The complex-type Padovan–p sequences

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Abstract. In this paper, we define the complex-type Padovan-p sequence and then give the relationships between the Padovan-p numbers and the complex-type Padovan-p numbers. Also, we provide a new Binet formula and a new combinatorial representation of the complex-type Padovan-p numbers by the aid of the nth power of the generating matrix of the complex-type Padovan-p sequence. In addition, we derive various properties of the complex-type Padovan-p numbers such as the permanental, determinantal and exponential representations and the finite sums by matrix methods.

1. Introduction and Preliminaries

The Padovan p-numbers \{Pap(n)\} for any given \(p (p = 2, 3, 4, \ldots)\) is defined [4] by the following homogeneous linear recurrence relation:

\[ \text{Pap}(n + p + 2) = \text{Pap}(n + p) + \text{Pap}(n), \]

for \(n \geq 1\), with initial conditions \(\text{Pap}(1) = \text{Pap}(2) = \cdots = \text{Pap}(p) = 0\), \(\text{Pap}(p + 1) = 1\) and \(\text{Pap}(p + 2) = 0\). When \(p = 1\) in (1), the Padovan p-numbers \{Pap(n)\} is reduced to the usual Padovan sequence \{P(n)\}.

The complex Fibonacci sequence \(\{F^*_n\}\) is defined [7] by a two-order recurrence equation:

\[ F^*_n = F_n + iF_{n+1}, \]

for \(n \geq 0\), where \(\sqrt{-1} = i\) and \(F_n\) is the \(n\)th Fibonacci number (cf. [1, 8]).

Kalman [10] mentioned that these sequences are special cases of a sequence which is defined recursively as a linear combination of the preceding \(k\) terms

\[ a_{n+k} = c_0a_n + c_1a_{n+1} + \cdots + c_{k-1}a_{n+k-1}, \]

where \(c_0, c_1, \ldots, c_{k-1}\) are real constants. In [10], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix.


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method as follows:

\[
A_k = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1}
\end{bmatrix}.
\]

Then by an inductive argument he obtained that

\[
A^n_k \begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{k-1}
\end{bmatrix} = \begin{bmatrix}
a_n \\
a_{n+1} \\
\vdots \\
a_{n+k-1}
\end{bmatrix}.
\]

In the literature, many interesting properties and applications of the recurrence sequences relevant to this paper have been studied by many authors, for example [9,11–13,16–22]. In particular, in [5] and [6], the authors defined the new sequences using the quaternions and complex numbers, and then they gave miscellaneous properties and many applications of the sequences defined. In this work, we define the complex-type Padovan-\(p\) sequence. Also, give the relationships between the Padovan-\(p\) numbers and the complex-type Padovan-\(p\) numbers, and then we obtain generating a matrix of the complex-type Padovan-\(p\) sequence. Furthermore, we produce the Binet formula for this defined sequence. Finally, we give various properties of the complex-type Padovan-\(p\) numbers such as the combinatorial, permanental, determinantal and exponential representations, and the finite sums by matrix methods.

2. The Main Results

Now we define a new sequence that we call the complex-type Padovan-\(p\) sequence \(\{P_{a_p}^{(i)}(n)\}\) as follows:

\begin{equation}
P_{a_p}^{(i)}(n + p + 2) = i^2 \cdot P_{a_p}^{(i)}(n + p) + i^{p+2} \cdot P_{a_p}^{(i)}(n),
\end{equation}

for any given \(p\) (\(p = 3, 5, 7, \ldots\)) and \(n \geq 1\), where \(P_{a_p}^{(i)}(1) = \cdots = P_{a_p}^{(i)}(p) = 0\), \(P_{a_p}^{(i)}(p + 1) = 1\), \(P_{a_p}^{(i)}(p + 2) = 0\) and \(\sqrt{-1} = i\). From the relations in the definitions of the complex-type Padovan-\(p\) numbers and the Padovan-\(p\) numbers, we derive the following relations:

\[
P_{a_p}^{(i)}(n) = \begin{cases}
i^{p+1} \cdot P_{ap}(n), & \text{for } n \equiv 0 \pmod{4}, \\
i^{p+2} \cdot P_{ap}(n), & \text{for } n \equiv 1 \pmod{4}, \\
i^{p+3} \cdot P_{ap}(n), & \text{for } n \equiv 2 \pmod{4}, \\
i^p \cdot P_{ap}(n), & \text{for } n \equiv 3 \pmod{4}.
\end{cases}
\]
From the equation (2), we can write the following companion matrix:

\[
D_p = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

The matrix \(D_p\) is said to be the complex-type Padovan-\(p\) matrix. Then we can write the following matrix relation:

\[
\begin{bmatrix}
P_a^{(i)}(n + p + 2) \\
P_a^{(i)}(n + p + 1) \\
\vdots \\
P_a^{(i)}(n + 2) \\
P_a^{(i)}(n + 1)
\end{bmatrix}
= D_p \cdot
\begin{bmatrix}
P_a^{(i)}(n + p + 1) \\
P_a^{(i)}(n + p) \\
\vdots \\
P_a^{(i)}(n + 1) \\
P_a^{(i)}(n)
\end{bmatrix}
\]

It can be readily established by mathematical induction that for \(n \geq p + 1\),

\[
(D_p)^n = \begin{bmatrix}
P_a^{(i)}(n + p + 1) & P_a^{(i)}(n + p + 2) & i^{p+2} \cdot P_a^{(i)}(n + 1) \\
P_a^{(i)}(n + p) & P_a^{(i)}(n + p + 1) & i^{p+2} \cdot P_a^{(i)}(n) \\
P_a^{(i)}(n + p - 1) & P_a^{(i)}(n + p) & i^{p+2} \cdot P_a^{(i)}(n - 1) \\
\vdots & \vdots & \vdots \\
P_a^{(i)}(n + 1) & P_a^{(i)}(n + 2) & i^{p+2} \cdot P_a^{(i)}(n - p + 1) \\
P_a^{(i)}(n) & P_a^{(i)}(n + 1) & i^{p+2} \cdot P_a^{(i)}(n - p)
\end{bmatrix}
\]

from which it is clear that \(\det D_p = i^{p+2}\). For more information on the companion matrices, see [14, 15].

Using the \((D_p)^n\) matrix, we determine the following relationships between complex-type Padovan-\(p\) numbers and the Padovan-\(p\) sequence for \(n \geq p + 1\).
such that every odd integer where \( p \geq 3 \):

\[
(D_p)^n = \begin{bmatrix}
  i^{n+4} \cdot Pap(n + p + 1) & i^{n+5} \cdot Pap(n + p + 2) & i^{n+2} \cdot Pap(n + 1) \\
  i^{n+3} \cdot Pap(n + p) & i^{n+4} \cdot Pap(n + p + 1) & i^{n+1} \cdot Pap(n) \\
  i^{n+2} \cdot Pap(n + p - 1) & i^{n+3} \cdot Pap(n + p) & i^n \cdot Pap(n - 1) \\
  \vdots & \vdots & \vdots \\
  i^{n-p+4} \cdot Pap(n + 1) & i^{n-p+5} \cdot Pap(n + 2) & i^{n-p+2} \cdot Pap(n - p + 1) \\
  i^{n-p+3} \cdot Pap(n) & i^{n-p+4} \cdot Pap(n + 1) & i^{n-p+1} \cdot Pap(n - p) \\
  i^{n-p+2} \cdot Pap(n - p + 2) & \cdots & i^{n+1} \cdot Pap(n) \\
  i^{n-p+3} \cdot Pap(n - p + 2) & \cdots & i^n \cdot Pap(n - 1)
\end{bmatrix}
\]

Now we concentrate on finding the Binet formulas for the complex-type Padovan-p numbers.

**Lemma 1.** Let \( p \) be a positive odd integer such that \( p \geq 3 \). The characteristic equation of the complex-type Padovan-p numbers \( x^{p+2} + x^p - i^{p+2} = 0 \) does not have multiple roots.

**Proof.** Let \( f (x) = x^{p+2} + x^p - i^{p+2} \). It is clear that \( f(0) \neq 0 \) and \( f(1) \neq 0 \) for all \( p \geq 3 \). Let \( \alpha \) be a multiple root of \( f (x) \), then \( \alpha \notin \{0, 1\} \). Since \( \alpha \) is a multiple root,

\[
f (\alpha) = \alpha^{p+2} + \alpha^p - i^{p+2} = 0
\]

and

\[
f' (\alpha) = (p + 2) \alpha^{p+1} + p \alpha^{p-1} = 0,
\]

hence

\[
f' (\alpha) = \alpha^{p-1} \left((p + 2) \alpha^2 + p\right) = 0.
\]

Thus we obtain \( \alpha = \pm \left(\frac{-p}{p+2}\right)^{\frac{1}{2}} \). Since \( p \) is a positive odd integer such that \( p \geq 3 \), \( f (\alpha) \neq 0 \), which is a contradiction. Thus, the equation \( f (x) = 0 \) does not have multiple roots. \( \square \)

Let \( f (x) \) be the characteristic polynomial of the matrix \( D_p \). Then we have \( f (x) = x^{p+2} + x^p - i^{p+2} \), which is a well-known fact from the companion matrices. If \( \delta_1, \delta_2, \ldots, \delta_{p+2} \) are roots of the equation \( x^{p+2} + x^p - i^{p+2} = 0 \),
then by Lemma 1, it is known that $\delta_1, \delta_2, \ldots, \delta_{p+2}$ are distinct. Define the $(p+2) \times (p+2)$ Vandermonde matrix $V^{p+2}$ as follows:

$$V^{p+2} = \begin{bmatrix}
(\delta_1)^{p+1} & (\delta_2)^{p+1} & \cdots & (\delta_{p+2})^{p+1} \\
(\delta_1)^p & (\delta_2)^p & \cdots & (\delta_{p+2})^p \\
\vdots & \vdots & \ddots & \vdots \\
\delta_1 & \delta_2 & \cdots & \delta_{p+2} \\
1 & 1 & \cdots & 1
\end{bmatrix}.$$

Assume that $V^{p+2}_{j,k}$ is a $(p+2) \times (p+2)$ matrix obtained from the Vandermonde matrix $V^{p+2}$ by replacing the $j^{th}$ column of $V^{p+2}$ by $W_t^p$, where $W_t^p$ is a $(p+2) \times 1$ matrix as follows

$$W_t^p = \begin{bmatrix}
(\delta_1)^{n+p+2-t} \\
(\delta_2)^{n+p+2-t} \\
\vdots \\
(\delta_{p+2})^{n+p+2-t}
\end{bmatrix}.$$

Then we can give the generalized Binet formula for the complex-type Padovan-$p$ numbers with the following theorem.

**Theorem 1.** Let $n \geq p+1$ and let $p$ be a positive odd integer such that $p \geq 3$, then

$$d_{j,k}^{Pa,p,n} = \frac{\det V^{p+2}_{j,k}}{\det V^{p+2}},$$

where $(D_p)^n = \left[d_{j,k}^{Pa,p,n}\right].$

**Proof.** Since the equation $x^{p+2} + x^p - i^{p+2} = 0$ does not have multiple roots for $p \geq 3$, when $p$ is a positive odd integer, the eigenvalues of the complex-type Padovan-$p$ matrix $D_p$ are distinct. Then, it is clear that $D_p$ is diagonalizable. Let $R_p = diag(\delta_1, \delta_2, \ldots, \delta_{p+2})$, then we may write $D_p V^{p+2} = V^{p+2} R_p$. Since the matrix $V^{p+2}$ is invertible, we obtain the equation $(V^{p+2})^{-1} D_p V^{p+2} = R_p$. Thus, $D_p$ is similar to $R_p$; hence, $(D_p)^n V^{p+2} = V^{p+2} (R_p)^n$ for $n \geq p+1$. Therefore we have the following linear system of equations:

$$\begin{cases}
d_{j,1}^{Pa,p,n} (\delta_1)^{p+1} + d_{j,2}^{Pa,p,n} (\delta_1)^p + \cdots + d_{j,p+2}^{Pa,p,n} = (\delta_1)^{n+p+2-t} \\
d_{j,1}^{Pa,p,n} (\delta_2)^{p+1} + d_{j,2}^{Pa,p,n} (\delta_2)^p + \cdots + d_{j,p+2}^{Pa,p,n} = (\delta_2)^{n+p+2-t} \\
\vdots \\
d_{j,1}^{Pa,p,n} (\delta_{p+2})^{p+1} + d_{j,2}^{Pa,p,n} (\delta_{p+2})^p + \cdots + d_{j,p+2}^{Pa,p,n} = (\delta_{p+2})^{n+p+2-t}.
\end{cases}$$

Then we conclude that

$$d_{j,k}^{Pa,p,n} = \frac{\det V^{p+2}_{j,k}}{\det V^{p+2}},$$

for each $j, k = 1, 2, \ldots, p+2$. \qed
Thus by Theorem 1 and the matrix \((D_p)^n\), we have the following useful result for the complex-type Padovan-\(p\) numbers.

**Corollary 1.** Let \(p\) be a positive odd integer such that \(p \geq 3\) and \(P_{a_p}^{(i)}(n)\) be the \(n\)th element of the complex-type Padovan-\(p\) number for \(n \geq p + 1\), then

\[
P_{a_p}^{(i)}(n) = \frac{\det V_{p+2}^2}{\det V_{p+2}}
\]

and

\[
P_{a_p}^{(i)}(n) = \frac{\det V_{2,3}^2}{i^{p+2} \cdot \det V_{p+2}} = \frac{\det V_{3,4}^2}{i^{p+2} \cdot \det V_{p+2}} = \cdots = \frac{\det V_{p+1,p+2}^2}{i^{p+2} \cdot \det V_{p+2}}.
\]

Let \(C(c_1, c_2, \ldots, c_v)\) be a \(v \times v\) companion matrix as follows:

\[
C(c_1, c_2, \ldots, c_v) = \begin{bmatrix}
1 & c_2 & \cdots & c_{v-1} & c_v \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}.
\]

**Theorem 2** (Chen and Louck [3]). The \((i, j)\) entry \(c_{i,j}^{(n)}(c_1, c_2, \ldots, c_v)\) in the matrix \(C^n(c_1, c_2, \ldots, c_v)\) is given by the following formula:

\[
c_{i,j}^{(n)}(c_1, c_2, \ldots, c_v) = \sum_{(t_1, t_2, \ldots, t_v)} t_i + t_j + 1 + \cdots + t_v \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} c_{t_1}^{t_1} \cdots c_{t_v}^{t_v},
\]

where the summation is over nonnegative integers satisfying \(t_1 + 2t_2 + \cdots + vt_v = n - i + j\), \(\binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} = \frac{(t_1 + \cdots + t_v)!}{t_1! \cdots t_v!}\) is a multinomial coefficient, and the coefficients in (3) are defined to be 1 if \(n = i - j\).

Here we investigate combinatorial representations for the complex-type Padovan-\(p\) numbers by the following corollary.

**Corollary 2.**

(i) For \(n \geq p + 1\),

\[
P_{a_p}^{(i)}(n) = \sum_{(t_1, t_2, \ldots, t_{p+2})} \binom{t_1 + t_2 + \cdots + t_{p+2}}{t_1, t_2, \ldots, t_{p+2}} (-1)^{t_2} (i^{p+2})^{t_{p+2}},
\]

where the summation is over nonnegative integers satisfying \(t_1 + 2t_2 + \cdots + (p + 2) t_{p+2} = n - p - 1\).

(ii) For \(n \geq p + 1\),

\[
P_{a_p}^{(i)}(n) = \frac{1}{i^{p+2}} \sum_{(t_1, t_2, \ldots, t_{p+2})} \binom{t_3 + t_4 + \cdots + t_{p+2}}{t_1 + t_2 + \cdots + t_{p+2}} \times \binom{t_1 + t_2 + \cdots + t_{p+2}}{t_1, t_2, \ldots, t_{p+2}} (-1)^{t_2} (i^{p+2})^{t_{p+2}}
\]

\[
= \frac{1}{i^{p+2}} \sum_{(t_1, t_2, \ldots, t_{p+2})} \binom{t_4 + t_5 + \cdots + t_{p+2}}{t_1 + t_2 + \cdots + t_{p+2}} \times \binom{t_1 + t_2 + \cdots + t_{p+2}}{t_1, t_2, \ldots, t_{p+2}} (-1)^{t_2} (i^{p+2})^{t_{p+2}}
\]

\[
= \frac{1}{i^{p+2}} \sum_{(t_1, t_2, \ldots, t_{p+2})} \binom{t_4 + t_5 + \cdots + t_{p+2}}{t_1 + t_2 + \cdots + t_{p+2}} \times \binom{t_1 + t_2 + \cdots + t_{p+2}}{t_1, t_2, \ldots, t_{p+2}} (-1)^{t_2} (i^{p+2})^{t_{p+2}}
\]
Theorem 3. Then we have the following theorem. Let us consider the matrix

\[
A_{p,m} = \begin{pmatrix} a_{k,j}^{(p,i,m)} \end{pmatrix}
\]

where the summation is over nonnegative integers satisfying

\[ t_1 + 2t_2 + \cdots + (p+2)t_{p+2} = n + 1. \]

Proof. In Theorem 2, if we take \( i = p+2 \) and \( j = 1 \) for the case (i), and \( i = p+1 \) and \( j = p+2 \) such that \( 3 \leq \varepsilon \leq p+2 \) for the case (ii), then we can directly see the conclusions from \((D_p)^n\). \( \square \)

Now we consider the permanental representations for the complex-type Padovan-\( p \) numbers.

Definition 1. A \( u \times v \) real matrix \( M = [m_{i,j}] \) is called a contractible matrix in the \( k \)-th column (resp. row) if the \( k \)-th column (resp. row) contains exactly two non-zero entries.

Suppose that \( x_1, x_2, \ldots, x_u \) are row vectors of the matrix \( M \). If \( M \) is contractible in the \( k \)-th column such that \( m_{i,k} \neq 0, m_{j,k} \neq 0 \) and \( i \neq j \), then the \((u-1) \times (v-1)\) matrix \( M_{ij:k} \) obtained from \( M \) by replacing the \( i \)-th row with \( m_{i,k}x_j + m_{j,k}x_i \) and deleting the \( j \)-th row. The \( k \)-th column is called the contraction in the \( k \)-th column relative to the \( i \)-th row and the \( j \)-th row.

In [2], Brualdi and Gibson obtained that \( \text{per}(M) = \text{per}(N) \) if \( M \) is a real matrix of order \( \alpha > 1 \) and \( N \) is a contraction of \( M \).

Now we concentrate on finding relationships among the complex-type Padovan-\( p \) numbers and the permanents of certain matrices that are obtained by using the generating matrix of the Padovan-\( p \) numbers. Let \( p \) be a positive odd integer such that \( p \geq 3 \) and let \( A_{p,m}^{(i)} = [a_{k,j}^{(p,i,m)}] \) be the \( m \times m \) super-diagonal matrix, defined by

\[
a_{k,j}^{(p,i,m)} = \begin{cases} i^{p+2}, & \text{if } k = \tau \text{ and } j = \tau + p + 1 \text{ for } 1 \leq \tau \leq m-p-1, \\ 1, & \text{if } k = \tau + 1 \text{ and } j = \tau \text{ for } 1 \leq \tau \leq m-1, \\ -1, & \text{if } k = \tau \text{ and } j = \tau + 1 \text{ for } 1 \leq \tau \leq m-1, \\ 0, & \text{otherwise.} \end{cases}
\]

Then we have the following theorem.

Theorem 3. For \( m \geq p+2 \) and \( p \geq 3 \),

\[
\text{per}A_{p,m}^{(i)} = P a_{p}^{(i)} (m + p + 1).
\]

Proof. Let us consider the matrix \( A_{p,m}^{(i)} \) and let the equation be hold for \( m \geq p+2 \). We prove by induction on \( m \). Then we show that the equation holds for \( m+1 \). If we expand the \( A_{p,m}^{(i)} \) by the Laplace expansion of permanent with respect to the first row, then we obtain

\[
\text{per}A_{p,m+1}^{(i)} = -\text{per}A_{m-1}^{(i,k)} + i^{p+2} \cdot \text{per}A_{p,m-p-1}^{(i)}.
\]
Since \( \text{per} A_{p,m}^{(i)} = Pa_p^{(i)}(m + p) \) and \( \text{per} A_{p,m-p}^{(i)} = Pa_p^{(i)}(m) \), it is clear that \( \text{per} A_{p,m+1}^{(i)} = Pa_p^{(i)}(m + p + 2) \). So the proof is complete. □

Let \( p \) be a positive odd integer such that \( p \geq 3 \) and let \( B_{p,m}^{(i)} = [b_{k,j}^{(p,i,m)}] \) be the \( m \times m \) super-diagonal matrix, defined by

\[
b_{k,j}^{(p,i,m)} = \begin{cases} 
  i^p + 2, & \text{if } k = \tau \text{ and } j = \tau + p + 1 \text{ for } 1 \leq \tau \leq m - p - 1, \\
  1, & \text{if } k = \tau + 1 \text{ and } j = \tau \text{ for } 1 \leq \tau \leq m - 4 \text{ and } k = \tau \text{ and } j = \tau + 1 \text{ for } m - 2 \leq \tau \leq m - 1, \\
  -1, & \text{if } k = \tau \text{ and } j = \tau + 1 \text{ for } 1 \leq \tau \leq m - 3 \text{ and } k = \tau + 1 \text{ and } j = \tau \text{ for } m - 3 \leq \tau \leq m - 1, \\
  0, & \text{otherwise,}
\end{cases}
\]

for \( m \geq p + 2 \).

Then we have the following theorem.

**Theorem 4.** For \( m \geq p + 2 \) and \( p \geq 3 \),

\[
\text{per} B_{p,m}^{(i)} = -Pa_p^{(i)}(m + p + 1). 
\]

**Proof.** Let us consider the matrix \( B_{p,m}^{(i)} \) and let the equation be hold for \( m \geq p + 2 \). We prove by induction on \( m \). Then we show that the equation holds for \( m + 1 \). If we expand the \( B_{p,m}^{(i)} \) by the Laplace expansion of permanent with respect to the first row, then we obtain

\[
\text{per} B_{p,m+1}^{(i)} = -\text{per} B_{m-1}^{(i)} + i^p + 2 \cdot \text{per} B_{p,m-p-1}^{(i)}. 
\]

Since \( \text{per} B_{p,m-1}^{(i)} = -Pa_p^{(i)}(m + p) \) and \( \text{per} B_{p,m-p-1}^{(i)} = -Pa_p^{(i)}(m) \), it is clear that \( \text{per} B_{p,m+1}^{(i)} = -Pa_p^{(i)}(m + p + 2) \). So the proof is complete. □

Assume next that \( C_{p,m} = [c_{k,j}^{(p,i,m)}] \) be the \( m \times m \) matrix, defined by

\[
C_{p,m}^{(i)} = \begin{bmatrix}
1 & \cdots & 1 & 0 & 0 \\
1 & 0 & \ddots & \ddots & \ddots \\
0 & & B_{m-1}^{(i)} & \ddots & \ddots \\
0 & & & \ddots & \ddots \\
0 & & & & \ddots & \ddots \\
\end{bmatrix}, \quad \text{for } m > p + 2,
\]

then we have the following results.

**Theorem 5.** For \( m > p + 2 \) and \( p \geq 3 \),

\[
\text{per} C_{p,m}^{(i)} = -\sum_{u=1}^{m+p} Pa_p^{(i)}(u). 
\]
Proof. If we extend $\text{per} C_{p,m}^{(i)}$ with respect to the first row, we write

$$\text{per} C_{p,m}^{(i)} = \text{per} C_{p,m-1}^{(i)} + \text{per} B_{p,m-1}^{(i)}.$$  

Thus, by the results and an inductive argument, the proof is easily seen. □

A matrix $M$ is called convertible if there is an $n \times n$ $(1, -1)$-matrix $K$ such that $	ext{per} M = \det (M \circ K)$, where $M \circ K$ denotes the Hadamard product of $M$ and $K$.

Now we give relationships among the complex-type Padovan-$p$ numbers and the determinants of certain matrices which are obtained by using the matrix $A_{p,m}^{(i)}$, $B_{p,m}^{(i)}$ and $C_{p,m}^{(i)}$. Let $m > p + 2$ and let $H$ be the $m \times m$ matrix, defined by

$$H = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ -1 & 1 & \ldots & 1 & 1 \\ 1 & -1 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & -1 & 1 \end{bmatrix}.$$  

Corollary 3. For $m > p + 2$ and $p \geq 3$,

$$\det \left( A_{p,m}^{(i)} \circ H \right) = P_{p}^{(i)} (m + p + 1),$$

$$\det \left( B_{p,m}^{(i)} \circ H \right) = -P_{p}^{(i)} (m + p + 1)$$

and

$$\det \left( C_{p,m}^{(i)} \circ H \right) = -\sum_{u=1}^{m+p} P_{p}^{(i)} (u).$$

Proof. Since $\text{per} A_{m}^{(i,k)} = \det \left( A_{p,m}^{(i)} \circ H \right)$, $\text{per} B_{m}^{(i,k)} = \det \left( B_{p,m}^{(i)} \circ H \right)$ and $\text{per} C_{m}^{(i,k)} = \det \left( C_{p,m}^{(i)} \circ H \right)$ for $m > p + 2$, by Theorem 3, Theorem 4 and Theorem 5, we have the conclusion. □

It is easy to see that the generating function of the complex-type Padovan-$p$ sequence $\{ P_{p}^{(i)} (n) \}$ is as follows:

$$g(x) = \frac{x^{p+1}}{1 + x^2 - (p+2) \cdot x^{p+2}},$$

where $p$ is a positive odd integer such that $p \geq 3$.

Now we are concerned about the exponential representation of the complex-type Padovan-$p$ numbers by the aid of the generating function with the following theorem.
Theorem 6. The complex-type Padovan-p sequence $\{P_{a_p}^{(i)}(n)\}$ have the following exponential representation:

$$g(x) = x^{p+1} \exp \left( \sum_{u=1}^{\infty} \left( \frac{x^u}{u} \left( -x + i^{p+2} \cdot x^{p+1} \right)^u \right) \right),$$

where $p$ is a positive odd integer such that $p \geq 3$.

Proof. Since

$$\ln g(x) = \ln x^{p+1} - \ln \left( 1 + x^2 - i^{p+2} \cdot x^{p+2} \right)$$

and

$$- \ln \left( 1 + x^2 - i^{p+2} \cdot x^{p+2} \right) = -\left[ -x \left( -x + i^{p+2} \cdot x^{p+1} \right) \right.
\left. - \frac{1}{2} x^2 \left( -x + i^{p+2} \cdot x^{p+1} \right)^2 - \ldots \right. \left. - \frac{1}{u} x^u \left( -x + i^{p+2} \cdot x^{p+1} \right)^u - \ldots \right]$$

it is clear that

$$g(x) = x^{p+1} \exp \left( \sum_{u=1}^{\infty} \left( \frac{x^u}{u} \left( -x + i^{p+2} \cdot x^{p+1} \right)^u \right) \right).$$

Thus we have the conclusion. \(\square\)

Now we give the sums of the complex-type Padovan-p numbers. Let

$$S_n = \sum_{u=1}^{n} P_{a_p}^{(i)}(u),$$

for $n \geq p+1$ and $p$ is a positive odd integer such that $p \geq 3$, and suppose that $R_p$ is the $(p+3) \times (p+3)$ matrix such that

$$R_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & D_p \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}.$$ 

If we use induction on $n$, then we obtain

$$\left( R_p \right)^n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ S_{n+p} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S_{n-1} & \cdots & \cdots & \left( D_p \right)^n \end{bmatrix}.$$
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