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Suzuki-type fixed point theorems in relational metric spaces with applications

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Abstract. In this paper, we establish a relation-theoretic version of the results presented by Kim et al. (Journal of Nonlinear and Convex Analysis, 16 (9) (2015), 1779–1786). To showcase the versatility of our results, we furnish some illustrative examples. Furthermore, we exhibit an application of our results to establish sufficient conditions for the existence of a positive definite common solution to a pair of nonlinear matrix equations.

1. Introduction

The Banach contraction principle (BCP) [5] is widely utilized in nonlinear analysis, offering a multitude of applications. Over time, various researchers have established several extensions of this principle. Within the realm of contraction mappings, a majority necessitates their validity for all points within the underlying space. Consequently, a natural inquiry arises: "Can we relax this requirement without compromising the outcomes of the theorem?" Answering this question affirmatively, Suzuki pioneered a new category of contraction mappings that need only hold for specific elements of the underlying space, rather than the entire space. As a result, he derived the subsequent generalization of the BCP.

Theorem 1 ([29]). *Let (W,ρ) be a complete metric space and $\theta : [0,1) \to (1/2,1]$ be a non-decreasing mapping defined by*

$$\theta(\kappa) = \begin{cases} 1, & \text{if } 0 \le \kappa \le (\sqrt{5}-1)/2, \\ (1-\kappa)\kappa^2, & (\sqrt{5}-1)/2 \le \kappa \le 2^{-1/2}, \\ (1+\kappa)^{-1}, & \text{if } 2^{-1/2} \le \kappa < 1. \end{cases} \tag{1}$$

Assume that $\mathcal{P} : W \to W$ be mappings such that

$$\theta(\kappa)\rho(\omega,\mathcal{P}\omega) \le \rho(\omega,\bar\omega) \Longrightarrow \rho(\mathcal{P}\omega,\mathcal{P}\bar\omega) \le \kappa\rho(\omega,\bar\omega),$$

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for all $\omega, \bar{\omega} \in W$ *and some* $\kappa \in [0, 1)$*. Then* $\mathcal{P}$ *has a unique fixed point in* W*.*

Several mathematicians have made significant contributions by generalizing and extending Suzuki's result [29]. For some recent advancement along this direction one can refer to [10, 12, 14, 21–23, 27, 28]. One such advancement was achieved by Kim et al. [18], who extended the result of Suzuki [29] for two self-mappings in metric spaces.

Theorem 2 ([18])**.** *Let* (W, ρ) *be a complete metric space and* $\mathcal{P}, \mathcal{Q} : W \to W$ *two self-mappings such that*

$$\theta(\kappa) \min\{\rho(\omega, \mathcal{P}\omega), \rho(\omega, \mathcal{Q}\omega)\} \leq \rho(\omega, \bar{\omega}) \text{ implies } M(\omega, \bar{\omega}) \leq \kappa\rho(\omega, \bar{\omega}),$$

for all $\omega, \bar{\omega} \in W$*, where* $\theta(\kappa)$ *is defined in* (1) *and*

$$M(\omega, \bar{\omega}) = \max\{\rho(\mathcal{P}\omega, \mathcal{P}\bar{\omega}), \rho(\mathcal{Q}\omega, \mathcal{Q}\bar{\omega}), \rho(\mathcal{P}\omega, \mathcal{Q}\bar{\omega}), \rho(\mathcal{P}\bar{\omega}, \mathcal{Q}\omega)\}.$$

Then $\mathcal{P}$ *and* $\mathcal{Q}$ *have a unique common fixed point* $\upsilon \in W$*.*

Shukla and Pant [28] further built upon the findings of Kim et al. [18] by extending their result to more general nonlinear contraction conditions. Their result unified the result of Boyd and Wong's [9] and Kim et al.'s [18] in metric space and extended the scope of these results.

Theorem 3 ([28])**.** *Let* (W, ρ) *be a complete metric space and* $Q, \mathcal{P} : W \to W$ *are two mappings such that for all* $\omega, \bar{\omega} \in W$*,*

$$\frac{1}{2} \min\{\rho(\omega, \mathcal{P}\omega), \rho(\omega, \mathcal{Q}\omega)\} \leq \rho(\omega, \bar{\omega}) \text{ implies } M(\omega, \bar{\omega}) \leq \psi(\rho(\omega, \bar{\omega})), \tag{2}$$

where $M(\omega, \bar{\omega})$ *is defined in the Theorem 2 and* $\psi : [0, \infty) \to [0, \infty)$ *is upper semi-continuous function from the right on* $[0, \infty)$ *such that* $\psi(t) < t$ *for all* $t > 0$*. Then there exists a common fixed point of* $\mathcal{P}$ *and* $\mathcal{Q}$ *in* W*.*

In 2015, Alam and Imdad [1] provided another affirmative response to the aforementioned problem. They introduced a relation-theoretic contraction that only needs to hold among elements related by a binary relation. Furthermore, they demonstrated that under the universal relation, the relation-theoretic contraction principle is equivalent to the BCP. This result was later undertaken by several researchers [3, 4, 6, 11, 13, 15–17, 25–27]. In 2020, Prasad et al. [23] extended this line of research by establishing an analogous version of Theorem 1 in a relational metric space, which not only generalizes the result of Alam and Imdad [1] but also open a new scope for Suzuki-type fixed point theorems to relational metric spaces.

The objective of the present paper is to establish a relation-theoretic version of Kim et al.'s result [18]. In doing so, we extend the results of Suzuki [29], Alam and Imdad [1], Paesano and Vetro [20] and many others. Additionally, we provide an illustrative example to highlight the significance of our main results. Finally, we derive some sufficient conditions for the existence of a positive definite common solution to a pair of nonlinear matrix equations.

2. Preliminaries

In this paper, we follow the following notation: $\Im$ represents a nonempty binary relation, $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{R}$ represents the set of all real numbers. These notations will be used consistently throughout the paper to clarify our discussion and mathematical expressions.

Definition 1 ([1, 2])**.** Let $\Im$ be a binary relation on a metric space (W, ρ). Then:

(i) two elements ω and $\bar{\omega}$ in W are $\Im$-comparable if either $(\omega, \bar{\omega}) \in \Im$ or $(\bar{\omega}, \omega) \in \Im$, denoted by $[\omega, \bar{\omega}] \in \Im$;

(ii) a sequence (ω_n) on W is said to be $\Im$-preserving if $(\omega_n, \omega_{n+1}) \in \Im$, $n \in \mathbb{N} \cup \{0\}$;

(iii) a metric space (W, ρ) is $\Im$-complete if every $\Im$-preserving Cauchy sequence in W converges;

(iv) $\Im$ is called ρ-self-closed, whenever $\Im$-preserving sequence (ω_n) on W such that $\omega_n \xrightarrow{\rho} \omega$ have a subsequence (ω_{n_l}) of (ω_n) and $[\omega_{n_l}, \omega_n] \in \Im$ for $l \in \mathbb{N} \cup \{0\}$;

(v) $\Im$ is called a transitive relation if for any $\omega, \bar{\omega}, \upsilon \in W$, $(\omega, \bar{\omega}) \in \Im$ and $(\bar{\omega}, \upsilon) \in \Im \Rightarrow (\omega, \upsilon) \in \Im$;

(vi) a path of length $l \in \mathbb{N}$, from ω to $\bar{\omega}$ in $\Im$, is a finite sequence $\upsilon_0, \upsilon_1, \ldots, \upsilon_l \subseteq W$ satisfying the conditions: (p_1) $\upsilon_0 = \omega$ and $\upsilon_l = \bar{\omega}$, and (p_2) $(\upsilon_i, \upsilon_{i+1}) \in \Im$, for all $i \in 0, 1, 2, \ldots, l-1$;

(vii) we denote the family of all paths from ω to $\bar{\omega}$ in $\Im$ by $\gamma(\omega, \bar{\omega}, \Im)$, and the set of all points ω of W for which $(\omega, \mathcal{P}\omega) \in \Im$ by $W(\mathcal{P}; \Im)$.

Definition 2 ([1, 2])**.** Let W be a nonempty set and let $\Im$ be a binary relation on W. Also, consider a self-mapping $\mathcal{P}$ on W. Then

(a) the binary relation $\Im$ is called $\mathcal{P}$-closed if for any $\omega, \bar{\omega} \in W$ with $(\omega, \bar{\omega}) \in \Im$, it follows that $(\mathcal{P}\omega, \mathcal{P}\bar{\omega}) \in \Im$;

(b) the mapping $\mathcal{P}$ is said to be $\Im$-continuous at a point $\omega \in W$ if, for any $\Im$-preserving sequence (ω_n) such that $\omega_n \xrightarrow{\rho} \omega$, we have $\mathcal{P}(\omega_n) \xrightarrow{\rho} \mathcal{P}(\omega)$. Moreover, if $\mathcal{P}$ is $\Im$-continuous at each point of W, then $\mathcal{P}$ is called $\Im$-continuous.

Proposition 1 ([1])**.** *Let $\Im$ be a binary relation on a nonempty set W and let $\mathcal{P}$ be a self-mapping on W. If $\Im$ is $\mathcal{P}$-closed, then $\Im$ is also $\mathcal{P}^n$-closed for all $n \in \mathbb{N} \cup 0$, where $\mathcal{P}^n$ denotes the n^{th} iterate of $\mathcal{P}$.*

3. Common fixed point results for Suzuki-type contractions

Firstly, we introduce the term "sequential limit property" as follows.

Definition 3. Let (W, ρ) be a metric space and $\Im$ be a binary relation on W. Then, $(W, \rho, \Im)$ has the sequential limit property, if for two $\Im$-preserving sequences $(\omega_n), (\bar{\omega}_n) \subset W$ such that $\omega_n \to \omega,\ \bar{\omega}_n \to \bar{\omega}$ and $(\omega_n, \bar{\omega}_n) \in \Im$, we have $(\omega, \bar{\omega}) \in \Im$.

Now, we state our main results.

Theorem 4. *Let (W, ρ) be a metric space equipped with a binary relation $\Im$ on W and $\mathcal{P}, \mathcal{Q} : W \longrightarrow W$ be two mappings such that the following conditions hold:*

(i) *$W(\mathcal{P}; \Im)$ is nonempty,*
(ii) *$\Im$ is $\mathcal{P}$-closed,*
(iii) *(W, ρ) is $\Im$-complete,*
(iv) *$\mathcal{P}$ is $\Im$-continuous or*
(v) *$\Im$ is ρ-self closed and $(W, \rho, \Im)$ has the sequential limit property,*
(vi) *there exists $\kappa \in [0, 1)$ such that*

$$\theta(\kappa) \min\{\rho(\omega, \mathcal{P}\omega), \rho(\omega, \mathcal{Q}\omega)\} \leq \rho(\omega, \bar{\omega}) \implies M(\omega, \bar{\omega}) \leq \kappa\rho(\omega, \bar{\omega}), \tag{3}$$

for all $\omega, \bar{\omega} \in W$ with $(\omega, \bar{\omega}) \in \Im$, where $M(\omega, \bar{\omega})$ is defined in Theorem 2 and $\theta(\kappa)$ is defined in (1).

Then $\mathcal{P}$ and $\mathcal{Q}$ have a common fixed point.

Proof. Firstly, we demonstrate that if υ is a fixed point of either $\mathcal{P}$ or $\mathcal{Q}$ in the relational metric space $(W, \rho, \Im)$, then υ is a common fixed point of both $\mathcal{P}$ and $\mathcal{Q}$. Let υ be a fixed point of $\mathcal{P}$, i.e., $\mathcal{P}\upsilon = \upsilon$. We will now establish that υ is also a fixed point of $\mathcal{Q}$. Since

$$0 = \theta(\kappa) \min\{\rho(\upsilon, \mathcal{P}\upsilon), \rho(\upsilon, \mathcal{Q}\upsilon)\} \leq \rho(\upsilon, \mathcal{P}\upsilon),$$

then it follows from (3),

$$\rho(\mathcal{Q}\upsilon, \upsilon) \leq \max\left\{\begin{array}{c} \rho(\mathcal{Q}\upsilon, QP\mu), \rho(\mathcal{P}\upsilon, \mathcal{P}^2\upsilon) \\ \rho(\mathcal{Q}\upsilon, \mathcal{P}^2\upsilon), \rho(QP\mu, \mathcal{P}\upsilon) \end{array}\right\} \leq \kappa d(\upsilon, \mathcal{P}\upsilon) = 0,$$

which means υ is a fixed point of $\mathcal{Q}$.

Since $W(\mathcal{P}; \Im)$ is nonempty, so we can pick an arbitrary point $\omega_0 \in W(\mathcal{P}; \Im)$ and construct a sequence $(\omega_n) \in W$ such that

$$\omega_{n+1} = \mathcal{P}\omega_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

As $\omega_0 \in W(\mathcal{P}; \Im)$, it follows that $(\omega_0, \mathcal{P}\omega_0) \in \Im$. Now, $\mathcal{P}$-closedness of $\Im$ and by Proposition 1, $\Im$ is $\mathcal{P}^n$-closed for all $n \in \mathbb{N} \cup \{0\}$. This implies $(\omega_n, \omega_{n+1}) \in \Im$ for all $n \in \mathbb{N} \cup \{0\}$. Hence, the (ω_n) is $\Im$-preserving sequence in W. Since $\theta(\kappa) \leq 1$ for all $\kappa \in [0, 1)$ and

$$\theta(\kappa) \min\{\rho(\omega_n \mathcal{P}\omega_n), \rho(\omega_n, \mathcal{Q}\omega_n)\} \leq \theta(\kappa)\rho(\omega_n, \omega_{n+1}) \leq \rho(\omega_n, \omega_{n+1}).$$

It follows from (3) that

$$\rho(\omega_{n+1}, \omega_{n+2}) = \rho(\mathcal{P}\omega_n, \mathcal{P}\omega_{n+1}) \leq \kappa\rho(\omega_n, \omega_{n+1}), \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Also, by triangle inequality for all $m \geq n$, we have

$$\begin{aligned}\rho(\omega_n, \omega_m) &\leq \rho(\omega_n, \omega_{n+1}) + \rho(\omega_{n+1}, \omega_{n+2}) + \cdots + \rho(\omega_{m-1}, \omega_m)\\ &\leq \kappa^n \rho(\omega_0, \omega_1) + \kappa^{n+1}\rho(\omega_0, \omega_1) + \ldots \kappa^{m-1}\rho(\omega_0, \omega_1)\\ &= \kappa^n(1 + \kappa + \cdots + \kappa^{m-n-1})\rho(\omega_0, \omega_1)\\ &\leq \frac{\kappa^n}{1-\kappa}\rho(\omega_0, \omega_1) \to 0, \text{ as } n \to \infty.\end{aligned}$$

So, we have

$$\lim_{n,m\to\infty} \rho(\omega_n, \omega_m) = 0.$$

Hence (ω_n) is $\Im$-preserving Cauchy sequence in W. Since W is $\Im$-complete, there exists $\upsilon \in W$ such that $\omega_n \xrightarrow{\rho} \upsilon$. If we assume that $\mathcal{P}$ is $\Im$-continuous then we have $\omega_{n+1} = \mathcal{P}\omega_n \xrightarrow{\rho} \mathcal{P}\upsilon$. Owing to the uniqueness of the limit, we obtain $\mathcal{P}\upsilon = \upsilon$, that is, υ is a fixed point of $\mathcal{P}$ and since $(\omega_n, \omega_{n+1}) \in \Im$ for $n \in \mathbb{N}\cup\{0\}$, in view of the limit sequential property, we conclude $(\upsilon, \upsilon) \in \Im$.

On the other hand, if we assume $\Im$ is ρ-self-closed then it guarantees the existence of the subsequence (ω_{n_l}) of (ω_n) with $[\omega_{n_l}, \omega] \in \Im$ for all $l \in \mathbb{N}\cup\{0\}$. By virtue of $\mathcal{P}$-closedness, we have $[\omega_{n_l+1}, \mathcal{P}\upsilon] \in \Im$ for all $l \in \mathbb{N}\cup\{0\}$. Taking limit $l \to \infty$, in light of the sequentially limit property, we have $[\upsilon, \mathcal{P}\upsilon] \in \Im$.

Now, we will prove that

$$\rho(\upsilon, \mathcal{P}\omega) \leq \kappa\rho(\upsilon, \omega), \quad \text{for } \omega \in W - \{\upsilon\}. \tag{4}$$

Since $\omega_{n_l} \xrightarrow{\rho} \upsilon$ and let $\omega \neq \upsilon \in W$ then there exists $\upsilon \in \mathbb{N}$ such that

$$\rho(\omega_{n_l}, \upsilon) < \rho(\omega, \upsilon)/3, \quad \text{for all } n \in \mathbb{N} \ \text{ with } \ n \geq \upsilon.$$

Then, we have

$$\begin{aligned}\theta(\kappa)\min\{\rho(\omega_{n_l}, \mathcal{Q}\omega_{n_l}), \rho(\omega_{n_l}, \mathcal{P}\omega_{n_l})\} &\leq \theta(\kappa)\rho(\omega_{n_l}, \omega_{n_l+1})\\ &\leq \rho(\omega_{n_l}, \upsilon) + \rho(\omega_{n_l+1}, \upsilon)\\ &\leq (2/3)\rho(\omega, \upsilon) = \rho(\omega, \upsilon) - \rho(\omega, \upsilon)/3\\ &\leq \rho(\omega, \upsilon) - \rho(\omega_{n_l}, \upsilon) = \rho(\omega_{n_l}, \omega).\end{aligned}$$

Hence, by hypothesis (3), we have

$$\rho(\omega_{n_l+1}, \mathcal{P}\omega) < \rho(\omega_{n_l}, \omega), \quad \text{for all } n \geq \upsilon \in \mathbb{N}.$$

Letting $l \to \infty$, we get

$$\rho(\upsilon, \mathcal{P}\omega) \leq \kappa\rho(\upsilon, \omega), \quad \text{for } \omega \in W - \{\upsilon\}.$$

Arguing by contraction, we assume that $\mathcal{P}^m\upsilon \neq \upsilon$, for all $m \in \mathbb{N}$. Then by (4), we have

$$\rho(\upsilon, \mathcal{P}^{m+1}\upsilon) \leq \kappa^m\rho(\upsilon, \mathcal{P}\upsilon), \quad \text{for } m \in \mathbb{N}. \tag{5}$$

We consider the following three cases.

Case 1: Let $0 \leq \kappa < \dfrac{\sqrt{5}-1}{2}$. Then, we note that $\kappa^2+\kappa-1 \leq 0$ and $2\kappa^2 \leq 1$. If we assume that $\rho(\upsilon, \mathcal{P}^2\upsilon) < \rho(\mathcal{P}^2\upsilon, \mathcal{P}^3\upsilon)$ then we have

$$\begin{aligned}
\rho(\upsilon, \mathcal{P}\upsilon) &\leq \rho(\upsilon, \mathcal{P}^2\upsilon) + \rho(\mathcal{P}\upsilon, \mathcal{P}^2\upsilon) \\
&< \rho(\mathcal{P}^2\upsilon, \mathcal{P}^3\upsilon) + \rho(\mathcal{P}\upsilon, \mathcal{P}^2\upsilon) \\
&\leq \kappa^2\rho(\upsilon, \mathcal{P}\upsilon) + \kappa\rho(\upsilon, \mathcal{P}\upsilon) < \rho(\upsilon, \mathcal{P}\upsilon).
\end{aligned}$$

This is a contradiction. So, we have

$$\rho(\upsilon, \mathcal{P}^2\upsilon) \geq \rho(\mathcal{P}^2\upsilon, \mathcal{P}^3\upsilon) \geq \theta(\kappa)\min\{\rho(\mathcal{P}^2\upsilon, \mathcal{P}^3\upsilon), \rho(\mathcal{P}^2\upsilon, \mathcal{Q}\mathcal{P}^2\upsilon)\}.$$

Then by hypothesis and (5), we have

$$\begin{aligned}
\rho(\upsilon, \mathcal{P}\upsilon) &\leq \rho(\upsilon, \mathcal{P}^3\upsilon) + \rho(\mathcal{P}^3\upsilon, \mathcal{P}\upsilon) \\
&\leq \kappa^2\rho(\upsilon, \mathcal{P}\upsilon) + \kappa\rho(\mathcal{P}^2\upsilon, \upsilon) \\
&\leq \kappa^2\rho(\upsilon, \mathcal{P}\upsilon) + \kappa^2\rho(\mathcal{P}\upsilon, \upsilon) = 2\kappa^2\rho(\upsilon, \mathcal{P}\upsilon) \\
&< \rho(\upsilon, \mathcal{P}\upsilon).
\end{aligned}$$

This is again a contradiction.

Case 2: Let $\dfrac{\sqrt{5}-1}{2} \leq \kappa < 2^{-1/2}$. Then, we note that $2\kappa^2 < 1$. If we assume that $\rho(\upsilon, \mathcal{P}^2\upsilon) < \theta(\kappa)\rho(\mathcal{P}^2\upsilon, \mathcal{P}^3\upsilon)$, then we have

$$\begin{aligned}
\rho(\upsilon, \mathcal{P}\upsilon) &\leq \rho(\upsilon, \mathcal{P}^2\upsilon) + \rho(\mathcal{P}\upsilon, \mathcal{P}^2\upsilon) \\
&< \theta(\kappa)\rho(\mathcal{P}^2\upsilon, \mathcal{P}^3\upsilon) + \rho(\mathcal{P}\upsilon, \mathcal{P}^2\upsilon) \\
&\leq \theta(\kappa)\kappa^2\rho(\upsilon, \mathcal{P}\upsilon) + \kappa\rho(\upsilon, \mathcal{P}\upsilon) = \rho(\upsilon, \mathcal{P}\upsilon).
\end{aligned}$$

This is a contradiction. So, we have

$$\rho(\upsilon, \mathcal{P}^2\upsilon) \geq \theta(\kappa)\rho(\mathcal{P}^2\upsilon, \mathcal{P}^3\upsilon) \geq \theta(\kappa)\min\{\rho(\mathcal{P}^2\upsilon, \mathcal{P}^3\upsilon), \rho(\mathcal{P}^2\upsilon, \mathcal{Q}\mathcal{P}^2\upsilon)\}.$$

Then by hypothesis and (5), we have

$$\begin{aligned}
\rho(\upsilon, \mathcal{P}\upsilon) &\leq \rho(\upsilon, \mathcal{P}^3\upsilon) + \rho(\mathcal{P}^3\upsilon, \mathcal{P}\upsilon) \\
&\leq \kappa^2\rho(\upsilon, \mathcal{P}\upsilon) + \kappa\rho(\mathcal{P}^2\upsilon, \upsilon) \\
&\leq \kappa^2\rho(\upsilon, \mathcal{P}\upsilon) + \kappa^2\rho(\mathcal{P}\upsilon, \upsilon) = 2\kappa^2\rho(\upsilon, \mathcal{P}\upsilon) \\
&< \rho(\upsilon, \mathcal{P}\upsilon).
\end{aligned}$$

This is again a contradiction.

Case 3: Let $2^{-1/2} \leq \kappa < 1$. Then, we will prove that for subsequence (ω_{n_l}) of (ω_n) with $[\omega_{n_l}, \upsilon]$, for all $n \in \mathbb{N} \cup \{0\}$, we have

(6)

$$\text{either} \quad \frac{1}{1+\kappa}\rho(\omega_{n_l}, \omega_{n_l+1}) \leq \rho(\omega_{n_l}, \upsilon) \quad \text{or} \quad \frac{1}{1+\kappa}\rho(\omega_{n_l+1}, \omega_{n_l+2}) \leq \rho(\omega_{n_l+1}, \upsilon).$$

Suppose that for some $n_l \in \mathbb{N}$,

$$\frac{1}{1+\kappa}\rho(\omega_{n_l},\omega_{n_l+1}) > \rho(\omega_{n_l},\upsilon) \quad \text{and} \quad \frac{1}{1+\kappa}\rho(\omega_{n_l+1},\omega_{n_l+2}) > \rho(\omega_{n_l+1},\upsilon).$$

Hence, we have

$$\begin{aligned}\rho(\omega_{n_l},\omega_{n_l+1}) &\leq \rho(\omega_{n_l},\upsilon)+\rho(\omega_{n_l+1},\upsilon)\\ &< \frac{1}{1+\kappa}\rho(\omega_{n_l},\omega_{n_l+1})+\frac{1}{1+\kappa}\rho(\omega_{n_l+1},\omega_{n_l+2})\\ &\leq \frac{1}{1+\kappa}\rho(\omega_{n_l},\omega_{n_l+1})+\frac{\kappa}{1+\kappa}\rho(\omega_{n_l},\omega_{n_l+1})\\ &= \rho(\omega_{n_l},\omega_{n_l+1}),\end{aligned}$$

which is not possible. Hence, for each (ω_{n_l}) with $[\omega_{n_l},\upsilon]$, we have (6) but then contraction condition (3) implies

$$\rho(\mathcal{P}\omega_{n_l},\mathcal{P}\upsilon) \leq \max\left\{\begin{array}{c}\rho(\mathcal{Q}\omega_{n_l},\mathcal{Q}\upsilon),\rho(\mathcal{P}\omega_{n_l},\mathcal{P}\upsilon),\\ \rho(\mathcal{Q}\omega_{n_l},\mathcal{P}\omega_{n_l}),\rho(\mathcal{Q}\upsilon,\mathcal{P}\upsilon)\end{array}\right\} \leq \kappa\rho(\omega_{n_l},\upsilon).$$

Letting $l \to \infty$, we get

$$\rho(\upsilon,\mathcal{P}\upsilon) \leq \kappa\rho(\upsilon,\upsilon) = 0,$$

which implies $\mathcal{P}\upsilon = \upsilon$. Therefore, we have proved that there exists a common fixed point υ of $\mathcal{P}$ and $\mathcal{Q}$ in W, that is $\mathcal{P}\upsilon = \upsilon = \mathcal{Q}\upsilon$. □

Similarly, interchanging the role of mapping $\mathcal{P}$ with mapping $\mathcal{Q}$ in Theorem 4, we get the following result.

Theorem 5. *Let (W,ρ) be a metric space equipped with a binary relation $\Im$ on W. Assume that $\mathcal{P},\mathcal{Q} : W \longrightarrow W$ be two mappings such that the following conditions hold:*

(i) *$W(\mathcal{Q};\Im)$ is nonempty,*
(ii) *$\Im$ is $\mathcal{Q}$-closed,*
(iii) *(W,ρ) is $\Im$-complete,*
(iv) *$\mathcal{Q}$ is $\Im$-continuous or*
(v) *$\Im$ is ρ-self closed and $(W,\rho,\Im)$ has the sequential limit property,*
(vi) *there exists $\kappa \in [0,1)$ such hat condition (3) holds.*

Then $\mathcal{P}$ and $\mathcal{Q}$ have a common fixed point in W.

Now, we establish corresponding uniqueness results for common fixed points of two self mappings.

Theorem 6. *Suppose that $\Im$ is a transitive relation on W and $\gamma(\omega,\bar{\omega},\Im)$ is nonempty, for all $\omega,\bar{\omega} \in Fix(\mathcal{P},\mathcal{Q}) := \{\upsilon \in W : \mathcal{P}\upsilon = \upsilon = \mathcal{Q}\upsilon\}$ in addition to the assumptions of Theorem 4 (respectively, Theorem 5). Then $\mathcal{P}$ and $\mathcal{Q}$ possess a unique common fixed point.*

Proof. To prove the uniqueness, assume that ω and $\bar{\omega}$ are two distinct common fixed points of $\mathcal{P}$ and $\mathcal{Q}$, that is $\mathcal{P}\omega = \omega = \mathcal{Q}\omega$, $\mathcal{P}\bar{\omega} = \bar{\omega} = \mathcal{Q}\bar{\omega}$ and $\omega \neq \bar{\omega}$. Since $\gamma(\omega, \bar{\omega}, \Im)$ is nonempty, there is a path say $\{\upsilon_0, \upsilon_1, \upsilon_2, \dots, \upsilon_l\}$ of some finite length l in $\Im$ from ω to $\bar{\omega}$, so that

$$\upsilon_0 = \omega, \upsilon_l = \bar{\omega} \quad \text{and} \quad (\upsilon_i, \upsilon_{i+1}) \in \Im, \quad \text{for each } i = 0, 1, 2, \dots, l-1.$$

By the transitivity of $\Im$, we get

$$(\omega, \upsilon_1) \in \Im, (\upsilon_1, \upsilon_2) \in \Im, \dots, (\upsilon_{l-1}, \bar{\omega}) \in \Im \implies (\omega, \bar{\omega}) \in \Im.$$

Since $0 = \theta(\kappa) \min\{\rho(\omega, \mathcal{P}\omega), \rho(\omega, \mathcal{Q}\omega)\} \leq \rho(\omega, \bar{\omega})$ then the condition (3.1) implies that

$$\rho(\omega, \bar{\omega}) = \rho(\mathcal{P}\omega, \mathcal{P}\bar{\omega}) \leq \kappa \rho(\omega, \bar{\omega})$$

which is possible only when $\rho(\omega, \bar{\omega}) = 0$ or $\omega = \bar{\omega}$. Thus $\mathcal{P}$ and $\mathcal{Q}$ have a unique common fixed point. $\square$

If we take $\Im = W^2$ or universal relation in Theorem 4 and 5 then as a consequence, we obtain Theorem 2. Similarly, if we take $\mathcal{P} = \mathcal{Q}$ in Theorem 4 (respectively in Theorem 5) then we get a relation theoretic version of Suzuki's [29] fixed point theorem.

Corollary 1. *Let (W, ρ) be a metric space and $\Im$ a binary relation on W. Assume that $\mathcal{P} : W \longrightarrow W$ be a mappings and the following conditions are satisfied:*

(i) *$W(\mathcal{P}; \Im)$ is nonempty,*
(ii) *$\Im$ is $\mathcal{P}$-closed,*
(iii) *(W, ρ) is $\Im$-complete,*
(iv) *$\mathcal{P}$ is $\Im$-continuous or*
(v) *$\Im$ is ρ-self closed and $(W, \rho, \Im)$ has the sequential limit property,*
(vi) *there exists $\kappa \in [0, 1)$ such that*

$$\theta(\kappa)\rho(\omega, \mathcal{P}\omega) \leq \rho(\omega, \bar{\omega}) \implies \rho(\mathcal{P}\omega, \mathcal{P}\bar{\omega}) \leq \kappa \rho(\omega, \bar{\omega}),$$

for all $\omega, \bar{\omega} \in W$ with $(\omega, \bar{\omega}) \in \Im$, where $\theta(\kappa)$ is defined in (1).

Then $\mathcal{Q}$ and $\mathcal{P}$ have a common fixed point.

Example 1. Let $W = \{(0, 0), (0, 2), (2, 0), (2, 3), (3, 2)\}$ equipped with binary relation $\Im$ defined as

$$\Im = \left\{ \begin{array}{c} ((0,0),(0,0)), (2,0),(0,0)), ((0,0),(2,0)), ((0,0),(0,2)), \\ ((2,0),(0,2)), ((2,3),(2,0)), ((3,2),(0,2)) \end{array} \right\}$$

and Euclidean metric ρ defined by

$$\rho\left((\beta_1, \beta_2), (\bar{\beta}_1, \bar{\beta}_2)\right) = \sqrt{(\beta_1 - \beta_2)^2 + (\bar{\beta}_1 - \bar{\beta}_2)^2}.$$

Then W is $\Im$-complete metric space. Define two mappings $\mathcal{P}, \mathcal{Q} : W \to W$ such that

$$\mathcal{P}(\beta_1, \beta_2) = \begin{cases} (\beta_1, 0), & \text{if } \beta_1 \leq \beta_2 \\ (0, \beta_2), & \text{if } \beta_1 > \beta_2 \end{cases} \text{ and } \mathcal{Q}(\beta_1, \beta_2) = \begin{cases} (0, \beta_1), & \text{if } \beta_1 \leq \beta_2, \\ (\beta_2, 0), & \text{if } \beta_1 > \beta_2. \end{cases}$$

We observe that following inequality is not true for $\beta = (\beta_1, \beta_2)$, $\bar{\beta} = (\bar{\beta}_1, \bar{\beta}_2) \in \{(2,3),(3,2)\}$,

$$\theta(\kappa)\min\{\rho(\beta, \mathcal{P}\beta), \rho(\beta, \mathcal{Q}\beta)\} > 3/2 > \sqrt{2} = \rho(\beta, \bar{\beta}).$$

Then, by routine calculation it can easily verify that $\mathcal{P}, \mathcal{Q}$ are $\Im$-continuous, $W(\mathcal{P}, \Im)$ is nonempty, $\Im$ is $\mathcal{P}$-closed, $(W, \rho, \Im)$ has the sequential limit property and $\mathcal{P}$ and $\mathcal{Q}$ satisfy condition (3). Thus all the assumptions of Theorem 4 are satisfied and $(0,0)$ is a unique common fixed point of $\mathcal{P}$ and $\mathcal{Q}$ in W.

4. Application to nonlinear matrix equations

In this section, we shall use the following notations: consider the sets $\mathcal{H}_n$, $\mathcal{P}_n$, and $\mathcal{H}_n^+$, which respectively represent the set of all $n \times n$ Hermitian matrices, the subset of $\mathcal{H}_n$ consisting of positive definite matrices, and the subset of $\mathcal{H}_n$ consisting of positive semi-definite matrices. To simplify notation, we use $U \succ 0$ to indicate that $U \in \mathcal{P}_n$ (i.e., U is positive definite), and $U \succeq 0$ to indicate that $U \in \mathcal{H}_n^+$ (i.e., U is positive semi-definite). Moreover, we define $U \succ V$ and $U \succeq V$ to mean that $U - V$ is positive definite and positive semi-definite, respectively.

Further, we introduce the symbol $\|.\|$ to represent two different metrics associated with matrices C. First, $\|C\|$ denotes the spectral norm of the matrix, defined as $\|C\| = \sqrt{\lambda^+(C^*C)}$. In this context, $\lambda^+(C^*C)$ refers to the largest eigenvalue of the product C^*C and C^* represents the conjugate transpose of C. Second, we use the same symbol $\|.\|$ to denote the metric induced by the trace norm, defined as $\|C\| = \sum_{j=1}^{n} s_j(C)$, where $s_j(C)$ represents the singular values of the matrix $C \in \mathcal{H}_n$.

It is worth noting that the metric space $(\mathcal{H}_n, \|.\|)$ is complete (see in [7], [8], and [24]). Additionally, $(\mathcal{H}_n, \|.\|)$ can be considered a relational metric space equipped with the partial order relation $\preceq$ on $\mathcal{H}_n$, where $U \preceq V$ is equivalent to $V \succeq U$.

Next, we will demonstrate how our main result can be utilized to establish the existence of common solution of a pair of nonlinear matrix equations of the form:

$$U = Q_1 + \sum_{i=1}^{n} A_i^* \left(\mathcal{G}(U)\right) A_i, \tag{7}$$

$$U = Q_2 + \sum_{i=1}^{n} A_i^* \left(\mathcal{F}(U)\right) A_i, \tag{8}$$

where $A_1, A_2, \ldots, A_n$ are arbitrary $n \times n$ matrices, Q_1 and Q_2 are a Hermitian positive definite matrices and $\mathcal{G}$, $\mathcal{F}$ are mappings from the set of Hermitian positive definite matrices to itself. These mappings are assumed to be $\preceq$-continuous, with $\mathcal{G}(0) = 0 = \mathcal{F}(0)$, and possess an order-preserving property.

Specifically, for any $U, V \in \mathcal{H}_n$ where $U \preceq V$, it follows that $\mathcal{G}(U) \preceq \mathcal{G}(V)$ and $\mathcal{F}(U) \preceq \mathcal{F}(V)$.

Lemma 1 ([24]). *Let $U, V \in \mathcal{H}_n$ and $U \succeq 0,\ V \succeq 0$. Then*

$$0 \leq tr(UV) \leq \|U\|\ tr(V).$$

Lemma 2 ([19]). *Let $U \in \mathcal{H}_n$ and $U \prec I$. Then $\|U\| < 1$.*

Theorem 7. *Let $Q \in \mathcal{P}_n$ and the following assertions are true:*
(i) *for $U, V \in \mathcal{H}_n$ such that $U \preceq V$, we have*

$$\theta(\kappa)\min\{|tr\,(V-\mathcal{G}(V))\,|, |tr\,(V-\mathcal{F}(V))\,|\} \leq |tr\,(V-U)\,|$$

implies

$$\max\left\{\begin{array}{l} |tr\,(\mathcal{G}(V)-\mathcal{G}(U)))\,|, |tr\,(\mathcal{F}(V)-\mathcal{F}(U)))\,| \\ |tr\,(\mathcal{G}(V)-\mathcal{F}(U)))\,|, |tr\,(\mathcal{F}(U)-\mathcal{F}(V)))\,| \end{array}\right\} \leq \kappa |tr\,(V-U)\,|, \tag{9}$$

where $\kappa \in [0,1)$ and $\theta(\kappa)$ is defined in (1).
(ii) $\sum_{i=1}^{n} A_i A_i^* \preceq I_n$ *and* $\sum_{i=1}^{n} A_i^* \mathcal{G}(U) A_i \succ 0$.
Then the pair of matrix equations given by (7) *and* (8) *have a common positive definite solution.*

Proof. We define a mapping $\mathcal{M}, \mathcal{N} : \mathcal{H}_n \to \mathcal{H}_n$ by

$$\mathcal{M}(U) = Q_1 + \sum_{i=1}^{n} A_i^* \mathcal{G}(U) A_i \quad \text{and} \quad \mathcal{N}(U) = Q_2 + \sum_{i=1}^{n} A_i^* \mathcal{F}(U) A_i,$$

for all $U \in \mathcal{H}_n$. Consider a set by

$$\mathcal{H}_n^+(\mathcal{M}, \preceq) = \{A \in \mathcal{H}^+ : A \preceq \mathcal{M}(A) \quad \text{or} \quad \mathcal{M}(A) - A \succeq 0\}.$$

Then $\mathcal{M}$ and $\mathcal{N}$ are well defined mappings on H_n, $\mathcal{H}_n^+(\mathcal{M}, \preceq)$ is a nonempty as $Q_1 \in \mathcal{H}_n^+,\ \mathcal{M}(Q_1) - Q_1 = \sum_{i=1}^{n} A_i^* \mathcal{G}(Q_1) A_i \succeq 0$. Also, $\preceq$ is $\mathcal{M}$-closed and transitive, and continuity of $\mathcal{F}$ implies $\mathcal{M}$ is also a $\preceq$-continuous mapping. Now, we will prove that $\mathcal{M}$ and $\mathcal{N}$ satisfy condition (3) on H_n. Then for $V, U \in H_n$ with $V \preceq U$ such that

$$\begin{aligned} \theta(\kappa)\min\{\|V-\mathcal{G}(V)\|, \|V-\mathcal{F}(V)\|\} &= \theta(\kappa)\min\left\{\begin{array}{l} |tr\,(V-\mathcal{G}(V))\,, \\ |tr\,(V-\mathcal{F}(V))\,| \end{array}\right\} \\ &\leq |tr(V-U)| = \|V-U\| \end{aligned}$$

by condition (9) and in view of Lemma 1, 2, we have

$$\begin{aligned} \|\mathcal{M}(V) - \mathcal{M}(U)\| &= tr\left(\sum_{i=1}^{n} A_i^*(\mathcal{G}(V) - \mathcal{G}(U))A_i\right) \\ &= \sum_{i=1}^{n} tr\left(A_i^*\left(\mathcal{G}(V) - \mathcal{G}(U)\right)A_i\right), \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} tr\left(A_i A_i^*(\mathcal{G}(V) - \mathcal{G}(U))\right) \\
&= tr\left([\mathcal{G}(V) - \mathcal{G}(U)] \sum_{i=1}^{n} A_i A_i^*\right) \\
&\leq \sum_{i=1}^{n} \|A_i A_i^*\| \; |tr\left(\mathcal{G}(V) - \mathcal{G}(U)\right)|, \\
&\leq |tr\left(\mathcal{G}(V) - \mathcal{G}(U)\right)| \\
&\leq \max\left\{ \begin{array}{c} \|\mathcal{M}(V) - \mathcal{M}(U)\|, \|\mathcal{N}(V) - \mathcal{N}(U)\| \\ \|\mathcal{M}(V) - \mathcal{N}(U)\|, \|\mathcal{M}(U) - \mathcal{N}(V)\| \end{array} \right\} \\
&\leq \kappa |tr(V - U)| = \kappa \|V - U\|
\end{aligned}
$$

implies

$$
\max\left\{ \begin{array}{c} \|\mathcal{M}(V) - \mathcal{M}(U)\|, \|\mathcal{N}(V) - \mathcal{N}(U)\| \\ \|\mathcal{M}(V) - \mathcal{N}(U)\|, \|\mathcal{M}(U) - \mathcal{N}(V)\| \end{array} \right\} \leq \kappa \|V - U\|.
$$

This shows that $\mathcal{M}$ and $\mathcal{N}$ satisfy condition (3) for $\mathcal{M}$ and $\mathcal{N}$ mappings on $\mathcal{H}_n$. Since, all the assumptions of Theorem 4 are satisfied therefore there exists $Z \in \mathcal{H}_n$ such that $\mathcal{M}(Z) = Z = \mathcal{N}(Z)$, that is, the pair of matrix equations (7) and (8) have a common solution. □

Example 2. Consider the following nonlinear equations:

$$
\begin{aligned}
\mathcal{M}(U) &= Q_1 + A_1^* \times \mathcal{G}(U) \times A_1 + A_2^* \times \mathcal{G}(U) \times A_2, \\
\mathcal{N}(U) &= Q_2 + A_1^* \times \mathcal{F}(U) \times A_1 + A_2^* \times \mathcal{F}(U) \times A_2.
\end{aligned}
$$

Consider matrices A_1, A_2, Q_1, Q_2 as

$$
A_1 = \begin{bmatrix} 0.4794 & 0.0168 & 0.0023 & 0.1262 \\ 0.1664 & 0.3435 & 0.1080 & 0.1719 \\ 0.0269 & 0.0648 & 0.3222 & 0.1948 \\ 0.0121 & 0.0603 & 0.0293 & 0.3803 \end{bmatrix},
$$

$$
A_2 = \begin{bmatrix} 0.3124 & 0.1240 & 0.1034 & 0.1390 \\ 0.0908 & 0.4299 & 0.1113 & 0.0853 \\ 0.0856 & 0.1440 & 0.3086 & 0.1673 \\ 0.1932 & 0.0694 & 0.1124 & 0.5014 \end{bmatrix},
$$

$$
Q_1 = Q_2 = \begin{bmatrix} 0.9345 & 0.5252 & 0.4987 & 0.7513 \\ 0.5252 & 1.4453 & 0.5916 & 0.5132 \\ 0.4987 & 0.5916 & 0.9779 & 0.7254 \\ 0.7513 & 0.5132 & 0.7254 & 1.8343 \end{bmatrix}.
$$

The initial matrices are

$$U_0 = \begin{bmatrix} 7.9831 & 2.6873 & 0.9830 & 1.3722 \\ 2.6873 & 6.2889 & 2.3754 & 2.2816 \\ 0.9830 & 2.3754 & 5.1872 & 2.1945 \\ 1.3722 & 2.2816 & 2.1945 & 5.5958 \end{bmatrix},$$

$$V_0 = 10^4 \times \begin{bmatrix} 1.7183 & 1.5016 & 0.9708 & 1.0817 \\ 1.5016 & 1.5514 & 1.0849 & 1.1748 \\ 0.9708 & 1.0849 & 0.8228 & 0.8681 \\ 1.0817 & 1.1748 & 0.8681 & 0.9634 \end{bmatrix},$$

$$W_0 = \begin{bmatrix} 765.8160 & 588.3909 & 351.6007 & 400.9216 \\ 588.3909 & 645.2194 & 440.0624 & 469.7429 \\ 351.6007 & 440.0624 & 370.5008 & 364.7142 \\ 400.9216 & 469.7429 & 364.7142 & 427.7357 \end{bmatrix}.$$

To test our algorithm, we take $n = 4$, $\kappa = 1/2$, $\theta(\kappa) = 3/4$, tolerance: tol=1e-20 and $\mathcal{G}(U) = \dfrac{1}{1111}U^{1/10}$, $\mathcal{F}(U) = \dfrac{1}{2000}U^{1/3}$.
After 3 successive iterations, we obtain the following coincidence point

$$Z = \begin{bmatrix} 1.8850 & -0.3509 & -0.3526 & -0.5345 \\ -0.3509 & 0.9884 & -0.4534 & 0.0465 \\ -0.3526 & -0.4534 & 1.8048 & -0.4424 \\ -0.5345 & 0.0465 & -0.4424 & 0.9260 \end{bmatrix},$$

$$\mathcal{M}(Z) = \mathcal{N}(Z) = \begin{bmatrix} 0.9348 & 0.5253 & 0.4988 & 0.7515 \\ 0.5253 & 1.4456 & 0.5917 & 0.5134 \\ 0.4988 & 0.5917 & 0.9781 & 0.7256 \\ 0.7515 & 0.5134 & 0.7256 & 1.8347 \end{bmatrix}.$$

The graphical view of convergence of Z are shown in Figure 1.

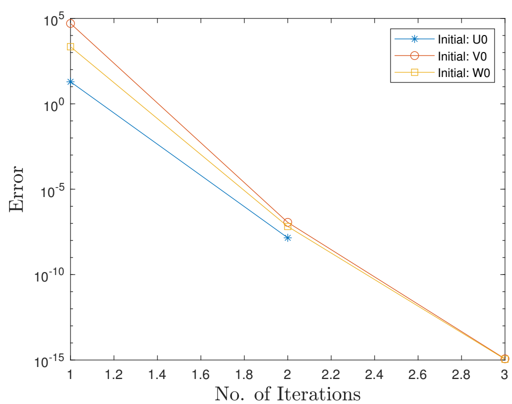

FIGURE 1. Convergence behaviour.

5. Conclusion

In conclusion, we have introduced the concept of the "sequential limit property" and used it to establish a relation-theoretic version of Kim et al.'s result [18]. Our result builds upon the foundational work of Suzuki [29] and its subsequent generalizations [13, 14, 20–22, 28]. The versatility of our theorems has been demonstrated through illustrative examples. The practical significance of our findings is emphasized by the derived sufficient conditions for the existence of positive definite common solutions to a pair of non-linear matrix equations. Looking ahead, there is potential for further extensions of our results. Future research could extend and generalize the findings to more general contraction conditions or to three or more self-mappings.

6. Conflict of interest

The authors declare that they have no conflict of interest.

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