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On approximation properties of functions by means of Fourier and Faber series in weighted Lebesgue spaces with variable exponent

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Abstract. In this paper the approximation of functions by linear means of Fourier series in weighted variable exponent Lebesgue spaces was studied. This result was applied to the approximation of the functions by linear means of Faber series in Smirnov classes with variable exponent defined on simply connected domain of the complex plane.

1. Introduction and main results

Let $\mathbb{T}$ denote the interval $[0,2\pi]$ and $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable functions on $\mathbb{T}$.

Let $\wp$ denote the class of Lebesgue measurable functions $p : \mathbb{T} \longrightarrow (1,\infty)$ such that

$$1 < p_* := \underset{x\in\mathbb{T}}{ess\inf}\, p(x) \leq p^* := \underset{x\in T}{ess\sup}\, p(x) < \infty.$$

The conjugate exponent of $p(x)$ is shown by $p'(x) := \frac{p(x)}{p(x)-1}$. For $p \in \wp$, we define a class $L^{p(\cdot)}(\mathbb{T})$ of 2π periodic measurable functions $f : \mathbb{T} \to \mathbb{R}$ satisfying the condition

$$\int_{\mathbf{T}} |f(x)|^{p(x)} dx < \infty.$$

This class $L^{p(\cdot)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{T})} := \inf\{\lambda > 0 : \int_{\mathbf{T}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\}.$$

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The spaces $L^{p(\cdot)}(\mathbb{T})$ are called generalized Lebesgue spaces with variable exponent. It is known that for $p(x) := p$ $(0 < p \leq \infty)$, the space $L^{p(x)}(\mathbb{T})$ coincides with the Lebesgue space $L^{p}(\mathbb{T})$. If $p^{*} < \infty$ then the spaces $L^{p(\cdot)}(\mathbb{T})$ represent a special case of the so-called Orlicz-Musielak spaces [37]. For the first time Lebesgue spaces with variable exponent were introduced by Orlicz [38]. Note that the generalized Lebesgue spaces with variable exponent are used in the theory of elasticity, in mechanics, especially in fluid dynamics for the modelling of electrorheological fluids, in the theory of differential operators, and in variational calculus [5,8,9,41,43]. The detailed information about properties of the Lebesque spaces with variable exponent can be found in [8, 10, 27, 33, 34, 42, 46]. Note that some of the fundamental problems of the approximation theory in the generalized Lebesgue spaces with variable exponent of periodic and non-periodic functions were studied and solved by Sharapudinov [47–49].

A function $\omega : \mathbb{T} \to [0, \infty]$ is called a *weight function* if ω is a measurable and almost everywhere (a.e.) positive.

Let ω be a 2π periodic weight function. We denote by $L^{p}_{\omega}(\mathbb{T})$ the weighted Lebesgue space of 2π periodic measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that $f\omega^{\frac{1}{p}} \in L^{p}(\mathbb{T})$. For $f \in L^{p}_{\omega}(\mathbb{T})$ we set

$$\left\|f\right\|_{L^{p}_{\omega}(\mathbb{T})} := \left\|f\omega^{\frac{1}{p}}\right\|_{L^{p}(\mathbb{T})}.$$

$L^{p(\cdot)}_{\omega}(\mathbb{T})$ stands for the class of Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that $\omega f \in L^{p(\cdot)}(\mathbb{T})$. $L^{p(\cdot)}_{\omega}(\mathbb{T})$ is called the weighted Lebesgue space with variable exponent. The space $L^{p(\cdot)}_{\omega}(\mathbb{T})$ is a Banach space with respect to the norm

$$\left\|f\right\|_{L^{p(\cdot)}_{\omega}(\mathbb{T})} := \left\|f\omega\right\|_{L^{p(\cdot)}(\mathbb{T})}.$$

It is known (see [28]) that the set of trigonometric polynomials is dense in $L^{p(\cdot)}_{\omega}(\mathbb{T})$, if $[\omega(x)]^{p(x)}$ is integrable on $\mathbb{T}$.

Let $\mathcal{B}$ be the class of all intervals in $\mathbb{T}$. For $B \in \mathcal{B}$ we set

$$p_{B} := \left(\frac{1}{|B|}\int\limits_{B}\frac{1}{p(x)}dx\right)^{-1}.$$

For given $p \in \wp$ the class of weights ω satisfying the condition

$$\left\|\omega^{p(x)}\right\|_{A_{p(.)}} := \sup_{B\in\mathcal{B}}\frac{1}{|B|^{p_{B}}}\left\|\omega^{p(x)}\right\|_{L^{1}(B)}\left\|\frac{1}{\omega^{p(x)}}\right\|_{L^{(p'(.)/p(.))}(B)} < \infty$$

will be denoted by $A_{p(.)}$ [1, 15, 23, 30, 32].

We say that the variable exponent $p(x)$ satisfies *Local log-Hölder continuity condition*, if there is a positive constant c_1 such that

(1) $$\left|p(x) - p(y)\right| \leq \frac{c_{1}}{\log\left(e + \frac{1}{|x-y|}\right)}, \quad \text{for all } x, y \in \mathbb{T}.$$

A function $p \in \wp$ is said to belong to the class $\wp^{\log}$, if the condition (1) is satisfied.

We denote by $E_n(f)_{L_\omega^{p(\cdot)}(\mathbb{T})}$ the best approximation of $f \in L_\omega^{p(\cdot)}(\mathbb{T})$ by trigonometric polynomials of degree not exceeding $n-1$, i.e.,

$$E_n(f)_{L_\omega^{p(\cdot)}(\mathbb{T})} = \inf\Big\{\left\|f - T_{n-1}\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} : T_{n-1} \in \Pi_{n-1}\Big\},$$

where Π_{n-1} denotes the class of trigonometric polynomials of degree at most $n-1$.

Let $1 < p < \infty$, $1/p + 1/p' = 1$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy *Muckenhoupt*'s A_p-*condition* on $\mathbb{T}$ [2, 3, 15, 17], if

$$\sup_J \left(\frac{1}{|J|}\int_J \omega^p(t)dt\right)^{1/p}\left(\frac{1}{|J|}\int_J \omega^{-p'}(t)dt\right)^{1/p'} < \infty,$$

where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length.

Let us denote by $A_p(\mathbb{T})$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on $\mathbb{T}$.

We use the constants $c, c_1, c_2, \ldots,$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest.

Let $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \wp^{\log}$ and $\omega \in A_{p(\cdot)}$. We define the modulus of smoothness as

$$\Omega(f,\delta)_{p(\cdot),\omega} := \sup_{0<h\le\delta}\left\|\frac{1}{h}\int_0^h \big(f(x+t) - f(x)\big)dt\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})}, \quad \delta > 0.$$

Note that according to [53, 54] $\Omega(f,\delta)_{p(\cdot),\omega} \le c(p)\|f\|_{L_\omega^{p(\cdot)}(\mathbb{T})}$. It can easily be shown that $\Omega(\cdot, f)_{p(\cdot),\omega}$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0}\Omega(f,\delta)_{p(\cdot),\omega} = 0,$$
$$\Omega(f+g,\delta)_{p(\cdot),\omega} \le \Omega(f,\delta)_{p(\cdot),\omega} + \Omega(g,\delta)_{p(\cdot),\omega}$$

for $f, g \in L_\omega^{p(\cdot)}(\mathbb{T})$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\big(a_k(f)\cos kx + b_k(f)\sin kx\big) \tag{2}$$

be the Fourier series of the function $f \in L_1(\mathbb{T})$, where $a_k(f)$ and $b_k(f)$ are the Fourier coefficients of the function f. Let (2) be the Fourier series of the function f.

For $f \in L_\omega^{p(.)}(\mathbb{T})$ we define the summability method by the tringular matrix $\Lambda = \{\lambda_{ij}\}_{i,j=0}^{j,\infty}$ by the linear means

$$U_n(x,f) = \lambda_{0n}\frac{a_0}{2} + \sum_{i=1}^{n} \lambda_{in}\big(a_i(f)\cos ix + b_i(f)\sin ix\big).$$

If the Fourier series of f is given by (2), then *Zygmund-Rieszmeans of order* k is defined as

$$Z_n^k(x,f) = \frac{a_0}{2} + \sum_{i=1}^{n}\left(1 - \frac{i^k}{(n+1)^k}\right)\big(a_i(f)\cos ix + b_i(f)\sin ix\big).$$

We denote by $E_n(f)_{p(.),\omega}$ the best approximation of $f \in L_\omega^{p(.)}(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_{p(.),\omega} = \inf\Big\{\big\|f - T_n\big\|_{L_\omega^{p(.)}(\mathbb{T})} : T_n \in \Pi_n\Big\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

Let $T_n \in \Pi_n$

$$T_n = \frac{c_0}{2} + \sum_{i=1}^{n}\big(c_i\cos ix + d_i\sin ix\big).$$

The conjugate polynomial $\widetilde{T_n}$ is defined by

$$\widetilde{T_n} = \sum_{i=1}^{n}\big(c_i\sin ix - d_i\cos ix\big).$$

We will say that the method of summability by the matrix Λ satisfies condition $b_{k,p(\cdot)}$ (respectively $b^*_{k,p(\cdot)}$) if for $T_n \in \Pi_n$ the inequality

$$\big\|T_n - U_n(T_n)\big\|_{L_\omega^{p(.)}(\mathbb{T})} \le c(n+1)^{-k}\big\|T_n^{(k)}\big\|_{L_\omega^{p(.)}(\mathbb{T})}$$

$$\Big(\big\|T_n - U_n(T_n)\big\|_{L_\omega^{p(.)}(\mathbb{T})} \le c(n+1)^{-k}\big\|\widetilde{T_n}^{(k)}\big\|_{L_\omega^{p(.)}(\mathbb{T})}\Big)$$

holds and the norms

$$\big\|\Lambda\big\|_1 := \int_0^{2\pi}\Big|\frac{\lambda_{0n}}{2} + \sum_{i=1}^{n}\lambda_{in}\cos it\Big|dt$$

are bounded.

In the present paper, the necessary and sufficient condition about the relationship between the approximation of functions by linear means of Fourier series and by Zygmund-Riesz means of order k was investigated in weighted Lebesgue spaces with variable exponent. Also, we investigate the approximation of functions by linear means of Fourier series in terms of the modulus of smoothness of these functions in weighted Lebesgue spaces with variable exponent. This result was applied to the approximation of the functions by linear means of Faber series in weighted Smirnov classes with variable exponent defined on simply connected domains of the complex plane. The similar problems in different spaces were investigated by several authors (see, for example, [1, 2, 4, 7, 12, 14, 16–26, 29–32, 35, 36, 39, 44, 50–57]).

The main results in the present work are the following theorems.

Theorem 1. *Let $f \in L_\omega^{p(\cdot)}(T)$, $p(\cdot) \in \wp^{\log}$ and $\omega \in A_{p(.)}$. In order that*

$$\big\|f - U_n(\cdot, f)\big\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \leq c_1 \big\|f - Z_n^k(\cdot, f)\big\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \tag{3}$$

it is sufficient and necessary that

$$\big\|T_n - U_n(\cdot, T_n)\big\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \leq c_2 \big\|T_n - Z_n^k(\cdot, T_n)\big\|_{L_\omega^{p(\cdot)}(\mathbb{T})}. \tag{4}$$

Theorem 2. *Let $f \in L_\omega^{p(\cdot)}(T)$, $p(\cdot) \in \wp^{\log}$ and $\omega \in A_{p(.)}$. If the summability method with the matrix Λ satisfies the condition $(b_{k,M})$ or $(b^*_{k,M})$, then the inequality*

$$\big\|f - U_n(\cdot, f)\big\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \leq c_3 \Omega\left(f, \frac{1}{n}\right)_{p(\cdot),\omega} \tag{5}$$

holds with a constant $c_3 > 0$ independent of n.

Theorem 3. *Let $f \in L_\omega^{p(\cdot)}(T)$, $p(\cdot) \in \wp^{\log}$ and $\omega \in A_{p(.)}$. If the summability method with the matrix Λ satisfies the condition $(b_{k,p(\cdot)})$ or $(b^*_{k,p(\cdot)})$, then the estimate*

$$\Omega\big(U_n(\cdot, f), \delta\big)_{p(\cdot),\omega} \leq c_4 \Omega(f, \delta)_{p(\cdot),\omega} \tag{6}$$

holds with a constant $c_4 > 0$ not depend on n, f and δ.

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ, and let $G^- := ext\Gamma$. Further let

$$\mathbb{T} := \big\{w \in \mathbb{C} : |w| = 1\big\}, \quad \mathbb{D} := int\,\mathbb{T}, \quad \mathbb{D}^- := ext\,\mathbb{T}.$$

Let $w = \phi(z)$ be the conformal mapping of G^- onto D^- normalized by

$$\phi(\infty) = \infty, \quad \lim_{z\to\infty} \frac{\phi(z)}{z} > 0,$$

and let ψ denote the inverse of ϕ.

Let $w = \phi_1(z)$ denote a function that maps the domain G conformally onto the disk $|w| < 1$. The inverse mapping of ϕ_1 will be denoted by ψ_1. Let Γ_r denote circular images in the domain G, that is, curves in G corresponding to circle $|\phi_1(z)| = r$ under the mapping $z = \psi_1(w)$.

Let us denote by E_p, where $p > 0$, the class of all functions $f(z) \neq 0$ which are analytic in G and have the property that the integral

$$\int_{\Gamma_r} |f(z)|^p |dz|$$

is bounded for $0 < r < 1$. We shall call the E_p-class the *Smirnov class*. If the function $f(z)$ belongs to E_p, then $f(x)$ has definite limiting values $f(z')$ almost everywhere on Γ, over all nontangential paths; $|f(z')|$ is summable on Γ; and

$$\lim_{r\to 1}\int\limits_{\Gamma_r}\left|f(z)\right|^p|dz| = \int\limits_{\Gamma}\left|f(z')\right|^p|dz|.$$

It is known that $\varphi' = E_1(G^-)$ and $\psi' \in E_1(D^-)$. Note that the general information about Smirnov classes can be found in the books [13, pp. 438-453] and [11, pp. 168-185].

Let $L_M(\mathbb{T},\omega)$ is a weighted Orlicz space defined on Γ. We define also the *ω-weighted Smirnov class* of variable exponent $E_{p(\cdot)}(G,\omega)$ as

$$E_{p(\cdot)}(G,\omega) := \left\{f \in E_1(G) : f \in L^{p(\cdot)}_{\omega}(\Gamma)\right\}.$$

For $f \in L^{p(\cdot)}_{\omega}(\Gamma)$ with $p \in \wp^{\log}$ we define the functions

$$f_0(t) := f\big(\psi(t)\big), \quad t \in \mathbb{T},$$
$$p_0(t) := p\big(\psi(t)\big), \quad t \in \mathbb{T}.$$

Let h be a continuous function on $[0, 2\pi]$. Its modulus of continuity is defined by

$$\omega(t,h) := \sup\big\{|h(t_1) - h(t_2)| : t_1, t_2 \in [0,2\pi], |t_1 - t_2| \le t\big\}, \quad t \ge 0.$$

The curve Γ is called *Dini-smooth* if it has a parameterization

$$\Gamma : \varphi_0(s), \quad 0 \le s \le 2\pi,$$

such that $\varphi_0'(s)$ is Dini-continuous, i.e.,

$$\int\limits_0^{\pi}\frac{\omega(t,\varphi_0')}{t}dt < \infty$$

and $\varphi_0'(s) \ne 0$ [40, p. 48]. If Γ is Dini-smooth curve, then there exist (see [58]) the constants c_5 and c_6 such that

$$0 \le c_5 \le |\psi'(t)| \le c_6 < \infty, \quad |t| > 1. \tag{7}$$

Note that if Γ is a Dini-smooth curve, then by (7) we have $f_0 \in L^{p(\cdot)}_{\omega_0}(\mathbb{T})$ for $f \in L^{p(\cdot)}_{\omega}(\Gamma)$. It is known (see [20]) that, if Γ is a Dini-smooth curve, then $p_0 \in \wp^{\log}(\mathbb{T})$ if and only if $p \in \wp^{\log}(\Gamma)$.

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'}$ and let ω be a weight function on Γ. ω is said to satisfy Muckenhoupt's A_p-condition on Γ, if

$$\sup_{z\in\Gamma}\sup_{r>0}\left(\frac{1}{r}\int\limits_{\Gamma\cap D(z,r)}|\omega(\tau)|^p|d\tau|\right)^{1/p}\left(\frac{1}{r}\int\limits_{\Gamma\cap D(z,r)}[\omega(\tau)]^{-p'}|d\tau|\right)^{1/p'} < \infty,$$

where $D(z,r)$ is an open disk with radius r and centered z.

Let us denote by $A_p(\Gamma)$ the set of all weight functions satisfying Muckenhoupt's A_p -condition on Γ. For a detailed discussion of Muckenhoupt weights on curves, see, e.g., [3].

Let Γ be a rectifiable Jordan curve and $f \in L_1(\Gamma)$. Then the function f^+ defined by

$$f^+(z) := \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(s)ds}{s-z}, \quad z \in G$$

is analytic in G. Note that if $p(\cdot) \in \wp^{\log}$, $\omega \in A_p(\Gamma)$ and $f \in L_\omega^{p(\cdot)}(\Gamma)$, then by Lemma 5 in [53] $f^+ \in E_{p(\cdot)}(G, \omega)$.

Let $\phi_k(z)$, $k = 0, 1, 2, \ldots$, be the Faber polynomials for G. The Faber polynomials $\phi_k(z)$, associated with $G \cup \Gamma$, are defined through the expansion

$$\frac{\psi'(w)}{\psi(w)-z} = \sum_{k=0}^{\infty} \frac{\phi_k(z)}{w^{k+1}}, \quad z \in G,\ w \in D^- \tag{8}$$

and the equalities

$$\begin{aligned} \phi_k(z) &= \frac{1}{2\pi i} \int\limits_{T} \frac{t^k \psi'(t)}{\psi(t)-z} dt, \quad z \in G, \\ \phi_k(z) &= \phi^k(z) + \frac{1}{2\pi i} \int\limits_{T} \frac{\phi^k(s)}{s-z} ds, \quad z \in G^- \end{aligned} \tag{9}$$

hold [45, p. 33-48].

Let $f \in E_{p(\cdot)}(G, \omega)$. Since $f \in E_1(G)$, we obtain

$$f(z) := \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(s)ds}{s-z} = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f(\psi(t))\psi'(t)}{\psi(t)-z} dt,$$

for every $z \in G$. Considering this formula and expansion (8), we can associate with f the formal series

$$f(z) \sim \sum_{i=0}^{\infty} a_i(f) \phi_i(z), \quad z \in G, \tag{10}$$

where

$$a_i(f) := \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f(\psi(t))}{t^{i+1}} dt, \quad i = 0, 1, 2, \ldots.$$

This series is called the *Faber series* expansion of f, and the coefficients $a_i(f)$ are said to be the *Faber coefficients* of f.

Let (10) be the Faber series of the function $f \in E_{p(\cdot)}(G, \omega)$. For the function f we define the summability method by the tringular matrix

$\Lambda = \{\lambda_{ij}\}_{i,j=0}^{j,\infty}$ by the linear means

$$U_n(z,f) = \sum_{i=0}^{n} \lambda_{in} a_i(f)\phi_i(z),$$

The n-the *partial sums* and *Zygmund means of order* k of the series (10) are defined, respectively, as

$$S_n(z,f) = \sum_{k=0}^{n} a_k(f)\phi_k(z),$$

$$Z_n^k(z,f) = \sum_{i=0}^{n} \left(1 - \frac{i^k}{(n+1)^k}\right) a_i(f)\phi_i(z).$$

Let Γ be a Dini-smooth curve. Using the nontangential boundary values of f_0^+ on $\mathbb{T}$ we define the *modulus of smoothness* of $f \in L_\omega^{p(\cdot)}(\Gamma)$ as

$$\Omega(f,\delta)_{p(\cdot),\Gamma,\omega} := \Omega(f_0^+,\delta)_{p_0(\cdot),\omega_0}, \quad \delta > 0.$$

The following theorem holds.

Theorem 4. *Let Γ be a Dini-smoth curve, $p(\cdot) \in \wp^{\log}$, $\omega \in A_p(\Gamma)$ and the summability method with the matrix Λ satisfies the condition $(b_{k,p(\cdot)})$ or $(b_{k,p(\cdot)}^*)$, then for $f \in E_{p(\cdot)}(G,\omega)$ the estimate*

$$\left\| f - U_n(\cdot,f)\right\|_{L_\omega^{p(\cdot)}(\Gamma)} \le c_7 \Omega\left(f,\frac{1}{n}\right)_{p(\cdot),\Gamma,\omega} \tag{11}$$

holds with a constant $c_7 > 0$, independent of n.

Let $\mathcal{P}$ be the set of all algebraic polynomials (with no restriction on the degree), and let $\mathcal{P}(\mathbb{D})$ be the set of traces of members of $\mathcal{P}$ on $\mathbb{D}$. We define the operator

$$T : \mathcal{P}(\mathbb{D}) \longrightarrow E_{p(\cdot)}(G,\omega)$$

as

$$T(P)(z) := \frac{1}{2\pi i}\int_T \frac{P(w)\psi'(w)}{\psi(w) - z} dw, \quad z \in G.$$

Then, from (9) we have

$$T\left(\sum_{k=0}^{n} \beta_k w^k\right) = \sum_{k=0}^{n} \beta_k \phi_k(z).$$

The following result holds for the linear operator T [53].

Theorem 5. *If Γ is a Dini-smooth curve, $p(\cdot) \in \wp^{\log}$ and $\omega \in A_p(\Gamma)$, then the operator*

$$T : E_{p_0(\cdot)}(\mathbb{D},\omega_0) \longrightarrow E_{p(\cdot)}(G,\omega)$$

is linear, bounded, one-to-one and onto. Moreover, $T(f_0^+) = f$ for every $f \in E_{p(\cdot)}(G,\omega)$.

2. Proof of the main results

Proof of Theorem 1. It is clear that the inequality (4) follows from the inequality (3).

Sufficiency. Let $f \in L_{\omega}^{p(.)}(T)$, $p(\cdot) \in \wp^{\log}$ and $\omega \in A_{p(.)}$ and let $T \in \Pi_n$ $(n = 0, 1, 2, \dots)$ be the polynomial of best approximation to f. Then

$$\begin{aligned}
&\left\|f - U_n(\cdot, f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\le \left\|f - T_n\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} - \left\|T_n - U(\cdot, T_n)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + \left\|U_n(\cdot, f - T_n)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\le E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + c_2\left\|T_n - Z_n^k(\cdot, T_n)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + c_8 E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\le c_9 E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + c_2\Big(\left\|T_n - f\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + \left\|f - Z_n^k(\cdot, f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\qquad\qquad + \left\|Z_n^k(\cdot, f - T_n)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}\Big) \\
&\le c_9 E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + c_2 E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\quad + c_2\left\|f - Z_n^k(\cdot, f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + c_2 c_{10} E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\le c_{11} E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + c_2\left\|f - Z_n^k(\cdot, f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\le c_{12}\left\|f - Z_n^k(\cdot, f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}.
\end{aligned}$$

The proof of Theorem 1 is completed. □

Proof of Theorem 2. We suppose that the condition $b_{k,p(\cdot)}^{*}$ is satisfed. Let $f \in L_{\omega}^{p(.)}(T)$, $p(\cdot) \in \wp^{\log}$, $\omega \in A_{p(.)}$ and $T_n \in \Pi_n$ be the polynomial of best approximation to f. Note that $U_n(f) = \Lambda_n * f$. Considering [6] the operator $U_n(f)$ is bounded in $L_{\omega}^{p(\cdot)}(\mathbb{T})$, i.e., $\left\|U_n(\cdot, f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \le c_5\left\|f\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}$. Then we have

$$\begin{aligned}
&\left\|f - U_n(\cdot, f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\le \left\|f - T_n\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + \left\|T_n - U(\cdot, T_n)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\qquad + \left\|U(\cdot, T_n) - U(\cdot, f)\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\le c_{13} E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + c_7 E_n(f)_{L_{\omega}^{p(\cdot)}(\mathbb{T})} + c_{14}(n+1)^{-1}\left\|\widetilde{T_n'}\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \\
&\le c_{15} E_n(f)_{M,\omega} + c_{16} n^{-1}\left\|\widetilde{T_n'}\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}.
\end{aligned} \tag{12}$$

Using boundedness of the linear operator $f \to \widetilde{f}$ in $L_{\omega}^{p(\cdot)}(\mathbb{T})$ into account [22, Lemma 1] we obtain

$$\left\|\widetilde{T_n'}\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})} \le c_{17}\left\|T_n'\right\|_{L_{\omega}^{p(\cdot)}(\mathbb{T})}, \tag{13}$$

where $\widetilde{f}$ is the conjugate function of $f \in L_\omega^{p(\cdot)}(\mathbb{T})$.
Use of (13) and [54] gives us

$$\begin{aligned} n^{-1}\left\|\widetilde{T_n'}\right\|_{L_M(\mathbb{T},\omega)} &\leq c_{19}n^{-1}\left\|T_n'\right\|_{L_M(\mathbb{T},\omega)} \\ &\leq c_{20}\Omega\left(f,\frac{1}{n}\right)_{p(\cdot),\omega}. \end{aligned} \tag{14}$$

Note that according to the direct theorem of approximation in $L_\omega^{p(\cdot)}(\mathbb{T})$ given in [21, Lemma 4] the inequality

$$E_n(f)_{M,\omega} \leq c_{21}\Omega\left(f,\frac{1}{n}\right)_{p(\cdot),\omega}. \tag{15}$$

holds. Taking into account the relations (12), (14) and (15), we obtain

$$\left\|f - U_n(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)} \leq c_{22}\Omega\left(f,\frac{1}{n}\right)_{p(\cdot),\omega}.$$

If the summability method with the matrix Λ satisfies condition $(b^*_{k,p(\cdot)})$, the proof is made anologously to the above.

The proof of Theorem 2 is completed. □

Proof of Theorem 3. By [21] the inequality

$$\Omega(U_n(f) - f,\delta)_{p(\cdot),\omega} \leq c_{23}\left\|U_n(\cdot,f) - f\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})}. \tag{16}$$

holds.

Let $\delta \geq (n+1)^{-1}$. By using Theorem 2 and (16) we have

$$\begin{aligned} \Omega(U_n(f),\delta)_{p(\cdot),\omega} &\leq \Omega(f,\delta)_{p(\cdot),\omega} + \Omega^r_{M,\omega}(U_n(\cdot,f) - f,\delta)_{p(\cdot),\omega} \\ &\leq \Omega(f,\delta)_{p(\cdot),\omega} + c_{24}\left\|U_n(\cdot,f) - f\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\ &\leq \Omega(f,\delta)_{p(\cdot),\omega} + c_{25}\Omega(f,\delta)_{p(\cdot),\omega} \\ &\leq c_{26}\Omega(f,\delta)_{p(\cdot),\omega}. \end{aligned} \tag{17}$$

Now we suppose that $\delta < (n+1)^{-1}$. Then considering [22, Lemma 3] and [54, Theorem 1.3] we conclude that

$$\Omega(U_n(\cdot,f),\delta)_{p(\cdot),\omega} \leq c_{27}\delta\left\|U_n'(\cdot,f)\right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \leq c_{28}\Omega(f,\delta)_{p(\cdot),\omega}. \tag{18}$$

The use of (17) and (18) gives us the inequality (6) of Theorem 3. □

Proof of Theorem 4. Let $f \in E_{p(\cdot)}(G,\omega)$. Then by virtue of Theoerm 5 the operator $T : E_{p_0(\cdot)}(\mathbb{D},\omega_0) \longrightarrow E_{p(\cdot)}(G,\omega)$ is linear, bounded, one-to-one and onto and $T(f_0^+) = f$. The function f has the following Faber series

$$f(z) \backsim \sum_{m=0}^{\infty} a_m(f)\phi_m(z).$$

Taking into account [53, relation (3) and lemma 5] we conclude that $f_0^+ \in E_{p_0(\cdot)}(\mathbb{D}, \omega_0)$. For the function f_0^+ the following Taylor series holds:

$$f_0^+(w) = \sum_{m=0}^{\infty} a_m(f) w^m.$$

Note that $f_0^+ \in E_1(\mathbb{D})$ and boundary function $f_0^+ \in L_{\omega_0}^{p_0(\cdot)}(\mathbb{T})$. Then by [11, Theorem, 3.4] for the function f_0^+ we have the following Fourier expansion:

$$f_0^+(w) \backsim \sum_{m=0}^{\infty} a_m(f) e^{imt}.$$

Hence, if we consider boundedness of the operator $T : E_{p_0(\cdot)}(\mathbb{D}, \omega_0) \longrightarrow E_{p(\cdot)}(G, \omega)$ and Theorem 2, we have

$$\begin{aligned}
\left\| f - U_n(.,f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} &= \left\| T(f_0^+) - T(U_n(.,f_0^+)) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \\
&\le c_{29} \left\| f_0^+ - U_n(.,f_0^+) \right\|_{L_{\omega_0}^{p_0(\cdot)}(\mathbb{T})} \\
&\le c_{30} \Omega \left(f_0^+, \frac{1}{n} \right)_{p_0(\cdot),\omega_0} \\
&= c_{31} \Omega \left(f, \frac{1}{n} \right)_{p(\cdot),\Gamma,\omega},
\end{aligned}$$

and (11) is proved. □

Remark 1. Let $f \in L_\omega^{p(\cdot)}(T)$, $p(\cdot) \in \wp^{\log}$ and $\omega \in A_{p(.)} L_M(\mathbb{T}, \omega)$. Then by virtue of Theorem 2 in [22] the inequality

$$\Omega \left(f, \frac{1}{n} \right)_{p(\cdot),\omega} \le \frac{c_{32}}{n} \sum_{m=0}^{n} E_m(f)_{L_{\omega_0}^{p_0(\cdot)}(\mathbb{T})} \tag{19}$$

holds, with a constant c_{32} independent of n. If the summability method with the matrix Λ satisfy the condition $(b_{k,p(\cdot)})$ or $(b^*_{k,p(\cdot)})$, then relation (5) and inequality (19) immediately yield

$$\left\| f - U_n(.,f) \right\|_{L_\omega^{p(\cdot)}(\mathbb{T})} \le \frac{c_{33}}{n} \sum_{m=0}^{n} E_m(f)_{L_{\omega_0}^{p_0(\cdot)}(\mathbb{T})}. \tag{20}$$

The inequality (20) holds for Zygmund-Riesz means of order k. Note that in the Lebesgue spaces $L_p(\mathbb{T})$, $1 < p \le \infty$, the inequality (20) was proved in [50].

3. CONCLUSION

In Theorem 1 of this work, the relationship between the linear means of Fourier and Zygmund means of Fourier series in weighted variable exponent Lebesgue spaces has been investigated. The necessary and sufficient condition has been found for this relationship.

In Theorem 2, the approximation of the function by the linear means of Fourier series in weighted variable exponent Lebesgue spaces was studied in terms of modulus of smoothness.

In Theorem 3, the modulus of smoothness of the linear means of Fourier series of the function has been estimated.

In Theorem 4, the result obtained in Theorem 2 was applied to the approximation of the functions by linear means of Faber series in Smirnov classes with variable exponent defined in the domains with a Dini-smooth boundary of the complex plane.

In Remark 1, the approximation of the function by linear means of Fourier series has been obtained in terms of the best approximation of the function.

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