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A geometric approach to the Proinov type contractions

Nihal Taş

Abstract. In this paper, we prove some fixed-circle, fixed-disc and fixed-ellipse results on metric spaces. To do this, we define the notions of Proinov type a_0-contraction and generalized Proinov type a_0-contraction. Also, we give some illustrative examples to show the validity of our obtained results. Finally, we present a nice application to exponential linear unit activation functions.

1. Introduction and motivation

What is the "*Fixed-circle problem*"?

Before we recall the definition of a fixed circle, we give a brief history of this problem.

Now, let us consider the following self-mappings $T_1 : \mathbb{R} \to \mathbb{R}$ and $T_2 : \mathbb{R} \to \mathbb{R}$ defined as

$$T_1 a = \frac{a}{1+e^{-a}}$$

and

$$T_2 a = \begin{cases} 0.01a, & \text{for } a < 0, \\ a, & \text{for } a \geq 0, \end{cases}$$

for all $a \in \mathbb{R}$. Then the fixed point set of T_1 is $Fix(T_1) = \{0\}$ and the fixed point set of T_2 is $Fix(T_2) = \{a \in \mathbb{R} : a \geq 0\}$. We note that the self-mapping T_1 has a unique fixed point and the self-mapping T_2 has infinitely number of fixed point.

Also, if we consider the self-mapping $T_3 : \mathbb{R} \to \mathbb{R}$ defined as

$$T_3 x = e^{-x^2},$$

for all $a \in \mathbb{R}$, then the fixed point set of T_3 is $Fix(T_3) = \emptyset$, that is, T_3 does not have a fixed point.

On the other hand, these self-mappings T_1, T_2 and T_3 are activation functions. T_1 is a Sigmoid linear unit activation function, T_2 is a Leakly rectified

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linear unit activation function and T_3 is a Gaussian activation function (see [6, 8, 11, 21]).

After the above motivations, "*Fixed-circle problem*" has been occurred as a geometric approach to the fixed-point theory when the self-mapping $T : X \to X$ has more than one fixed point [14]. Now, we recall the notion of a fixed circle.

Let (X, d) be a metric space and $T : X \to X$ a self-mapping. Then the circle is defined by

$$C_{a_0,\mu} = \{a \in X : d(a, a_0) = \mu\}.$$

If $Ta = a$ for every $a \in C_{a_0,\mu}$, then $C_{a_0,\mu}$ is called the fixed circle of T (see [14]).

After that, the notion of a fixed circle was generalized to the notion of a fixed disc as follows.

Let (X, d) be a metric space and $T : X \to X$ a self-mapping. Then the disc is defined by

$$D_{a_0,\mu} = \{a \in X : d(a, a_0) \leq \mu\}.$$

If $Ta = a$ for every $a \in D_{a_0,\mu}$, then $D_{a_0,\mu}$ is called the fixed disc of T (see [12] and [15]).

In many studies, some solutions were presented using various approaches, techniques and contractive conditions (see, for example, [12, 13, 15, 17, 18, 19, 22, 23] and the references therein). Recently, the notions of a fixed circle and a fixed disc were generalized the notion of a fixed figure.

A geometric figure $\mathcal{F}$ (a circle, an ellipse, a hyperbola, a Cassini curve, etc.) contained in the fixed point set $Fix\,(T)$ is called a *fixed figure* (a fixed circle, a fixed ellipse, a fixed hyperbola, a fixed Cassini curve, etc.) of the self-mapping T (see [16]). In this context, some fixed-ellipse theorems were obtained using different aspects (see, for example, [7, 9, 16] for more details).

The main goal of this paper is to obtain new solutions to the fixed-circle problem on metric spaces. To do this, we inspire some known techniques given in [20]. We introduce the notions of Proinov type a_0-contraction and generalized Proinov type a_0-contraction. Using these new notions, we prove two fixed-circle theorems and two fixed-disc results. On the other hand, using the similar approaches, we obtain two fixed-ellipse theorems. The obtained results are supported by some necessary examples. Finally, we give an application to exponential linear unit activation functions.

2. Main results

In this section, we present new solutions to the fixed-circle problem using Proinov type contractions.

Definition 1. Let (X, d) be a metric space and $T : X \to X$ a self-mapping. If there exists $a_0 \in X$ such that

$$d(Ta, a) > 0 \Longrightarrow \psi\,(d(Ta, a)) \leq \varphi\,(d(a, a_0)),$$

for all $a \in X$, where $\psi, \varphi : (0, \infty) \to \mathbb{R}$ are two functions with $\varphi(t) < \psi(t)$ for $t > 0$ and ψ is nondecreasing, then T is called Proinov type a_0-contraction.

Proposition 1. *Let (X, d) be a metric space and $T : X \to X$ Proinov type a_0-contraction with $a_0 \in X$. Then we get $Ta_0 = a_0$.*

Proof. Suppose $d(Ta_0, a_0) > 0$, that is, $Ta_0 \neq a_0$. Then using the Proinov type a_0-contractive condition, we get

$$\psi\left(d(Ta_0, a_0)\right) \leq \varphi\left(d(a_0, a_0)\right) = \varphi(0),$$

which is a contradiction with the definition of φ. So, it should be $Ta_0 = a_0$. □

Using Definition 1 and Proposition 1, we prove the following fixed-circle theorem.

Theorem 1. *Let (X, d) be a metric space, $T : X \to X$ Proinov type a_0-contraction with $a_0 \in X$ and μ defined as*

$$\mu = \inf\left\{d(Ta, a) : Ta \neq a, a \in X\right\}. \tag{1}$$

Then T fixes the circle $C_{a_0,\mu}$.

Proof. Now we consider the following two cases:

Case 1: Let $\mu = 0$. Hence we get $C_{a_0,\mu} = \{a_0\}$. Using Proposition 1, we have $Ta_0 = a_0$, that is, $C_{a_0,\mu}$ is a fixed circle of T.

Case 2: Let $\mu > 0$ and $a \in C_{a_0,\mu}$ be any point such that $d(Ta, a) > 0$. Using the Proinov type a_0-contraction hypothesis, we obtain

$$\psi\left(d(Ta, a)\right) \leq \varphi\left(d(a, a_0)\right) = \varphi\left(\mu\right) < \psi\left(\mu\right) \leq \psi\left(d(Ta, a)\right),$$

a contradiction. It should be $d(Ta, a) = 0$, that is, $Ta = a$.

Under the above cases, we see that T fixes the circle $C_{a_0,\mu}$. □

Corollary 1. *Let (X, d) be a metric space, $T : X \to X$ Proinov type a_0-contraction with $a_0 \in X$ and μ defined as in* (1). *Then T fixes the disc $D_{a_0,\mu}$.*

Proof. By the similar arguments used in the proof of Theorem 1, the proof can be easily obtained. □

Now we give the following examples for Proposition 1, Theorem 1 and Corollary 1.

Example 1. Let $X = \mathbb{R}$ be the usual metric space with the usual metric d defined as

$$d(a, b) = |a - b|,$$

for all $a, b \in \mathbb{R}$. Let us define the function $T : \mathbb{R} \to \mathbb{R}$ as

$$Ta = \begin{cases} a, & \text{for } |a| \leq 2, \\ a + 1, & \text{for } |a| > 2, \end{cases}$$

for all $a \in \mathbb{R}$. Then the function T is Proinov type a_0-contraction with $a_0 = 0$, the function $\psi : (0, \infty) \to \mathbb{R}$ defined by $\psi(t) = 3t$ and the function $\varphi : (0, \infty) \to \mathbb{R}$ defined by $\varphi(t) = 2t$. Consequently, we have $\mu = 1$ and so T fixes the circle $C_{0,1} = \{-1, 1\}$ and the disc $D_{0,1} = [-1, 1]$.

In the following example we can see that the selection of a point $a_0 \in X$ doesn't have to be unique.

Example 2. Let $X = \mathbb{R}$ be the usual metric space with the usual metric d. Let us define the function $T : \mathbb{R} \to \mathbb{R}$ as

$$Ta = \begin{cases} a, & \text{for } a \in [-3, \infty), \\ a+1, & \text{for } a \in (-\infty, -3), \end{cases}$$

for all $a \in \mathbb{R}$. Then the function T is Proinov type a_0-contraction with both $a_0 = 0$ and $a_0 = 6$, the function $\psi : (0, \infty) \to \mathbb{R}$ defined by $\psi(t) = 3t$ and the function $\varphi : (0, \infty) \to \mathbb{R}$ defined by $\varphi(t) = 2t$. Consequently, we have $\mu = 1$ and so T fixes the circles $C_{0,1} = \{-1, 1\}$, $C_{6,1} = \{5, 7\}$ and the discs $D_{0,1} = [-1, 1]$, $D_{6,1} = [5, 7]$.

The following theorem can be considered as a fixed-ellipse theorem. Let $E_\mu(a_1, a_2)$ be the ellipse defined as

$$E_\mu(a_1, a_2) = \{a \in X : d(a, a_1) + d(a, a_2) = \mu\}.$$

Theorem 2. *Let (X, d) be a metric space, $T : X \to X$ a self-mapping and μ defined as in (1). If there exist $a_1, a_2 \in X$ such that*

$$d(Ta, a) > 0 \Longrightarrow \psi(d(Ta, a)) \leq \varphi(d(a, a_1) + d(a, a_2)), \tag{2}$$

for all $a \in X$, where $\psi, \varphi : (0, \infty) \to \mathbb{R}$ are two functions with $\varphi(t) < \psi(t)$ for $t > 0$ and ψ is nondecreasing, then $E_\mu(a_1, a_2) \subseteq Fix(T)$.

Proof. We show $E_\mu(a_1, a_2) \subseteq Fix(T)$ under the following cases:

Case 1: Let $\mu = 0$. Then we get

$$E_\mu(a_1, a_2) = C_{a_1,\mu} = C_{a_2,\mu} = \{a_1\} = \{a_2\}.$$

Assume that $d(Ta_1, a_1) = d(Ta_2, a_2) > 0$. Using the inequality (2), we have

$$\psi(d(Ta_1, a_1)) \leq \varphi(d(a_1, a_1) + d(a_1, a_1)) = \varphi(0),$$

which is a contradiction with the definition of φ. Hence, it should be $Ta_1 = a_1 = a_2 = Ta_2$.

Case 2: Let $\mu > 0$ and $a \in E_\mu(a_1, a_2)$ be any point such that $Ta \neq a$, that is, $d(Ta, a) > 0$. Using the inequality (2), we obtain

$$\psi(d(Ta, a)) \leq \varphi(d(a, a_1) + d(a, a_2)) = \varphi(\mu) < \psi(\mu) \leq \psi(d(Ta, a)),$$

a contradiction. So, it should be $Ta = a$.

Consequently, we prove $E_\mu(a_1, a_2) \subseteq Fix(T)$. □

We give the following example to show the validity of Theorem 2. Also, in this example, we see that the selection of points $a_1, a_2 \in X$ and the fixed ellipse $E_\mu(a_1, a_2)$ are not to be unique.

Example 3. Let $X = \{-2, -1, 1, 2, 8, 12\}$ be a metric space with the usual metric d. Let us define the function $T : X \to X$ as

$$Ta = \begin{cases} a + 4, & \text{for } a = 8, \\ a, & \text{for } a \in X - \{8\}, \end{cases}$$

for all $a \in X$. Then T satisfies the conditions of Theorem 2 with $a_1 = -1$, $a_2 = 1$, the function $\psi : (0, \infty) \to \mathbb{R}$ defined by $\psi(t) = 4t$ and the function $\varphi : (0, \infty) \to \mathbb{R}$ defined by $\varphi(t) = \frac{3}{2}t$. We have $\mu = 4$ and so $E_4(-1, 1) = \{-2, 2\} \subseteq Fix(T) = X - \{8\}$. Also, T satisfies the conditions of Theorem 2 with $a_1 = -2$, $a_2 = 2$, the function $\psi : (0, \infty) \to \mathbb{R}$ defined by $\psi(t) = 4t$ and the function $\varphi : (0, \infty) \to \mathbb{R}$ defined by $\varphi(t) = \frac{3}{2}t$. Consequently, we get $E_4(-2, 2) = \{-2, -1, 1, 2\} \subseteq Fix(T) = X - \{8\}$.

Now, we give a new notion of a generalized contraction.

Definition 2. Let (X, d) be a metric space and $T : X \to X$ a self-mapping. If there exists $a_0 \in X$ such that

$$d(Ta, a) > 0 \Longrightarrow \psi(d(Ta, a)) \leq \varphi(m(a, a_0)),$$

for all $a \in X$, where $\psi, \varphi : (0, \infty) \to \mathbb{R}$ are two functions with $\varphi(t) < \psi(t)$ for $t > 0$, ψ is nondecreasing and

$$m(a, b) = \max\left\{d(a, b), d(a, Ta), d(b, Tb), \frac{d(a, Tb) + d(b, Ta)}{2}\right\},$$

then T is called generalized Proinov type a_0-contraction.

Proposition 2. *Let (X, d) be a metric space and $T : X \to X$ generalized Proinov type a_0-contraction with $a_0 \in X$. Then we get $Ta_0 = a_0$.*

Proof. Suppose $d(Ta_0, a_0) > 0$. Using the generalized Proinov type a_0-contractive condition and the symmetry property, we obtain

$$m(a_0, a_0) = d(a_0, Ta_0)$$

and

$$\psi(d(Ta_0, a_0)) \leq \varphi(m(a_0, a_0)) = \varphi(d(a_0, Ta_0)) = \varphi(d(Ta_0, a_0)) < \psi(d(Ta_0, a_0)),$$

a contradiction. Hence, it should be $Ta_0 = a_0$. □

We prove the following fixed-circle theorem using the notion of generalized Proinov type a_0-contraction.

Theorem 3. *Let (X, d) be a metric space, $T : X \to X$ generalized Proinov type a_0-contraction with $a_0 \in X$ and μ defined as in (1). If $d(Ta, a_0) \leq \mu$, then T fixes the circle $C_{a_0,\mu}$.*

Proof. Let us consider the following cases:

Case 1: Let $\mu = 0$. Then we get $C_{a_0,\mu} = \{a_0\}$. From Proposition 2, we have $Ta_0 = a_0$.

Case 2: Let $\mu > 0$ and $a \in C_{a_0,\mu}$ be any point such that $d(Ta, a) > 0$. Using the hypothesis and symmetry property, we get

$$m(a, a_0) \leq d(Ta, a)$$

and

$$\psi\left(d(Ta, a)\right) \leq \varphi\left(m(a, a_0)\right) < \psi\left(m(a, a_0)\right) \leq \psi\left(d(Ta, a)\right),$$

a contradiction. It should be $Ta = a$.

Consequently, T fixes the circle $C_{a_0,\mu}$. □

Corollary 2. *Let (X, d) be a metric space, $T : X \to X$ generalized Proinov type a_0-contraction with $a_0 \in X$ and μ defined as in* (1)*. If $d(Ta, a_0) \leq \mu$, then T fixes the disc $D_{a_0,\mu}$.*

Proof. From the similar arguments used in the proof of Theorem 3, it can be easily proved. □

We give the following example for Proposition 2, Theorem 3 and Corollary 2. From this example, we can say that the selection of a point $a_0 \in X$ doesn't have to be unique.

Example 4. Let $X = \mathbb{R}$ be the usual metric space with the usual metric d. Let us define the function $T : \mathbb{R} \to \mathbb{R}$ as

$$Ta = \begin{cases} a, & \text{for } a \in [-5, \infty), \\ a + 2, & \text{for } a \in (-\infty, -5), \end{cases}$$

for all $a \in \mathbb{R}$. Then the function T is generalized Proinov type a_0-contraction with both $a_0 = 0$ and $a_0 = 4$, the function $\psi : (0, \infty) \to \mathbb{R}$ defined by $\psi(t) = 5t$ and the function $\varphi : (0, \infty) \to \mathbb{R}$ defined by $\varphi(t) = 3t$. Consequently, we have $\mu = 2$ and so T fixes the circles $C_{0,2} = \{-2, 2\}$, $C_{4,2} = \{2, 6\}$ and the discs $D_{0,2} = [-2, 2]$, $D_{4,2} = [2, 6]$.

Now we prove the following fixed-ellipse theorem.

Theorem 4. *Let (X, d) be a metric space, $T : X \to X$ a self-mapping, μ defined as in* (1) *and the number $m(a, b)$ defined as in Definition 2. Suppose that there exist $a_1, a_2 \in X$ such that*

$$d(Ta, a) > 0 \Longrightarrow \psi\left(d(Ta, a)\right) \leq \varphi\left(\frac{m(a, a_1) + m(a, a_2)}{2}\right), \tag{3}$$

for all $a \in X$, where $\psi, \varphi : (0, \infty) \to \mathbb{R}$ are two functions with $\varphi(t) < \psi(t)$ for $t > 0$, ψ is nondecreasing. If

$$d(Ta, a_1) \leq \mu \text{ and } d(Ta, a_2) \leq \mu, \tag{4}$$

for $a \in E_\mu(a_1, a_2)$, then $E_\mu(a_1, a_2) \subseteq Fix(T)$.

Proof. We investigate the following cases:

Case 1: Let $\mu = 0$. Then we get

$$E_\mu(a_1, a_2) = C_{a_1,\mu} = C_{a_2,\mu} = \{a_1\} = \{a_2\}.$$

Suppose that $d(Ta_1, a_1) > 0$. Using the inequality (3) and the condition $a_1 = a_2$, we obtain

$$\begin{aligned} \psi\left(d(Ta_1, a_1)\right) &\leq \varphi\left(\frac{m(a_1, a_1) + m(a_1, a_1)}{2}\right) = \varphi\left(m(a_1, a_1)\right) \\ &= \varphi\left(d(Ta_1, a_1)\right) < \psi\left(d(Ta_1, a_1)\right), \end{aligned}$$

a contradiction. So, it should be

$$Ta_1 = a_1 = a_2 = Ta_2. \tag{5}$$

Case 2: Let $\mu > 0$ and $a \in E_\mu(a_1, a_2)$ be any point such that $Ta \neq a$. Using the inequalities (3), (4) and the equality (5), we get

$$\psi\left(d(Ta, a)\right) \leq \varphi\left(\frac{m(a,a_1)+m(a,a_2)}{2}\right) < \psi\left(\frac{m(a,a_1)+m(a,a_2)}{2}\right) \leq \psi\left(d(Ta, a)\right),$$

a contradiction. Hence, it should be $Ta = a$.

Under the above cases, we see $E_\mu(a_1, a_2) \subseteq Fix(T)$. □

In the following example, we see that an example shows the validity of Theorem 4 and the selection of points $a_1, a_2 \in X$ doesn't have to be unique.

Example 5. Let $X = \{-4, -1, 1, 4, 6, 14\}$ be a metric space with the usual metric d. Let us define the function $T : X \to X$ as

$$Ta = \begin{cases} a + 8, & \text{for } a = 6, \\ a, & \text{for } a \in X - \{6\}, \end{cases}$$

for all $a \in X$. Then T satisfies the conditions of Theorem 4 with $a_1 = -1$, $a_2 = 1$, the function $\psi : (0, \infty) \to \mathbb{R}$ defined by $\psi(t) = 3t$ and the function $\varphi : (0, \infty) \to \mathbb{R}$ defined by $\varphi(t) = \frac{5}{2}t$. We have $\mu = 8$ and so $E_8(-1, 1) = \{-4, 4\} \subseteq Fix(T) = X - \{8\}$. Also, T satisfies the conditions of Theorem 4 with $a_1 = -4$, $a_2 = 4$, the function $\psi : (0, \infty) \to \mathbb{R}$ defined by $\psi(t) = 3t$ and the function $\varphi : (0, \infty) \to \mathbb{R}$ defined by $\varphi(t) = \frac{5}{2}t$. Consequently, we get $E_8(-4, 4) = \{-4, -1, 1, 4\} \subseteq Fix(T) = X - \{8\}$.

If we inspire the notion of a Proinov type a_0-contraction and Amini-Harandi and Petruşel's fixed point theorem given in [1], then we get the following theorem.

Theorem 5. *Let (X, d) be a metric space, $T : X \to X$ a self-mapping and μ defined as in* (1). *If there exists $a_0 \in X$ such that*

$$d(Ta, a) > 0 \Longrightarrow \psi\left(d(Ta, a)\right) \leq \varphi\left(d(a, a_0)\right),$$

for all $a \in X$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$, $\varphi(t) < \psi(t)$ for $t > 0$ with $\psi(0) = \varphi(0) = 0$ and ψ is nondecreasing, then we have $C_{a_0,\mu} \subseteq Fix(T)$ (especially, $D_{a_0,\mu} \subseteq Fix(T)$).

Proof. By the similar approaches used in the proofs of Proposition 1 and Theorem 1, it can be easily seen. □

We note that Theorem 5 can be considered as an Amini-Harandi and Petruşel type fixed-circle (resp. Amini-Harandi and Petruşel type fixed-disc) theorem.

Theorem 6. *Let (X, d) be a metric space, $T : X \to X$ a self-mapping and μ defined as in (1). If there exist $a_1, a_2 \in X$ such that*

$$d(Ta, a) > 0 \Longrightarrow \psi\left(d(Ta, a)\right) \leq \varphi\left(d(a, a_1) + d(a, a_2)\right),$$

for all $a \in X$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$, $\varphi(t) < \psi(t)$ for $t > 0$ with $\psi(0) = \varphi(0) = 0$ and ψ is nondecreasing, then we have $E_\mu\left(a_1, a_2\right) \subseteq Fix(T)$.

Proof. By the similar arguments used in the proof of Theorem 2, it can be easily proved. □

We note that Theorem 6 can be considered as an Amini-Harandi and Petruşel type fixed-ellipse theorem.

From the above results with inspiring some known fixed-point theorems given in [2, 3, 4], we obtain the following important remarks.

Remark 1. (1) If we take $\psi(t) = t$ and $\varphi(t) = \lambda t$ $(0 \leq \lambda < 1)$ in Definition 1, then we obtain the notion of Banach type a_0-contraction. Theorem 1 can be considered as a Banach type fixed-circle theorem under this case. Similarly, Corollary 1 can be considered as a Banach type fixed-disc result and Theorem 2 can be considered as a Banach type fixed-ellipse theorem.

(2) If we take $\psi(t) = t$ and $\varphi(t) = \lambda t$ $(0 \leq \lambda < 1)$ in Definition 2, then we obtain the notion of Ćirić type a_0-contraction. Under this case, Theorem 3 (resp. Corollary 2 and Theorem 4) can be considered as a Ćirić type fixed-circle theorem (resp. a Ćirić type fixed-disc result and a Ćirić type fixed-ellipse theorem).

(3) If we take $\psi(t) = t$ in Definition 1, then we have the notion of Boyd-Wong type a_0-contraction. In this case, Theorem 1 (resp. Corollary 1 and Theorem 2) can be considered as a Boyd-Wong type fixed-circle theorem (resp. a Boyd-Wong type fixed-disc result and a Boyd-Wong type fixed-ellipse theorem).

3. An application to exponential linear unit activation functions

In this section, we obtain an application to activation functions to show the applicable of the obtained results.

What are activation functions? What is the importance of them?

Activation functions are mathematical functions used to convert an input signal of a node in the neural networks to an output signal. The importance of them is that they are used for constructing a neural network to learn and make sense of something. In the literature, a lot of activation functions are used in the neural networks. One of them is "*Scaled exponential linear unit* *activation function (SELU)*" and another one is "*Exponential linear unit activation function (ELU)*" (see [5] and [10]).

What is the significance of the activation functions for the fixed-circle problem?

Activation functions are important by means of an application to the fixed-circle problem. The obtained theoretical results related to the fixed-circle problem are applicable to various activation functions. Some authors gave some nice applications for this subject (see, for example, [15, 17, 18, 19, 22]).

In the light of the above questions, we give an application to exponential linear unit activation functions as follows.

At first, we recall the definitions of the activation functions SELU and ELU.

$$SELU(a) = \lambda \begin{cases} \alpha\left(e^a - 1\right), & \text{for } a < 0, \\ a, & \text{for } a \geq 0, \end{cases}$$

where $\lambda = 1.0507$, $\alpha = 1.67326$ and

$$ELU(a) = \begin{cases} \alpha\left(e^a - 1\right), & \text{for } a < 0, \\ a, & \text{for } a \geq 0, \end{cases}$$

where $\alpha \geq 0$.

Let us take $\alpha = 0$ in $ELU(a)$. Then we get

$$Ta = \begin{cases} 0, & \text{for } a < 0, \\ a, & \text{for } a \geq 0. \end{cases}$$

Now let $X = \{-e, 0, e, 3-e, 3, 3+e\}$ and d be the usual metric on X. Then T satisfies the condition of Theorem 1 with $a_0 = 3$ and the functions $\psi(t) = t + 1$, $\varphi(t) = t$. Consequently, T fixes the circle $C_{3,e} = \{3-e, 3+e\}$ and the disc $D_{3,e} = \{3-e, e, 3, 3+e\}$.

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Nihal Taş
Balikesir University
Department of Mathematics
10145 Balikesir
Turkey
E-mail address: `nihaltas@balikesir.edu.tr`