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Generalized fixed point theorems on metric spaces

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Abstract. In this paper, we establish some fixed point theorems for single valued and multi-valued mappings on a complete metric space. Suzuki's and some other fixed point theorems are generalized by taking a more general contractive condition for single valued mappings. It is also proved that our result characterizes the completeness of the metric space. Further, taking generalized contractive condition, a fixed point theorem is also established for multi-valued mappings.

1. Introduction

The Banach contraction principle (BCP), because of its wide applications in different areas of study, has played an important role in various fields of mathematical analysis. The BCP states that: *Let* (X, d) *be a complete metric space then a self mapping* T *on* X *such that, for each* $x, y \in X$, *there exists* $0 \leq a < 1$ *satisfying the condition* $d(Tx, Ty) \leq a\, d(x, y)$ *has a unique fixed point.* This result is considered as a main source of metric fixed point theory. However, the BCP does not characterize the completeness of the metric space. This can be easily seen from an example due to Connell (Example 3, [6]), see also [20]. In fact, for the continuous mappings, the fixed point property does not ensure the completeness of the metric space.

In 1969, Kannan [10] proved the following result.

Theorem 1 ([10]). *Let* (X, d) *be a complete metric space, and* $T : X \to X$. *If there exists* $r \in [0, \frac{1}{2})$ *such that for all* $x, y \in X$,

$$d(Tx, Ty) \leq r\,[d(x, Tx) + d(y, Ty)]. \tag{1}$$

Then T *has a unique fixed point.*

In 1972, Chatterjea [5] proved the following result.

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Theorem 2 ([5]). *Let (X,d) be a complete metric space, and $T : X \to X$. If there exists $r \in [0, \frac{1}{2})$ such that for all $x, y \in X$,*

$$d(Tx,Ty) \le r\,[d(x,Ty) + d(y,Tx)]. \tag{2}$$

Then T has a unique fixed point.

It is to be noted that the Kannan's theorem is not a generalization of the BCP. Also, the Kannan's result is important because Subrahmanyam [19] proved that a metric space is complete if and only if each Kannan type contraction on it has a fixed point.

In the last decade (2008), Suzuki [20] gave a simple but important generalization of the BCP which also preserves the metric completeness of the space. In fact, Suzuki proved the following.

Theorem 3 ([20]). *Let (X,d) be a complete metric space, and $T : X \to X$. Define a non-increasing function $\theta : [0,1) \to (\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \le r \le \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2}, & \text{if } \frac{\sqrt{5}-1}{2} \le r \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that for each $x, y \in X$,

$$\theta(r) d(x,Tx) \le d(x,y) \quad \text{implies} \quad d(Tx,Ty) \le r\, d(x,y). \tag{3}$$

Then T has a unique fixed point $z \in X$. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

In the same paper, Suzuki has been shown that his condition (3) is independent from Kannan's condition (1) with some examples.

Using Suzuki type contraction, several generalizations of the BCP and other results, not only in metric spaces but also in other settings of the spaces, have been obtained by many researchers (see, [2–4, 8, 9, 11, 12, 14, 16–18, 20] and references therein).

Let (X,d) be a metric space, and $T : X \to X$. Then for all $x, y \in X$, we denote

$$\begin{aligned} m(Tx,Ty) = a\, d(x,y) &+ b \max\{d(x,Tx), d(y,Ty)\} \\ &+ c[d(x,Ty) + d(y,Tx)], \end{aligned} \tag{4}$$

where a, b and c are non-negative reals such that $a+b+2c = r$ with $r \in [0,1)$.

Now, we consider the following generalized contractive condition

$$\begin{aligned} &\theta(r) \min\{d(x,Tx), d(x,Ty)\} \le d(x,y) \\ &\quad \text{implies} \quad d(Tx,Ty) \le m(Tx,Ty). \end{aligned} \tag{5}$$

It is remarkable that condition (5) is a generalization of the condition (22) and several other conditions mentioned in [15].

2. Main results

Theorem 4. *Let (X,d) be a complete metric space, and $T : X \to X$. Assume that there exists $r \in [0,1)$ such that the condition (5) is satisfied for each $x, y \in X$, where $\theta : [0,1) \to (\frac{1}{2}, 1]$ is as defined in Theorem 3. Then T has a unique fixed point $z \in X$. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.*

Proof. If $\min\{d(x,Tx), d(x,Ty)\} = d(x,Tx)$, then $\theta(r)d(x,Tx) \le d(x,Tx)$. Also, if $\min\{d(x,Tx), d(x,Ty)\} = d(x,Ty)$, then $d(x,Ty) \le d(x,Tx)$, which implies $\theta(r)d(x,Ty) \le \theta(r)d(x,Tx) \le d(x,Tx)$. Thus, in either cases, we find

$$\theta(r)\min\{d(x,Tx), d(x,Ty)\} \le d(x,Tx),$$

so, using (5), we have

$$\begin{aligned} d(Tx,T^2x) &\le m(Tx,T^2x) \\ &\le a\,d(x,Tx) + b\,\max\{d(x,Tx), d(Tx,T^2x)\} \\ &\qquad + c\,[d(x,Tx) + d(Tx,T^2x)] \\ &\le (a+b+2c)\max\{d(x,Tx), d(Tx,T^2x)\} \\ &= r\,\max\{d(x,Tx), d(Tx,T^2x)\}, \end{aligned}$$

which provides that

$$d(Tx,T^2x) \le r\,d(x,Tx), \quad \forall\, x \in X. \tag{6}$$

Consider an arbitrary point $u_0 = u \in X$ and define a sequence $\{u_n\}$ in X such that $u_n = T^n u$. Then using (6), we get $d(u_n, u_{n+1}) \le r^n\, d(u, Tu)$, and so $\sum_{n=1}^{\infty} d(u_n, u_{n+1}) < \infty$. Thus, $\{u_n\}$ is a Cauchy sequence in X and completeness implies that $\{u_n\}$ converges to a point $z \in X$. Now, we show that

$$d(Tx,z) \le r\,\max\{d(x,z), d(x,Tx)\} \quad \forall\, x \in X\backslash\{z\}. \tag{7}$$

For if, $x \in X\backslash\{z\}$, there exists an $n_0 \in \mathbb{N}$ such that $d(u_n, z) \le \frac{1}{3}d(x,z)$ for all $n \in \mathbb{N}$ with $n \ge n_0$. Then, we obtain

$$\begin{aligned} \theta(r)\min\{d(u_n,Tu_n), d(u_n,Tx)\} &\le d(u_n,Tu_n) \\ &\le d(u_n,z) + d(u_{n+1},z) \\ &\le \frac{2}{3}d(z,x) = d(x,z) - \frac{1}{3}d(x,z) \\ &\le d(x,z) - d(u_n,z) \\ &\le d(u_n,x), \end{aligned}$$

and using (5), we have $d(Tu_n, Tx) \le m(Tu_n, Tx)$ for $n \ge n_0$. Taking the limit as $n \to \infty$, we have

$$\begin{aligned} d(Tx,z) &\leq a\,d(x,z)+b\,d(x,Tx)+c\,[d(z,Tx)+d(x,z)] \\ &\leq (a+b+2c)\max\{d(x,z),d(x,Tx),d(Tx,z)\} \\ &= r\,\max\{d(x,z),d(x,Tx)\}, \end{aligned}$$

which is the condition (7).

Furthermore, we show that

$$T^k z = z, \text{ for some } k\in\mathbb{N}. \tag{8}$$

To see, let us assume that it is not happening, i.e., $T^k z\neq z$ for all $k\in\mathbb{N}$. Then by induction, first we show that

$$d(T^{k+1}z,z)\leq r^k\,d(z,Tz), \quad \forall\,k\in\mathbb{N}. \tag{9}$$

Now, from (7), we get

$$\begin{aligned} d(z,T^2z) &\leq r\,\max\{d(z,Tz),d(Tz,T^2z)\} \\ &\leq r\,\max\{d(z,Tz),r\,d(z,Tz)\}=r\,d(z,Tz). \end{aligned}$$

Also, suppose that $d(z,T^{k+1}z)\leq r^k\,d(z,Tz)$, then we have

$$\begin{aligned} d(z,T^{k+2}z) &\leq r\,\max\{d(z,T^{k+1}z),d(T^{k+1}z,T^{k+2}z)\} \\ &\leq r\,\max\{r^k\,d(z,Tz),r^{k+1}d(z,Tz)\} \\ &= r\cdot r^k\,d(z,Tz)=r^{k+1}\,d(z,Tz), \end{aligned}$$

hence by induction we establish (9).

Moreover, applying the condition (9) to appropriate situations, we find a contradiction to our assumption in following cases:

(i) For $0\leq r<\frac{\sqrt{5}-1}{2}$, we have $\theta(r)=1$, $r^2+r-1<0$ and $2r^2<1$. Now if $d(T^2z,z)<\theta(r)d(T^2z,T^3z)$, we get

$$\begin{aligned} d(z,Tz) &\leq d(z,T^2z)+d(T^2z,Tz) \\ &< \theta(r)d(T^2z,T^3z)+d(T^2z,Tz) \\ &\leq d(T^2z,T^3z)+d(T^2z,Tz) \\ &\leq r^2\,d(z,Tz)+r\,d(z,Tz) \\ &< d(z,Tz), \end{aligned}$$

which is a contradiction. Hence,

$$d(T^2z,z)\geq\theta(r)\min\{d(T^2z,T^3z),d(T^2z,Tz)\},$$

which implies

$$\begin{aligned} d(T^3z,Tz) &\leq m(T^3z,Tz) \\ &\leq a\,r\,d(z,Tz)+b\,\max\{r^2d(z,Tz),d(z,Tz)\} \\ &\quad +c\,[rd(z,Tz)+r^2d(z,Tz)] \\ &\leq a\,d(z,Tz)+b\,d(z,Tz)+2c\,d(z,Tz) \\ &= r\,d(z,Tz). \end{aligned}$$

Thus

$$\begin{aligned} d(z,Tz) &\le d(z,T^3z)+d(T^3z,Tz)\\ &\le r^2\,d(z,Tz)+r\,d(z,Tz)<d(z,Tz), \end{aligned}$$

which is again a contradiction.

(ii) For $\frac{\sqrt{5}-1}{2}\le r<\frac{1}{\sqrt{2}}$, we have $\theta(r)=\frac{1-r}{r^2}$ and $2r^2<1$. If

$$d(z,T^2z)<\theta(r)\,\min\{d(T^2z,T^3z),d(T^2z,Tz)\},$$

we get

$$\begin{aligned} d(z,Tz) &\le d(z,T^2z)+d(T^2z,Tz)\\ &< \theta(r)\min\{d(T^2z,T^3z),d(T^2z,Tz)\}+d(T^2z,Tz)\\ &\le \theta(r)\,d(T^2z,T^3z)+d(T^2z,Tz)\\ &\le \frac{1-r}{r^2}\,r^2\,d(z,Tz)+r\,d(z,Tz)\\ &= d(z,Tz) \end{aligned}$$

which is a contradiction. Hence,

$$\theta(r)\,\min\{d(T^2z,T^3z),d(T^2z,Tz)\}\le d(z,T^2z),$$

which implies

$$\begin{aligned} d(Tz,T^3z) &\le m(Tz,T^3z)\\ &\le a\,r\,d(z,Tz)+b\,\max\{d(z,Tz),r^2d(z,Tz)\}\\ &\qquad +c[r^2\,d(z,Tz)+r\,d(z,Tz)]\\ &\le (a+b+2c)\,d(z,Tz)\\ &= r\,d(z,Tz). \end{aligned}$$

Thus

$$d(z,Tz)\le d(z,T^3z)+d(T^3z,Tz)\le 2r^2\,d(z,Tz)<d(z,Tz),$$

which is again a contradiction.

(iii) For $\frac{1}{\sqrt{2}}\le r<1$, we have $\theta(r)=\frac{1}{1+r}$. Now, we claim that

$$\begin{aligned} &\text{either}\quad \theta(r)\min\{d(u_{2n},u_{2n+1}),d(u_{2n},Tz)\} \le d(u_{2n},z)\\ &\text{or}\quad \theta(r)\min\{d(u_{2n+1},u_{2n+2}),d(u_{2n+1},Tz)\} \le d(u_{2n+1},z). \end{aligned}\tag{10}$$

Suppose inequality (10) does not hold, then

$$\begin{aligned} d(u_{2n},u_{2n+1}) &\le d(u_{2n},z)+d(u_{2n+1},z)\\ &< \theta(r)[\min\{d(u_{2n},u_{2n+1}),d(u_{2n},Tz)\}\\ &\qquad +\min\{d(u_{2n+1},u_{2n+2}),d(u_{2n+1},Tz)\}]\\ &\le \theta(r)[d(u_{2n},u_{2n+1})+d(u_{2n+1},u_{2n+2})]\\ &\le \theta(r)(1+r)d(u_{2n},u_{2n+1})\\ &= d(u_{2n},u_{2n+1}), \end{aligned}$$

which is a contradiction. Hence, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\theta(r)\min\{d(u_{n_k},u_{n_k+1}),d(u_{n_k},Tz)\}\leq d(u_{n_k},u_{n_k+1})\leq d(u_{n_k},z).$$

Then (5) will imply

$$d(Tu_{n_k},Tz)\quad\leq\quad m(Tu_{n_k},Tz),$$

making limit as $n\to\infty$, we get

$$\begin{aligned} & d(z,Tz) &\leq\ & b\,d(z,Tz)+c\,d(z,Tz)\leq r\,d(z,Tz)\\ \Rightarrow\quad & d(z,Tz) &=\ & 0,\ \text{that is } Tz=z, \end{aligned}$$

which is again contrary to our assumption.

Thus, we conclude that the condition (8) is true, i.e., there exists some $l\in\mathbb{N}$ such that $T^l z=z$. Moreover $\{T^n z\}$ is a Cauchy sequence, so we have $Tz=z$, i.e., T has a fixed point in X.

To prove the uniqueness of the fixed point, let $Tz=z$ and $Tz'=z'$ with $z\neq z'$. Since $0=\theta(r)\min\{d(z,Tz),d(z,Tz')\}\leq d(z,z')$,

$$\begin{aligned} d(z,z')=d(Tz,Tz') &\leq a\,d(z,z')+b\,\max\{d(z,Tz),d(z',Tz')\}\\ &\qquad +c\,[d(z,Tz')+d(z',Tz)]\\ &\leq (a+b+2c)d(z,z')=r\,d(z,z') \end{aligned}$$

which is a contradiction. This completes the proof. □

Here, we give some examples which are showing that the condition (5) is generalization of condition (3) as well as the conditions due to Kannan [10] and Chatterjea [5].

Example 1 ([20])**.** Let $X=\{-1,0,1,2\}$ with usual metric $d(x,y)=|x-y|$ and T on X is defined by

$$Tx=\begin{cases}\ \ 0, & \text{if } x\neq 2,\\ -1, & \text{if } x=2.\end{cases}$$

Then T satisfies condition (5) and (1) but does not satisfy condition (3).

Proof. In this example, we only show that condition (5) is satisfied with $a=c=0$ and $b=\frac{1}{3}$. To see this, we have the following:

(i) If $x=-1$ and $y=0$, then

$$\theta(r)\min\{d(x,Tx),d(x,Ty)\}=\theta(r)\cdot 1\leq d(-1,0)\quad\text{and}$$
$$d(Tx,Ty)=0\leq m(Tx,Ty)=\tfrac{1}{3}.$$

(ii) If $x=-1$ and $y=1$, then

$$\theta(r)\min\{d(x,Tx),d(x,Ty)\}=\theta(r)\cdot 1\leq d(-1,1)\quad\text{and}$$
$$d(Tx,Ty)=0\leq m(Tx,Ty)=\tfrac{1}{3}.$$

(iii) If $x = -1$ and $y = 2$, then

$$\theta(r)\min\{d(x,Tx), d(x,Ty)\} = \theta(r)\cdot 0 \le d(-1,2) \quad \text{and}$$
$$d(Tx,Ty) = 1 \le m(Tx,Ty) = 1.$$

(iv) If $x = 1$ and $y = 0$, then

$$\theta(r)\min\{d(x,Tx), d(x,Ty)\} = \theta(r)\cdot 1 \le d(1,0) \quad \text{and}$$
$$d(Tx,Ty) = 0 \le m(Tx,Ty) = \tfrac{1}{3}.$$

(v) If $x = 1$ and $y = -1$, then

$$\theta(r)\min\{d(x,Tx), d(x,Ty)\} = \theta(r)\cdot 1 \le d(1,-1) \quad \text{and}$$
$$d(Tx,Ty) = 0 \le m(Tx,Ty) = \tfrac{1}{3}.$$

(vi) If $x = 1$ and $y = 2$, then

$$\theta(r)\min\{d(x,Tx), d(x,Ty)\} = \theta(r)\cdot 1 \le d(1,2) \quad \text{and}$$
$$d(Tx,Ty) = 1 \le m(Tx,Ty) = 1.$$

(vii) If $x = 0$ and $y = 1$, then

$$\theta(r)\min\{d(x,Tx), d(x,Ty)\} = \theta(r)\cdot 0 \le d(0,1) \quad \text{and}$$
$$d(Tx,Ty) = 0 \le m(Tx,Ty) = \tfrac{1}{3}.$$

(viii) If $x = 0$ and $y = -1$, then

$$\theta(r)\min\{d(x,Tx), d(x,Ty)\} = \theta(r)\cdot 0 \le d(0,-1) \quad \text{and}$$
$$d(Tx,Ty) = 0 \le m(Tx,Ty) = \tfrac{1}{3}.$$

(ix) If $x = 0$ and $y = 2$, then

$$\theta(r)\min\{d(x,Tx), d(x,Ty)\} = \theta(r)\cdot 0 \le d(0,2) \quad \text{and}$$
$$d(Tx,Ty) = 1 \le m(Tx,Ty) = 1.$$

(x) If $x = 2$ and $y = 0$, then

$$\theta(r)\min\{d(x,Tx), d(x,Ty)\} = \theta(r)\cdot 2 \le d(2,0) \quad \text{and}$$
$$d(Tx,Ty) = 1 \le m(Tx,Ty) = 1.$$

(xi) If $x = 2$ and $y = 1$, then

$$\theta(r)\min\{d(x,Tx), d(x,Ty)\} = \theta(r)\cdot 2 \le d(2,1)$$
$$\text{implies } \theta(r) \le \tfrac{1}{2}, \text{ which is not possible.}$$

(xii) If $x = 2$ and $y = -1$, then

$$\theta(r)\min\{d(x,Tx), d(x,Ty)\} = \theta(r)\cdot 2 \le d(2,-1) \quad \text{and}$$
$$d(Tx,Ty) = 1 \le m(Tx,Ty) = 1.$$

Also, T does not satisfy condition (3) but Kannan's condition (1) is satisfied (see Example 2 in [20]). $\square$

Example 2. Let $X = \{0, \frac{1}{3}, 1\}$ with usual metric $d(x,y) = |x-y|$ and T on X is defined by

$$Tx = \begin{cases} 0, & \text{if } x \neq 1, \\ \frac{1}{3}, & \text{if } x = 1. \end{cases}$$

Then T satisfies condition (5) and (3) but does not satisfy Kannan's condition (1).

Proof. In this example T does not satisfy Kannan's condition (1). To see this, if $x = 0$, $y = 1$. Then

$$d(Tx, Ty) = \frac{1}{3} \text{ and } r[d(x,Tx) + d(y,Ty)] = \frac{2}{3}r,$$

which implies $r \geq \frac{1}{2}$ (not possible). But, the condition (5) (with $a = c = 0$ and $b = \frac{1}{2}$) and Suzuki's condition (3) (with $r = \frac{1}{2}$) are satisfied. □

Example 3. Let $X = \{-1, 0, \frac{1}{3}, 1\}$ with usual metric $d(x,y) = |x-y|$ and T on X is defined by

$$Tx = \begin{cases} 0, & \text{if } x \neq \frac{1}{3}, \\ -1, & \text{if } x = \frac{1}{3}. \end{cases}$$

Then T satisfies condition (5) but does not satisfy the conditions (1), (2) and (3).

Proof.

(K) If $x = 0, y = \frac{1}{3}$, then $d(Tx,Ty) = 1$ and $r[d(x,Tx) + d(y,Ty)] = \frac{4}{3}r$. So, $d(Tx,Ty) \leq r[d(x,Tx) + d(y,Ty)]$ implies $r \geq \frac{3}{4} > \frac{1}{2}$, which is not possible. Hence, condition (1) is not satisfied.

(S) If $x = 0, y = \frac{1}{3}$, then $\theta(r)d(x,Tx) = 0 \leq d(0, \frac{1}{3})$, but $1 = d(Tx,Ty) \leq rd(x,y) = \frac{1}{3}r$ implies that $r \geq 3$, which is not possible. Hence, condition (3) is not satisfied.

(C) If $x = 0, y = \frac{1}{3}$, then $d(Tx,Ty) = 1$ and $r[d(x,Ty) + d(y,Tx)] = \frac{4}{3}r$. So, $d(Tx,Ty) \leq r[d(x,Ty) + d(y,Tx)]$ implies $r \geq \frac{3}{4} > \frac{1}{2}$, which is not possible. Hence, condition (2) is not satisfied.

However, T satisfies the condition (5) with $a = c = 0$ and $b = \frac{3}{4}$. □

Remark 1. Now it is obvious that Suzuki's condition (3) implies condition (5) but not conversely, see Example 1. Thus Theorem 4 is the generalization of Theorem 3 due to Suzuki [20].

Taking $b = c = 0$ in Theorem 4, we obatin the following generalization of Theorem 3 due to Suzuki [20].

Corollary 1. *Let (X,d) be a complete metric space and $T : X \to X$. Define a non-increasing function $\theta : [0,1) \to (\frac{1}{2}, 1]$ as in Theorem 3. Assume that there exists $a \in [0,1)$ such that for each $x, y \in X$*

$$\theta(a) \min\{d(x,Tx), d(x,Ty)\} \leq d(x,y) \ \text{ implies } \ d(Tx,Ty) \leq a\, d(x,y).$$

Then T has a unique fixed point in X. Moreover, $\lim_{n\to\infty} T^n x = z$, for all $x \in X$.

Taking $a = c = 0$ in Theorem 4, we have the generalization of Kannan's Theorem 1, which is the following.

Corollary 2. *Let (X,d) be a complete metric space and $T : X \to X$. Define a non-increasing function $\theta : [0,1) \to (\frac{1}{2}, 1]$ as in Theorem 3. Assume that there exists $b \in [0,1)$ such that for each $x, y \in X$*

$$\begin{aligned} \theta(b) \min\{d(x,Tx), d(x,Ty)\} &\leq d(x,y) \\ \text{implies } \ d(Tx,Ty) &\leq b \max\{d(x,Tx), d(y,Ty)\} \end{aligned}$$

Then T has a unique fixed point in X.

Taking $a = b = 0$ in Theorem 4, we have the following generalization of Chatterjea's Theorem 2.

Corollary 3. *Let (X,d) be a complete metric space and $T : X \to X$. Define a non-increasing function $\theta : [0,1) \to (\frac{1}{2}, 1]$ as in Theorem 3. Assume that there exists $c \in [0,1)$ such that for each $x, y \in X$*

$$\begin{aligned} \theta(c) \min\{d(x,Tx), d(x,Ty)\} &\leq d(x,y) \\ \text{implies } \ d(Tx,Ty) &\leq c\,[d(x,Ty) + d(y,Tx)] \end{aligned}$$

Then T has a unique fixed point in X.

Now, we prove the following result to discuss the completeness of metric spaces.

Theorem 5. *Let (X,d) be a metric space, and $A_{r,\eta}$ be the family of mappings T on X satisfying the condition:*

(a) *For $r \in [0,1)$ and $x, y \in X$,*
$\eta \min\{d(x,Tx), d(x,Ty)\} \leq d(x,y)$ implies $d(Tx,Ty) \leq r\, d(x,y)$,
where $\eta \in (0, \theta(r)]$ and θ is as defined in Theorem 3.

Let $B_{r,\eta}$ be the family of mappings T on X satisfying condition (a) with $T(X)$ is countably infinite and every subset of $T(X)$ is closed. Then the following are equivalent:

(i) *X is complete.*
(ii) *Every mapping $T \in A_{r,\theta(r)}$ has a fixed point for all $r \in [0,1)$.*
(iii) *There exist $r \in (0,1)$ and $\eta \in (0, \theta(r)]$ such that every mapping $T \in B_{r,\eta}$ has a fixed point.*

Proof. The proof is almost same as in Theorem 4 of Suzuki [20]. We just prove the part $(iii) \Rightarrow (i)$. For the clarity, we follow some of the steps of proof of Theorem 4 of Suzuki [20]. Consider that X is not complete. Then there is a Cauchy sequence $\{u_n\}$ which does not converge. Define a function $f : X \to [0, \infty)$ by $f(x) = \lim_{n\to\infty} d(x, u_n)$ for $x \in X$. Since $\{d(x, u_n)\}$ is a Cauchy sequence for every $x \in X$, f is well defined. We have following observations for f:

- $f(x) - f(y) \leq d(x, y) \leq f(x) + f(y)$ for $x, y \in X$;
- $f(x) > 0$ for all $x \in X$;
- $\lim_{n\to\infty} f(u_n) = 0$.

Define a mapping T on X as follows: For each $x \in X$, since $f(x) > 0$ and $\lim_{n\to\infty} f(u_n) = 0$, there exists $\nu \in \mathbb{N}$ satisfying $f(u_\nu) \leq \frac{\eta r}{3+2\eta r} f(x)$. We put $Tx = u_\nu$. Then it is obvious that $f(Tx) \leq \frac{\eta r}{3+2\eta r} f(x)$ and $Tx \in \{u_n : n \in \mathbb{N}\}$ for all $x \in X$.
Then $Tx \neq x$ for all $x \in X$ because $f(Tx) < f(x)$. Since $T(X) \subset \{u_n : n \in \mathbb{N}\}$. Which implies $T(X)$ is countably infinite and it is easy to prove that every subset of $T(X)$ is closed. Let us prove condition (a). Fix $x, y \in X$ with $\eta \min\{d(x, Tx), d(x, Ty)\} \leq d(x, y)$. If $\min\{d(x, Tx), d(x, Ty)\} = d(x, Tx)$, then the proof follows from Suzuki (Theorem 4, [20]).
If $\min\{d(x, Tx), d(x, Ty)\} = d(x, Ty)$, then $\eta d(x, Ty) \leq d(x, y)$. Now, in the case where $f(y) > 2f(x)$, we have

$$\begin{aligned}
d(Tx, Ty) &\leq f(Tx) + f(Ty) \\
&\leq \frac{\eta r}{3 + 2\eta r}(f(x) + f(y)) \\
&\leq \frac{r}{3}(f(x) + f(y)) \\
&\leq \frac{r}{3}(f(x) + f(y)) + \frac{2r}{3}(f(y) - 2f(x)) \\
&= r(f(y) - f(x)) \leq r\, d(x, y).
\end{aligned}$$

In the other case, where $f(y) \leq 2f(x)$, we have

$$\begin{aligned}
d(x, y) \geq \eta d(x, Ty) &\geq \eta(f(x) - f(Ty)) \\
&\geq \eta(f(x) - \frac{\eta r}{3 + 2\eta r} f(y)) \\
&\geq \eta(f(x) - \frac{2\eta r}{3 + 2\eta r} f(x)) \\
&= \frac{3\eta}{3 + 2\eta r} f(x)
\end{aligned}$$

and hence

$$\begin{aligned} d(Tx,Ty) &\leq f(Tx)+f(Ty) \\ &\leq \frac{\eta r}{3+2\eta r}(f(x)+f(y)) \\ &\leq \frac{3\eta r}{3+2\eta r}f(x) \\ &\leq rd(x,y). \end{aligned}$$

Therefore we have shown (a), that is, $T \in B_{r,\eta}$. By (iii), T has a fixed point which yields a contradiction. Hence we obtain that X is complete. $\square$

3. Multi-valued mappings and fixed point theorems

Let (X,d) be a metric space and $CB(X)(resp.\, CL(X))$ the collection of all nonempty closed and bounded subsets (resp. closed subsets) of X. The Hausdorff metric H on $CB(X)$(resp. $CL(X)$) induced by the metric d is given by

$$H(A,B)=\max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$$

for $A,B \in CB(X)(resp.\, CL(X))$, where $d(x,A)=\inf_{y\in A} d(x,y)$. Throughout this section, we will use the following notation:

$$M(Tx,Ty)=\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Nadler [13] introduced the concept of multi-valued contraction and proved that in a complete metric space X multi-valued contraction $T: X \to CB(X)$ has a fixed point, i.e., there exists $z \in X$ such that $z \in Tz$.

Đorić et al. [8] obtained a result for Ćirić type generalized multi-valued mappings [1] in Suzuki type context.

Theorem 6 ([8]). *Let (X,d) be a complete metric space and $T: X \to CB(X)$. Define a non-increasing function $\phi: [0,1) \to (\frac{1}{2},1]$ by*

$$\phi(r)=\begin{cases} 1, & \text{if } 0 \leq r < \frac{1}{2}, \\ 1-r, & \text{if } \frac{1}{2} \leq r < 1. \end{cases}$$

Assume there exists $r \in [0,1)$ such that

$$\phi(r)d(x,Tx) \leq d(x,y) \ \text{ implies } \ H(Tx,Ty) \leq r\, M(Tx,Ty),$$

for all $x,y \in X$. Then, there exists $z \in X$ such that $z \in Tz$.

There are a large number of results obtained by many researchers for multi-valued mapping in Suzuki type contraction, (see, [2, 7, 8, 12, 18] and references therein). Here we prove a result which is the generalization of result due to Đorić et al. [8] and many other well known results for multi-valued mappings. Our result also gives a new direction to many concepts about Suzuki type contraction for multi-valued mappings.

Theorem 7. *Let (X, d) be a complete metric space, and $T : X \to CL(X)$. If there exists $r \in [0, 1)$ such that for each $x, y \in X$,*

$$\phi(r) \min\{d(x, Tx), d(x, Ty)\} \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq r\, M(Tx, Ty), \tag{11}$$

where $\phi : [0, 1) \to (\frac{1}{2}, 1]$ is non-increasing function defined as in Theorem 6. Then there exists a point $z \in X$ such that $z \in Tz$.

Proof. If $M(Tx, Ty) = 0$, then obviously $x \in Tx$. So, we may take without loss of generality that $M(Tx, Ty) > 0$ for distinct $x, y \in X$. Let $\epsilon > 0$ be such that $\beta = r + \epsilon < 1$. Take an arbitrary point $u_0 \in X$ and $u_1 \in Tu_0$, then there exists $u_2 \in Tu_1$ such that

$$d(u_1, u_2) \leq H(Tu_1, Tu_0) + \epsilon\, M(Tu_1, Tu_0)$$

similarly there exists $u_3 \in Tu_2$ such that

$$d(u_3, u_2) \leq H(Tu_2, Tu_1) + \epsilon\, M(Tu_2, Tu_1)$$

Continuing in this manner, we can find a sequence $\{u_n\}$ in X such that $u_n \in Tu_{n-1}$ and

$$d(u_{n+1}, u_n) \leq H(Tu_n, Tu_{n-1}) + \epsilon\, M(Tu_n, Tu_{n-1}).$$

Since $\phi(r) \leq 1$,

$$\phi(r) \min\{d(u_n, Tu_n), d(u_n, Tu_{n+1}) \leq d(u_n, u_{n+1})$$

by assumption, we get

$$H(Tu_n, Tu_{n+1}) \leq r\, M(Tu_n, Tu_{n+1}).$$

Thus

$$\begin{aligned}
d(u_{n+1}, u_{n+2}) &\leq H(Tu_n, Tu_{n+1}) + \epsilon\, M(Tu_n, Tu_{n+1}) \\
&\leq r\, M(Tu_n, Tu_{n+1}) + \epsilon\, M(Tu_n, Tu_{n+1}) \\
&\leq \beta \max\Big\{d(u_n, u_{n+1}), d(u_n, Tu_n), d(u_{n+1}, Tu_{n+1}), \\
&\qquad\qquad \frac{d(u_n, Tu_{n+1} + d(u_{n+1}, Tu_n)}{2}\Big\} \\
&\leq \beta \max\Big\{d(u_n, u_{n+1}), d(u_n, u_{n+1}), \\
&\qquad\qquad d(u_{n+1}, u_{n+2}), \frac{1}{2} d(u_n, u_{n+2})\Big\} \\
&\leq \beta \max\{d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2})\}
\end{aligned}$$

which yields, $d(u_{n+1}, u_{n+2}) \leq \beta\, d(u_n, u_{n+1})$.

Therefore $\{u_n\}$ is a Cauchy sequence and has a limit in $z \in X$.

Let $x \in X\backslash\{z\}$, then $d(x,z) > 0$. As $u_n \to z$, there exists $n_0 \in \mathbb{N}$ such that

$$d(z,u_n) \le \frac{1}{3}d(z,x), \quad \text{for all } n \ge n_0. \tag{12}$$

Then

$$\begin{aligned}
\phi(r)\min\{d(u_n,Tu_n),d(u_n,Tx)\} &\le d(u_n,u_{n+1}) \\
&\le d(u_n,x)+d(u_{n+1},x) \\
&\le \frac{2}{3}d(z,x) = d(z,x) - \frac{1}{3}d(z,x) \\
&\le d(z,x) - d(u_n,z) \le d(u_n,x)
\end{aligned}$$

hence by the assumption

$$\begin{aligned}
H(Tu_n,Tx) &\le r\,M(Tu_n,Tx) \\
\Rightarrow \quad d(u_{n+1},Tx) &\le H(Tu_n,Tx) \\
&\le r\ \max\Big\{d(u_n,x),d(u_n,u_{n+1}),d(x,Tx), \\
&\qquad \frac{d(u_n,Tx)+d(x,u_{n+1})}{2}\Big\}.
\end{aligned}$$

Making $n \to \infty$, we have

$$\begin{aligned}
d(z,Tx) &\le r\ \max\Big\{d(z,x),d(x,Tx),\frac{d(z,Tx)+d(z,x)}{2}\Big\} \\
&\le r\ \max\{d(z,x),d(x,Tx),d(z,Tx)\}
\end{aligned} \tag{13}$$

$$d(z,Tx) \le r\ \max\{d(z,x),d(x,Tx)\}$$

Now, we consider the following two cases:

(i) In the case where $0 \le r < \frac{1}{2}$, we note that $2r < 1$. Suppose $z \notin Tz$, let $a \in Tz$ be such that $2r\,d(a,z) < d(z,Tz)$. Also $a \in Tz \Rightarrow a \ne z$, then by (13) we get

$$d(z,Ta) \le r\ \max\{d(z,a),d(a,Ta)\}. \tag{14}$$

On the other hand

$$\phi(r)\min\{d(z,Tz),d(z,Ta)\} \le d(z,Tz) \le d(z,a)$$

then by hypothesis, we get

$$\begin{aligned}
d(a,Ta) &\le H(Ta,Tz) \le r\,M(Ta,Tz) \\
&= r\ \max\Big\{d(a,z),d(a,Ta),d(z,Tz),\frac{1}{2}d(z,Ta)\Big\} \\
&\le r\ \max\{d(a,z),d(a,Ta),d(z,Tz)\} \\
\Rightarrow \quad d(a,Ta) &\le r\ \max\{d(a,z),d(z,Tz)\}.
\end{aligned}$$

Because $d(z,Tz) \le d(z,a) + d(a,Tz) = d(a,z)$, then using (14) we obtain

$$d(a,Ta) \le H(Ta,Tz) \le r\, d(a,z) < d(a,z)$$

condition (14) will imply $d(z,Ta) \le r\, d(a,z)$. Thus

$$\begin{aligned} d(z,Tz) &\le d(z,Ta) + H(Ta,Tz) \\ &\le r\, d(a,z) + r\, d(a,z) = 2r\, d(a,z) \\ &< d(z,Tz) \end{aligned}$$

which is a contradiction. Hence $z \in Tz$.

(ii) In case where $\frac{1}{2} \le r < 1$, first we will obtain that

$$H(Tx,Tu) \le r\, M(Tx,Tu), \quad \text{for all } x \in X.$$

If $x = z$, then previous obviously holds. So, assume that $x \neq z$, then for each $n \in \mathbb{N}$, there exists $z_n \in Tx$ such that $d(z,z_n) \le d(z,Tx) + \frac{1}{n} d(x,z)$. Therefore

$$\begin{aligned} d(x,Tx) &\le d(x,z_n) \le d(x,z) + d(z,z_n) \\ &\le d(x,z) + d(z,Tx) + \frac{1}{n} d(x,z) \end{aligned}$$

using (13) we get

$$d(x,Tx) \le d(x,z) + r\,\max\{d(x,z), d(x,Tx)\} + \frac{1}{n} d(x,z). \tag{15}$$

If $d(x,Tx) \le d(x,z)$, then (15) implies

$$\begin{aligned} d(x,Tx) &\le d(x,z) + r\, d(x,z) + \frac{1}{n} d(x,z) \\ &= (1 + r + \frac{1}{n}) d(x,z) \end{aligned}$$

making $n \to \infty$, we get $d(x,Tx) \le (1+r) d(x,z)$. Thus

$$\begin{aligned} \phi(r) \min\{d(x,Tx), d(x,Tz)\} &\le \phi(r) d(x,Tx) = (1-r) d(x,Tx) \\ &\le \frac{1}{1+r} d(x,Tx) \le d(x,z) \end{aligned}$$

then by assumption, $H(Tx,Tz) \le r\, M(Tx,Tz)$.
If $d(x,z) < d(x,Tx)$, then (15) implies

$$\begin{aligned} d(x,Tx) &\le d(x,z) + r\, d(x,z) + \frac{1}{n} d(x,z) \\ \Rightarrow \quad (1-r) d(x,Tx) &\le d(x,z). \end{aligned}$$

So,

$$\begin{aligned}\phi(r)\min\{d(x,Tx),d(x,Tz)\} &\leq \phi(r)d(x,Tx)\\ = (1-r)d(x,Tx) &\leq d(x,z),\end{aligned}$$

then by assumption, we get $H(Tx,Tz) \leq r\,M(Tx,Tz)$.

Taking $x = u_n$, we get

$$d(u_{n+1},Tu) \leq H(Tu_n,Tu) \leq r\,M(Tu_n,Tu)$$

passing the limit as $n \to \infty$, we get

$$d(z,Tz) \leq r\,\max\{d(z,Tz),\frac{1}{2}d(z,Tz)\} = r\,d(z,Tz)$$

which implies $d(z,Tz) = 0$. Since Tz is closed, $z \in Tz$.

Thus, we have shown that $z \in Tz$ in all cases, which completes the proof. □

Taking the mapping T as single valued in Theorem 7, we get following corollary like Theorem 4 for the function $\phi(r)$.

Corollary 4. *Let (X,d) be a complete metric space and T be a mapping on X. Define a non-increasing function $\phi : [0,1) \to (\frac{1}{2},1]$ as in Theorem 6. If there exists $r \in [0,1)$ such that for each $x,y \in X$,*

$$\begin{aligned}\phi(r)\min\{d(x,Tx),d(x,Ty)\} &\leq d(x,y)\\ \textit{implies}\quad d(Tx,Ty) &\leq r\,M(Tx,Ty).\end{aligned}$$

Then T has a unique fixed point in X.

From Theorem 7, we get the following result as corollary which is also a generalization of Nadler's result.

Corollary 5. *Let (X,d) be a complete metric space, and $T : X \to CL(X)$. Define a non-increasing function $\phi : [0,1) \to (\frac{1}{2},1]$ as in Theorem 6. If there exists $r \in [0,1)$ such that for each $x,y \in X$,*

$$\begin{aligned}\phi(r)\min\{d(x,Tx),d(x,Ty)\} &\leq d(x,y)\\ \textit{implies}\quad H(Tx,Ty) &\leq r\,\max\{d(x,y),d(x,Tx),d(y,Ty)\}.\end{aligned}$$

Then there exists a fixed point of T in X.

4. Conclusion

We have established fixed point theorems for single valued and multi-valued mappings in Suzuki type setting which generalize Theorem 1, Theorem 2, Theorem 3 and many others, see examples provided in section 2. Furthermore, it has been proved that our result also characterizes the completeness of the metric space, see Theorem 5. Henceforth, our theorems open a direction to new fixed point results and applications.

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