)^2 y \sin^2 \alpha \tan^2 \alpha \cot^2 \beta}{(1 + M_2)(1 + M_1 \sin^2 \alpha)}, \right) \\
 &\cdot \left(\frac{(1 + M_2)^2 x \csc^2 \alpha \sin^4 \beta (1 + M_1 \sin^2 \alpha)}{(1 + M_1)(1 + M_2 \sin^2 \beta)^2} \right) d\alpha d\beta, \\
 &\quad \left(\operatorname{Re}(a) > 0, \operatorname{Re}(a') > 0, \operatorname{Re}(b) > 0, \right. \\
 &\quad \left. \operatorname{Re}(b') > 0, M_1 > -1, M_2 > -1 \right).
 \end{aligned}$$

$$\begin{aligned}
 G_3(a, b; x, y) &= \frac{\Gamma(a + a')}{\Gamma(a)\Gamma(a')} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2} + \alpha \right)^{a'-1} \left(\frac{1}{2} - \alpha \right)^{a-1} \\
 (14) \quad &\times H_6\left(b, 1 - a', a + a'; -\left(\frac{1 + 2\alpha}{1 - 2\alpha}\right)x, -\frac{(1 - 2\alpha)^2 y}{2(1 + 2\alpha)}\right) d\alpha, \\
 &\quad \left(\operatorname{Re}(a) > 0, \operatorname{Re}(a') > 0 \right),
 \end{aligned}$$

Theorem 2.4. *The following integral representations hold:*

$$\begin{aligned}
 H_1(a, b, c; d; x, y) &= \frac{4M_1^a M_2^b \Gamma(2d)}{\Gamma(a)\Gamma(2d - a)} \\
 &\cdot \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{(\sin^2 \alpha)^{a-\frac{1}{2}} (\cos^2 \alpha)^{2d-a-\frac{1}{2}} (\sin^2 \beta)^{b-\frac{1}{2}} (\cos^2 \beta)^{c-\frac{1}{2}}}{(\cos^2 \alpha + M_1 \sin^2 \alpha)^{2d} (\cos^2 \beta + M_2 \sin^2 \beta)^{b+c}} \\
 (15) \quad &\cdot H_3\left(b + c, d + \frac{1}{2}; 2d - a; \frac{M_2 y \cot^2 \alpha \sin^2 2\beta}{4M_1 (\cos^2 \beta + M_2 \sin^2 \beta)^2}, \right. \\
 &\quad \left. \left(\frac{M_1 M_2 x \sin^2 2\alpha \sin^2 \beta}{(\cos^2 \alpha + M_1 \sin^2 \alpha)^2 \cos^2 \beta + M_2 \sin^2 \beta} \right) d\alpha d\beta, \right. \\
 &\quad \left. \left(\operatorname{Re}(a) > 0, \operatorname{Re}(2d - a) > 0, M_1 > 0, M_2 > 0 \right). \right.
 \end{aligned}$$

$$\begin{aligned}
 H_1(a, b, c; d; x, y) &= \frac{\Gamma(a+a')\Gamma(b+c)}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(c)} \\
 (16) \quad &\cdot \int_0^\infty \int_0^\infty \frac{\alpha^{a-1}}{(1+\alpha)^{a+a'}} \frac{\beta^{b-1}}{(1+\beta)^{b+c}} \times \\
 &\times H_4\left(b+c, a+a'; a', d; \frac{\beta y}{\alpha(1+\beta)^2}, \frac{\alpha\beta x}{(1+\alpha)(1+\beta)}\right) d\alpha d\beta, \\
 &(\operatorname{Re}(a) > 0, \operatorname{Re}(a') > 0, \operatorname{Re}(b) > 0, \operatorname{Re}(c) > 0),
 \end{aligned}$$

$$\begin{aligned}
 H_1(a, b, c; d; x, y) &= \frac{\Gamma(a+c)\Gamma(b+b')}{\Gamma(a)\Gamma(c)\Gamma(b)\Gamma(b')} \\
 (17) \quad &\cdot \int_0^\infty \int_0^\infty (e^{-\alpha})^{b'} (1-e^{-\alpha})^{b-1} (e^{-\beta})^a (1-e^{-\beta})^{c-1} \times \\
 &\times H_2\left(1-b', a+c, \frac{b+b'}{2}, \frac{b+b'+1}{2}; d; xe^{(\alpha-\beta)}(e^{-\alpha}-1)\right) \cdot \\
 &\cdot \left(4ye^{-(\alpha-\beta)}(e^{-\alpha}-1)(1-e^{-\beta})\right) d\alpha d\beta, \\
 &(\operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0, \operatorname{Re}(b') > 0, \operatorname{Re}(c) > 0),
 \end{aligned}$$

$$\begin{aligned}
 H_1(a, b, c; d; x, y) &= \frac{\Gamma(b+c)}{2^{b+c-1}\Gamma(b)\Gamma(c)} \int_{-1}^1 (1+\alpha)^{b-1} (1-\alpha)^{c-1} \times \\
 (18) \quad &\times H_5\left(b+c, a; d; \frac{(1+\alpha)(1-\alpha)}{4}y, \frac{(1+\alpha)}{2}x\right) d\alpha, \\
 &(\operatorname{Re}(b) > 0, \operatorname{Re}(c) > 0).
 \end{aligned}$$

CONCLUSION

We concluding our investigation by remarking that in our present investigation, we have presented some new Euler-type integral representations for the Horn's functions of two variables G_1, G_2, G_3 and H_1 . As above-noted, special functions and their integral representations have appeared in and been applied to a diversity of research subjects such as statistics, theoretical physics, theory of group representations, and number theory. In this connection, the newly introduced Euler-type integral representations presented here and the other uninvestigated ones (if any) are hoped and believed to have potential applications to various research subjects including the above-mentioned ones.

ACKNOWLEDGEMENTS

The authors would like to thank the anonymous referees for their valuable comments and suggestions. Shilpi Jain very thankful to SERB (project number: MTR/2017/000194) for providing necessary facility and Praveen Agarwal thanks the SERB (project TAR/2018/000001), DST (projects DST/INT/DAAD/P-21/2019 and INT/RUS/RFBR/308), and NBHM (DAE) (project 02011/12/2020 NBHM (R.P)/RD II/7867).

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