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# Homotopy extension property for multi-valued functions

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Abstract. In this study, we introduce some well-known definitions and properties for multi-valued functions. We present new definitions such as, m-retraction, m-section, m-homeomorphism, m-HEP and reducible function. We give a new result on the relation between multi-homotopy groups and m-homeomorphism. We also deal with some properties of m-HEP.

## 1. Introduction

There is a meaningful relation between multi-valued and single-valued functions. So, many concepts defined on single-valued function have also been tried to be given on multi-valued function. On the other hand, some notions are still not covered for multi-valued functions. Therefore, an important question comes to mind : Are there any differences on multi-valued application corresponding to same concept for single-valued function?

Some properties of multi-valued functions have been studied various author [4, 11, 14]. Smithson [11] gives generalization of some definition like monotone functions and nonaltering functions. Borges [4] determines which topological properties are preserved by multi-valued functions. The concept of continuity of multi-valued functions studied diverse ways. Strother [13] also studies on the continuity of a multi-valued function in his Doctoral dissertation. He shows that some of definitions of continuity are equivalent. Also, homotopy concepts are defined by Strother [14]. Rhee [9] determine that $M\pi_n$ is a functor. Also, Karaca, Denizaltı and Temizel [7] have studied homotopy of multi-valued functions. Lee [8] has investigated m-homotopy groups and absolute m-homotopy extension property for m-functions.

In this study, we first give some background for multi-valued functions. Then we express multi-category and define m-retraction and m-section in this category. We also investigate some properties with this expression.

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Later, we define the m-homeomorphism by using m-retraction and m-section. We also give a relation between m-homeomorphism and multi-homotopy group. We generalize the homotopy extension propery, m-HEP, for multi-valued functions and investigate some properties in the section 5. Then we give a definition of reducible multi-valued function. At the end of the section 5, we show the relation between HEP and m-HEP by using the reducible multi-valued function.

## 2. PRELIMARINES

Throughout the paper $X$, $Y$ and $Z$ will be topological spaces and $x$, $y$ and $z$ elements of these spaces, respectively. For a topological space $X$ we denote the set of all non-empty closed subset of $X$ by $S(X)$. We denote an identical function over $X$ by $1_X$ unless otherwise stated.

A *multi-valued function* $F\colon X \to Y$ maps every point $x$ to non-empty subset of $Y$ [3]. The function $F\colon X \to Y$ is *n-th valued* if it maps a point to subset $n$ points of $Y$ for all $x \in X$ [5]. We say that $n$ is degree of $F$ and we denote it by $D(F)$. Clearly, if $D(F) = 1$, then $F$ is a single-valued function. A single-valued function $f\colon X \to Y$ is a *selection* of $F$ if and only if $f(x) \in F(x)$ for every $x \in X$ [3]. If $X_0 \subset X$, then $F(X_0) = \bigcup_{x \in X_0} F(x)$. For a fixed subset $Y_0 \subset Y$, a multi valued function $F\colon X \to Y$ is called *constant* if $F(x) = Y_0$ for all $x \in X$ [3]. A multi-valued function $F|_A\colon A \to X$ called *restiriction* and defined by $F(x) = F|_A(x)$ for all $x \in X$, where $F\colon X \to Y$ is a multi-valued function and $A$ is a subset of $X$ [11]. $F\colon X \to Y$ is an *extension* of $G\colon A \to Y$ if $F|_A = G$ [12]. A multi-valued function $F$ is said to be *surjective* if there exists a point $x \in X$ such that $y \in F(x)$ for every $y \in Y$ [3]. The function $F$ is called *one-to-one* if, $x \neq x'$ implies that $F(x) \neq F(x')$ [13]. $F^{-1}(y)$ is defined to be the set of all $X$ such that $y \in F(x)$ and it follows that $(F^{-1})^{-1} = F$ [13]. We define upper and lower inverse $F^+(V)$, $F^-(V)$ as follows

$$F^+(V) = \{x \in X | F(x) \subset V\} \quad \text{and} \quad F^-(V) = \{x \in X | F(x) \cap V \neq \emptyset\},$$

where $V$ is an open subset of $Y$. The function $F$ is called *upper* (*lower*) semicontinuous if, for every open $V \subset Y$, $F^+(V) \subset X$ ($F^-(V) \subset X$) is open. $F$ is *continuous* if $F$ is both upper and lower semicontinuous [4]. A multi-valued function $F$ is called a *homeomultimorphism* if $F$ and $F^{-1}$ are continuous and one-to-one [13]. If $F$, and $G\colon X \to Y$ are two continuous multi(single)-valued functions, then *union map* $F \cup G\colon X \to Y$ defined by $F \cup G(x) = F(x) \cup G(x)$ for all $x \in X$ [3].

Smithson [11] shows that being compact, Hausdorff space are required for following result.

**Lemma 1** ([11]). *If $F\colon X \to Y$ and $G\colon Y \to Z$ are continuous and if $X$, $Y$ and $Z$ are compact, Hausdorff space, then $G \circ F$ is continuous.*

"Strother [14]" defined multi-homotopy and multi-homotopy group $M_{\pi_n}(Y)$. $I^n$ denotes *the product* of $n$ unit intervals. The boundary of $I^n$ is denoted by $B^{n-1}$.

**Definition 1** ([14])**.** Let $F, G \colon X \to Y$ be multi-valued function. Then $F$ is said to be *m-homotopic* to $G$ if there exists a continuous multi-valued function $H \colon X \times I \to Y$ such that $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$.

If $F$ is $m$-homotopic to a constant multi-valued function, then we say that $F$ is *null m-homotopic* [12]. If $F$ and $G$ are $m$-homotopic, then we denote the $m$-homotopy between $F$ and $G$ by $F \simeq_m G$.

**Definition 2** ([14])**.** Two continuous multi-valued functions $F, G \colon X \to Y$ are said to be *m-homotopic* relative to $A \subset X$ and $B \subset Y$ if there exists an $m$-homotopy $H$ connecting $F$ and $G$ such that $x \in A$ and $t \in I$ imply that $H(x, t) = B$.

**Definition 3** ([14])**.** For positive integer $n$ and closed subset $Y_0$ of $Y$, we define

$$MQ(n, Y, Y_0) = \{F \colon I^n \to Y | F \ is \ continuous, F(x) = Y_0 \ for \ all \ x \in B^{n-1}\}.$$

If $Y$ is a compact, Hausdorff space, then the functions in $MQ(n, Y, Y_0)$ are divided into m-homotopy classes relative to $(B^{n-1}, Y_0)$ [14, p. 284].

**Theorem 1** ([14])**.** *Let $Y$ be a compact Hausdorff space. Then the m-homotopy classes of the continuous functions in $MQ(n, Y, Y_0)$ form a group $M\pi_n(Y, Y_0)$.*

**Theorem 2** ([14])**.** *Let $Y$ be a compact, Hausdorff space. Then we have $\pi_n(S(Y), Y_0) \cong M\pi_n(Y, Y_0)$.*

A *cone* $CX$ over a topological space $X$ is quotient space $X \times I/_\sim$, where the equivalence relation $\sim$ defined by $(x, t) \sim (x', t')$ if $t = t' = 1$ [10].

**Definition 4** ([7])**.** A space $X$ is $m$-contractible if $1_X$ is null $m$-homotopic.

## 3. MULTI-CATEGORY

In this section, we consider not only multi-category but also section and retraction on multi-valued functions. Initially, we give a definition and example for multi-category.

**Definition 5** ([11])**.** A category is a triple $C = (O, M, \circ)$ consisting of

(1) a class $O$, whose members are called $C$-objects,
(2) for each pair $A, B \in O$, a set $M$ (or $Mor(A, B)$), the elements of $M$ called as a $C$-morphism from $A$ to $B$(the sets $Mor(A, B)$ are pairwise disjoint),
(3) for each $A, B, C \in O$, a map $\circ \colon Mor(A, B) \times Mor(B, C) \to Mor(A, C)$, called composition and denoted $(f, g) \mapsto g \circ f$, such that

(a) composition is associative,
(b) for each $A \in O$ there exists $1_A \in Mor(A, A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $C$-morphism $f$ and $g$.

Let $C$ be a category with morphism class $M$. If $M$ consist of multi-valued function, then we called $C$ as a multi-category(m-category) and denote m-$C$.

**Example 1.** (1) m-SET category:
Objects: Sets
Morphisms: multi-valued functions
Operation: Composition on multi-valued functions
(2) m-cHTop category:
Objects: Compact, Hausdorff topological spaces
Morphisms: Continious multi-valued functions
Operation: Composition on multi-valued functions

It is clear that $M_C \subset M_{m-C}$ and $O(C) = O(\text{m-}C)$ for all category $C$. Thus, a category $C$ can be seen as a subcategory of m-$C$.

**Definition 6.** Let $F\colon X \to Y$ be a multi-valued function. If there exists a multi-valued function $G\colon Y \to X$ such that following holds for all $y \in Y$:

$$\{y\} = \bigcap_{x \in G(y)} F(x),$$

then we say that $F$ is an m-retraction and $G$ is called as a cross retraction of $F$.

If $F$ and $G$ are single-valued functions, then we obtain a retraction definition. Here, $y = \cap F(x) = F(x) = F(G(y)) = F \circ G(y)$ where $x = G(y)$. It means that there exists a single-valued function $G\colon Y \to X$ such that $F \circ G = 1_Y$ for all $y \in Y$.

Presently, we touch on an important proposition about selections. For the following proposition surjectivity is essential, and so, we assume that multi-valued functions which our mentioned are surjective.

**Proposition 1.** *Let $F\colon X \to Y$ be an m-retraction. Then there exists a single-valued selection of $F$.*

*Proof.* Assume that $F\colon X \to Y$ be an m-retraction. The function $H$ is defined by $H(y) = A_i \subset X$ for all $y \in Y$. For some $i$, $A_i$ has two elements at least. Since $H$ is surjective, we can construct a single-valued function $f\colon X \to Y$ such that $f(x) = \cap F(x_i)$, where $x_i \in A_i'$ for a set $A_i'$ containig $x$ and if $x_i \in A_i$ and $x_i \in A_j$, then $x_i \in A_i'$ and $x_i \notin A_j'$ for $i \neq j$. Here, we have $f(x) \in F(x)$ for all $x \in X$. Hence, $f$ is a single-valued selection of $F$. ∎

**Remark 1.** Selection of an m-retraction is a surjective function. However, it may not be injective.

Now, we give a following example in which $G$ is an m-retraction.

**Example 2.** Let $F\colon \mathbb{R}^+ \to \mathbb{R}$ be a multi-valued function defined by

$$F(x) = \{\sqrt{x}, -\sqrt{x}\}$$

for all $x \in \mathbb{R}^+$. Let's define a multi-valued function $G\colon \mathbb{R} \to \mathbb{R}^+$ such that

$$G(y) = \begin{cases} \{y^2, 1\}, & \text{if } y > 0, \\ \{y^2, 0\}, & \text{if } y \leq 0, \end{cases}$$

for all $y \in \mathbb{R}$.
Let $x_0 \in \mathbb{R}^+$ be arbitrary. Then we have $F(x_0) = \{\sqrt{x_0}, -\sqrt{x_0}\}$. Here

$$\begin{aligned} G(\sqrt{x_0}) \cap G(-\sqrt{x_0}) &= \{(\sqrt{x_0})^2, 1\} \cap \{(-\sqrt{x_0})^2, 0\} \\ &= \{x_0, 1\} \cap \{x_0, 0\} \\ &= \{x_0\}. \end{aligned}$$

Since

$$\{x\} = \bigcap_{y \in F(x)} G(y)$$

always hold for all $x \in \mathbb{R}^+$, we can say that $G$ is an m-retraction. Morover, from Propositon 1 we have a selection $g\colon \mathbb{R} \to \mathbb{R}^+$ such that $g(y) = y^2$ for all $y \in \mathbb{R}$.

In contrary to general, we can not guarantee that the composition of two m-retractions is an m-retraction without any assumption.

**Lemma 2.** *The composite of m-retraction and retraction is an m-retraction.*

*Proof.* Taking a retraction $h\colon X \to Y$ and an m-retraction $F\colon Y \to Z$, we see that $F \circ h\colon X \to Z$ is an m-retraction from using definition. □

We can define an m-section in a similar logic to the m-retraction.

**Definition 7.** Let $F\colon X \to Y$ be a multi-valued function. If there exist a multi-valued function $G\colon Y \to X$ such that the following hold for all $x \in X$:

$$\{x\} = \bigcap_{y \in F(x)} G(y),$$

then we say that $F$ is an m-section and $G$ is called as a cross section of $F$.

If $F$ and $G$ are single-valued function, then we derive a section definition like the m-retraction. So, $x = \cap G(y) = G(y) = G(F(x)) = G \circ F(x)$, where $y = F(x)$. It means that there exists a single-valued function $G\colon Y \to X$ such that $G \circ F = 1_X$ for all $x \in X$.

**Lemma 3.** *The composite of two m-sections is an m-section.*

*Proof.* Let $F\colon X \to Y$ and $G\colon Y \to Z$ be a two m-sections. Then there is a multi-valued function $H\colon Y \to X$ such that for all $x \in X$:

$$\{x\} = \bigcap_{y \in F(x)} H(y). \tag{1}$$

Let $y \in F(x) \subset Y$ be an arbitrary point. Since G is an m-section, we can write

$$\{y\} = \bigcap_{z \in G(x)} K(z), \tag{2}$$

where $K$ is an m-retraction,i,e., there is a single-valued selection function $k\colon Z \to Y$. So, we can replace $k(z)$ with $y \in F(x)$. From the definition of a selection we have $F(x) = k(G \circ F(x))$, i.e., $H \circ F(x) = H \circ k(z)$, where $z \in G \circ F(x)$. Therefore, the intersection of $H \circ k(z)$ equals to $\{x\}$ for all $z \in G \circ F(x)$. It means that $G \circ F$ is an m-section. □

Now, we can give a few categorical proposition and conclusion. We see that $M_{\pi_n}$ is a functor from m-pTop to Grp (see [9]). From now on, we assume that all of topological spaces are compact, Hausdorff spaces. Firstly, we give a functor example which called Selection functor. Before the following proposition denote that m-pTop$_*$ is a multi-category with restiriction of $M$ of m-pTop, multi-pointed topological spaces category, to $M'$, where the elements of $M'$ are continuous multi-valued function having a continuous selection.

**Proposition 2.** *Let $S_F$ be defined by*

$$\begin{aligned} S_F\colon \textbf{\textit{m-pTop}}_* &\to \textbf{\textit{pTop}} \\ (X, X_0) &\mapsto (X, x_0) \\ F &\mapsto S_F(F) = f, \end{aligned}$$

*where $f$ is a selection of $F$ and $X_0$ is a topological space with a base point $x_0$. Then $S_F$ is a covariant functor.*

*Proof.* It is obvious that $S_F(1_X) = 1_X$ and $S_F(F \circ G) = f \circ g = S_F(F) \circ S_F(G)$. So, $S_F$ is a covariant functor. □

**Corollary 1.** *Let $F\colon X \to Y$ be a continuous multi-valued function with a continuous selection $f$. Then $M_{\pi_n}(F) = \pi_n \circ S_F(F)$.*

## 4. MULTI-VALUED HOMEOMORPHISM

At the beginning of this section, we introduce a homeomultimorphism in the sense of the m-retraction and the m-section.

**Definition 8.** Let $F\colon X \to Y$ be a multi-valued function. If the following conditions hold, then we say that $F$ is an m-homeomorphism and show $X \approx_m Y$:

(1) $F$ is a continuous;
(2) $F$ is m-section and m-retraction;
(3) The cross section and the cross retraction of $F$ are continuous.

If there exists an m-homeomorphism between two topological space $X$ and $Y$, then we say that these spaces are m-homeomorphic.

If $F$ is single-valued function, then we can tought m-sect and m-retract of $F$ as inverse of $F$. So, above conditions becomes that $F$ and inverse of $F$ is continuous. It implies that $F$ is a homeomorphism.

**Corollary 2.** *Let $F$ be a m-homeomorphism between $X$ and $Y$, then there exists continuous single-valued selections $f\colon X \to Y$ and $g\colon Y \to X$.*

**Example 3.** Let $X = \{a, b, c, d, e\}$ and $Y = \{0, 1, 2, 3, 4\}$ with topologies $\tau_X = \{X, \emptyset, \{c\}, \{b, c, d\}\}$ and $\tau_Y = \{Y, \emptyset, \{2, 3\}, \{0, 2, 3\}, \{1, 2, 3, 4\}\}$. A map $F\colon X \to Y$ defined with

$$F(a) = \{0, 1\}, F(b) = \{1, 2\}, F(c) = \{2, 3\}, F(d) = \{3, 4\}, F(e) = \{0, 4\}$$

is an m-homeomorphism. It is easy to see that $F$ is continuous. Now, defined $G\colon Y \to X$ by

$$G(0) = \{a, e\}, G(1) = \{a, b\}, G(2) = \{b, c\}, G(3) = \{c, d\}, G(4) = \{d, e\},$$

where $G$ is continuous. Also, for all $x \in X$ and $y \in Y$ the condition of being the m-retraction and the m-section holds, e.g., $a = G(0) \cap G(1)$ and $3 = F(c) \cap F(d)$.

**Remark 2.** If $f\colon X \to Y$ is a homeomorphism, then $f'\colon X/\{x\} \to Y/\{f(x)\}$ is also a homeomorphism. Nevertheless, this condition is not provided for an m-homeomorphism.

For the following proposition we assume that $F$ is $n$-th valued.

**Proposition 3.** *Every homeomultimorphism is an m-homeomorphsim.*

*Proof.* Assume that $F\colon X \to Y$ be a homeomultimorphism. From the definition of homeomultimorphism we say that $F$ and $F^{-1}$ are one-to-one and continuous. It is clear that $x \in F^{-1}(y)$, where $y \in F(x)$. Suppose that $\{x, x'\} = \cap F^{-1}(y)$, where $y \in F(x)$. So, $F$ maps $x'$ to all of $y \in F(x)$. Since $D(F) = n$, we have that $F(x) = F(x')$. It contradict to being one-to-one, hence $F$ is an m-section. Similarly, we can observe $F$ is an m-retraction because of $(F^{-1})^{-1} = F$. □

Actually, we give a more general homeomorphism concept on multi-valued function than Strother's definition [14].

**Corollary 3.** *Let $F$ and $G$ be two m-retractions such that $(G \circ F)^{-1}$ is one-to-one and continuous then $G \circ F$ is an m-retraction.*

The next corollory is a natural consequence of Definition 8, Lemma 3 and Corollary 3.

**Corollary 4.** *The composition of two m-homeomorphism is an m-homeomorphism provided that their composition is an m-retraction.*

Before stating a good result, we give the following remark.

**Remark 3.** Let $F\colon (X, X_0) \to (Y, Y_0)$ be an m-homeomorphism and let $f$ and $g$ be continuous selections such that $f(x_0) = y_0$ and $g(y_0) = x_0$ for base points $x_0, y_0$. Then the following diagram exists.

$$\begin{array}{ccccccc}
(S(X), x_0) & \xleftarrow[i_1]{} & (X, x_0) & \underset{g}{\overset{f}{\rightleftarrows}} & (Y, y_0) & \xrightarrow{i_2} & (S(Y), y_0) \\
\downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_n} \\
\pi_n(S(X), x_0) & \xleftarrow[i_{1*}]{} & \pi_n(X, x_0) & \underset{g_*}{\overset{f_*}{\rightleftarrows}} & \pi_n(Y, y_0) & \xrightarrow{i_{2*}} & \pi_n(S(Y), y_0)
\end{array}$$

**Theorem 3.** *Let $F$ be a m-homeomorphism between $X$ and $Y$. Then we have $M_{\pi_n}(X, X_0) \cong M_{\pi_n}(Y, Y_0)$ if selections are injective and continuous.*

*Proof.* By Theorem 2, we have $\pi_n(S(X), X_0) \cong M\pi_n(X, X_0)$ for every compact, Haussdorf space $X$. Assume that $F\colon (X, X_0) \to (Y, Y_0)$ be an m-homeomorphism and let $h_1\colon \pi_n(S(X), X_0) \to M\pi_n(X, X_0)$ and $h_2\colon \pi_n(S(Y), Y_0) \to M\pi_n(Y, Y_0)$ be isomorphisms. We have a selection $f\colon (X, x_0) \to (Y, y_0)$, by Proposition 1. Suppose that $f$ and $g$ are injective and continuous. Then, we have $f^{-1} = g$, where $g\colon (Y, y_0) \to (X, x_0)$ is the other selection. So, $f$ is a homeomorphism. Let $\psi\colon M\pi_n(X, X_0) \to M\pi_n(Y, Y_0)$ be a group homomorphism defined by $\psi([H]) = h_2 i_{2*} f_* i_{1*}^{-1} h_1^{-1}([H])$. Since $f$ is a homeomorphisim, $f_*$ is a isomorphism. Hence, $\psi$ is a group isomorphism. Consequently, $M_{\pi_n}(X, X_0) \cong M_{\pi_n}(Y, Y_0)$. □

## 5. HOMOTOPY EXTENSION PROPERTY FOR MULTI-VALUED FUNCTIONS

In this part of this study, we first introduce a relation between Homotopy Extention Property for the multi-valued function and the m-retraction. We also introduce some $m$-HEP properties. Then we define a reducible function. First, we give an original definition of Homotopy Extention Property.

**Definition 9.** [2] Let $A$ is a subset of a topological space $X$. Then we say that the pair $(X, A)$ has the *homotopy extension property* with respect to $Y$, if every continuous function $f\colon X \to Y$ and every homotopy $G\colon A \times I \to Y$ that starts with $f|_A$, we can extend $G$ to a homotopy $H\colon X \times I \to Y$ that starts with $f$.

**Definition 10.** A subspace $A$ of $X$ is an $m$-retract of $X$ if there is an m-retraction $R\colon X \to A$ such that $R \circ i(x) = \{x\}$ for all $x \in A$, where $i$ is an inclusion map.

Here, if $R\colon X \to A$ is an m-retraction, then there exists a single-valued selection $r\colon X \to A$. Since $r(x) \in R(x)$ for all $x \in X$ we have $r \circ i(x) = x$.

If $r$ is a continuous function, then it is a retraction. Thereby, we have following results.

**Corollary 5.** (1) *If $A$ is a retract of $X$ and $r\colon X \to A$ is a retraction, then there is an multi-valued extention of $r$ such that $R\colon X \to A$ is an m-retraction;*
(2) *A subspace $A$ is an m-retract of $X$ and single-valued selection $r$ is a continuous function if and only if $A$ is a retract of $X$.*

Now, we give a definition of Homotopy Extension Property for multi-valued functions.

**Definition 11.** For a subspace $A$ of $X$, a pair $(X, A)$ is said to be have an $m$-Homotopy Extension Property ($m$-HEP) if an arbitrary continuous multi-valued function $G\colon X \to Y$ and an m-homotopy function $G'\colon A \times I \to Y$ such that $G(x) = G'(x, 0)$ for all $x \in A$, there exists an m-homotopy function $F\colon X \times I \to Y$ such that $F(x, 0) = G(x)$ and $F|_{A\times I} = G'$.

**Proposition 4.** *[6] For a subspace $A \subset X$, $X \times \{0\} \cup A \times I$ is an m-retract of $X \times I$ if a pair $(X, A)$ has the m-HEP.*

*Proof.* For a special map (identity map) $id\colon X\times\{0\}\cup A\times I \to X\times\{0\}\cup A\times I$, the $m$-HEP implies that the identical map $id$ extends to an extension map $r\colon X \times I \to X \times \{0\} \cup A \times I$, so $r$ is a retraction. By the Corollary 5-($i$), $X \times \{0\} \cup A \times I$ is an m-retract of $X \times I$. □

**Proposition 5** ([1])**.** *Suppose that $(X, A)$ has the m-HEP and that two continuous multi-valued functions $F_0, F_1\colon A \to Y$ are m-homotopic. Then $F_0$ has a continuous extension if and only if $F_1$ has a continuous extension.*

*Proof.* Let $F_0 \simeq_m F_1$. Then there exists a continuous m-homotopy function $F\colon A \times I \to Y$ such that $F(x, 0) = F_0$ and $F(x, 1) = F_1$ for all $x \in A$. Let $F_0'\colon X \to Y$ be a continuous extension of $F_0$. So, $F(x, 0) = F_0 = F_0'$ for all $x \in A$. Since $(X, A)$ has the $m$-HEP, there exists an m-homotopy function $H\colon X \times I \to Y$ such that $H(x, 0) = F_0$ and $H|_{A\times I} = F$. A multi-valued function $F_1'\colon X \to Y$ is defined by $F_1'(x) = H(x, 1)$ for all $x \in X$. $F_1'$ is a continuous multi-valued function and $F_1'(x) = H(x, 1) = F(x, 1) = F_1$ and $F_1'|_A = H|_A = F|_A = F_1$ for all $x \in A$. Hence, $F_1$ has a continuous extension. □

**Proposition 6.** *Let $X \subseteq Y \subseteq Z$. If both pairs $(Y, X)$ and $(Z, Y)$ have the m-HEP, then $(Z, X)$ has also an m-HEP.*

*Proof.* Let $K\colon Z \to T$ be a continuous multi-valued function. Assume that $K'\colon X \times I \to T$ be an m-homotopy such that $K(z) = K'(z, 0)$ for all $z \in X$. Since $(Y, X)$ has $m$-HEP, for a restiriction map $K|_Y$ there exists an m-homotopy $F\colon Y \times I \to T$ such that $F(y, 0) = K|_Y$. From our assumption $(Z, Y)$ has the $m$-HEP, so we can extend an m-homotopy $F$ to an m-homotopy $H\colon Z \times I \to T$, where $H|_{X\times I} = F|_{X\times I} = K'$ and $H(z, 0) = K(z)$. Therefore, $(Z, X)$ has the $m$-HEP. □

**Theorem 4.** *Let $CX$ be a cone of any topological space $X$. Then the pair $(CX, X)$ has the m-HEP.*

*Proof.* Let $F\colon CX \to Y$ be a continuous multi-valued function with an m-homotopy $H\colon X \times I \to Y$ such that $F(\bar{x}) = H(x, 0)$, where $\bar{x}$ denotes the image of $x \in X$ in $CX$. Let $G\colon CX \times I \to Y$ be an m-homotopy defined by

$$G(\overline{(x,t)}, t') = \begin{cases} F\overline{(x, 1-t'')}, & \text{if } t'' \leq 1, \\ H(x, t''-1), & \text{if } t'' \geq 1, \end{cases}$$

where $t'' = (1-t)(1-t')$. Here, $G((x,t),0) = F(\bar{x})$ and $G|_{X\times I} = H$. Then $(CX, X)$ has the $m$-HEP. □

**Definition 12.** Let $F\colon X \to Y$ be a continuous multi-valued function. If $F = \cup f_i$ for some continuous single-valued function $f_i\colon X \to Y$, then we say that $F$ is a reducible multi-valued function.

**Proposition 7.** *Every constant multi-valued function is a reducible.*

*Proof.* Let $F\colon X \to Y$ be a constant multi-valued function defined by $F(x) = Y_0$ for all $x \in X$, where $Y_0 \subset Y$. Assume that $Y_0 = \{y_1, y_2, \ldots, y_n\}$. Then there exists continuous constant functions $f_n\colon X \to Y_0$ defined by $f_n(x) = y_n$ for all $x \in X$. $F = \cup f_i$, where $1 \leq i \leq n$. □

**Remark 4.** If $F$ is a constant multi-valued function, then $F$ has an extension.

Before giving a relation between extension of multi-valued functions and $m$-contiractible, we need the following theorem.

**Theorem 5.** *[7] Let $Y$ be an m-contractible space. Then every continuous multi-valued function $F\colon X \to Y$ is null m-homotopic.*

**Theorem 6.** *If $Y$ is an m-contractible, then every continuous multi-valued function has the continuous extension.*

*Proof.* Let $Y$ be an m-contractible and $F\colon X \to Y$ be a continuous multi-valued function. From Theorem 5 $F$ is a null m-homotopic. Then there exists an m-homotopy between $F$ and constant multi-valued function $C$, i.e., $F \simeq_m C$. By the Remark 4 and Proposition 5, we conclude that $F$ has the continuous extension. □

Now, we can refer the relation between $m$-HEP and HEP.

**Theorem 7.** *If every continuous multi-valued function is a reducible from arbitrary space $X$ to $Y$, then the pair $(X, A)$ has the HEP if and only if $(X, A)$ has the m-HEP.*

*Proof.* Assume that $(X, A)$ has the HEP and $F\colon X \to Y$ be a continuous reducible multi-valued function such that $F = \cup f_n$. Since $(X, A)$ has the HEP, there exists homotopies $G_n\colon A\times I \to Y$ and extensions $H_n\colon X\times I \to Y$

for all $f_n$. Let two $m$-homotopies $G' \colon A \times I \to Y$ and $H' \colon X \times I \to Y$ defined by $G' = \cup G_n$ and $H' = \cup H_n$, respectively. So, $H|_{A \times I} = G'$ and $G'(x) = \cup f_n(x) = F(x)$ for all $x \in A$. Hence, the pair $(X, A)$ has the $m$-HEP. Conversely, if we choose a special continuous reducible multi-valued function $F$ such that $F = f$, then the $m$-HEP requires the HEP. □

## 6. Conclusion

We can consider single-valued functions as a special version of multi-valued functions. Many researchers have worked on multi-valued functions with this idea. In this paper, we give a definition of multi-category and also investigate some properties related m-retraction and selection. Moreover, we generalize a homeomorphism concept in general topology and give theorems related to the homotopy in algebraic topology.

The purpose of this paper is to cover some properties to multi-valued functions using algebraic topology. Therefore, we determine whether there is any difference in multi-valued function version.

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