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Perturbed functional fractional differential equation of Caputo-Hadamard order

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Abstract. In this paper, we investigate the existence of solution and extremal solutions for an initial-value problem of perturbed functional fractional differential equations with Caputo-Hadamard derivative. Our analysis relies on the fixed point theorem of Burton and Kirk and the concept of upper and lower solutions combined with a fixed point theorem in ordered Banach space established by Dhage and Henderson.

1. Introduction

The theory of fractional differential equations has received much attention over the past years and has become an important field of investigation due to its extensive applications in numerous branches of physics, chemistry, aerodynamics, electrodynamics of complex medium, palaeethnology, etc. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials. As a consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [6, 8, 10, 11, 23, 36] and the references therein.

There has been a significant development in fractional differential equations in recent years; see the monographs of Kilbas et al. [29], Miller and Ross [32], Podlubny, Samko et al. [35] and the papers of Delbosco and Rodino [18], Diethelm et al. [20–22], El-Sayed [23], Kilbas and Marzan [28], Mainardi [31], Podlubny et al. [33] and Agrawal et al. [2–5, 34].

There are sevral definitions of fractional derivatives, the definitions of Riemann-Liouville (1832), Riemann (1849), Caputo (1997), Grunwald-Letnikov (1867), Hadamard (1891, [25]), and the Caputo Hadamard derivative which is a new approach obtained from the Hadamard derivative by changing the order of its differential and integral parts. Despite the different requirements on the function itself, the main difference between the Caputo Hadamard fractional derivative and the Hadamard fractional derivative is

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that the Caputo Hadamard derivative of a constant is zero [26]. The most important advantage of Caputo Hadamard is that it brings a new definition through which the integer order initial conditions can be defined for fractional.

Belarbi, Benchohra, Hamani and Ntouyas studied the following Initial value problem [9]

$$(1) \qquad {}^cD^\alpha y(t) = f(t, y_t) + g(t, y_t), \text{ for a.e. } t \in J = [0, b], \quad 0 < \alpha \leq 1,$$

$$(2) \qquad y(t) = \phi(t),$$

where ${}^cD^\alpha$ is Caputo fractional derivatives, $f, g : [0, b] \times C([-r, b], \mathbb{R}) \to \mathbb{R}$ is a given function and $\phi \in C([-r, b], \mathbb{R}) \to \mathbb{R}$.

For any continuous function y defined on $[-r, b]$ and any $t \in J$, we denote y_t the element of$C([-r, b], \mathbb{R})$ defined by

$$y_t(\theta) = y_t(t + \theta), \quad t \in [-r, b].$$

Hence $y_t(.)$ represents the history of the state from time $t - r$ up to the present time t.

Motivated by the work above, in this paper, we consider of the following initial value problem:

$$(3) \qquad {}^c_HD^\alpha y(t) = f(t, y_t) + g(t, y_t), \quad \text{for a.e. } t \in J = [1, T],\ 0 < \alpha \leq 1,$$

$$(4) \qquad y(t) = \phi(t),$$

where ${}^c_HD^\alpha$ is Caputo-Hadamard fractional derivatives, $f, g : [1, T] \times C([-r, T], \mathbb{R}) \to \mathbb{R}$ are a given functions and $\phi \in C([-r, T], \mathbb{R}) \to \mathbb{R}$.

For any continuous function y defined on $[-r, T]$ and any $t \in J$, we denote y_t the element of $C([-r, T], \mathbb{R})$ defined by

$$y_t(\theta) = y_t(t + \theta), \quad t \in [-r, T].$$

Hence $y_t(.)$ represents the history of the state from time $t - r$ up to the present time t. In this paper, we shall prove the existence of solutions, as well as, the existence of extremal solutions of (3)-(4).

For the existence of solutions, our approach is based on a new fixed point theorem of Burton and Kirk, and for the existence of extremal solutions it is based on the concept of upper and lower solutions combined with a fixed point theorem in ordered Banach space established by Dhage and Henderson [19]. These results can be considered as a contribution to this emerging field.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts that will be used further in the paper.

Let $C([1, T], \mathbb{R})$ be the Banach space of all continuous functions from $[1, T]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 1 \leq t \leq T\},$$

and $L^1(J,E)$ as the Banach space of Lebesgue integrable functions $y : J \longrightarrow E$ with the norm

$$\|y\|_{L^1} = \int_J |y(t)|dt,$$

Let the space $C([-r,1],\mathbb{R})$ be endowed with the norm $\|.\|_C$ defined by

$$\|\phi\|_C = \sup\{|\phi(\theta)| : -r \leq \theta \leq 1\},$$

and

$$AC^n_\delta(J,\mathbb{R}) = \{h : J \to \mathbb{R} : \delta^{n-1}h(t) \in AC(J,\mathbb{R})\},$$

where $\delta = t\dfrac{d}{dt}$ is the Hadamard derivative and $AC(J,\mathbb{R})$ is the space of absolutely continuous functions on J.

Let us recall some definitions and properties of Hadamard fractional integration and differentiation.

Definition 1 ([29])**.** The Hadamard fractional integral of order $r > 0$ for a function $h \in L^1([1,+\infty),\mathbb{R})$ is defined as

$$^H I^r h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds,$$

provided the integral exists for a.e. $t > 1$.

Example 1 ([26])**.** Let $q > 0$. Then

$$^H I^q_1 \ln t = \frac{1}{\Gamma(2+q)}(\ln t)^{1+q}; \ \textit{for a.e. } t \in [1,+\infty).$$

Definition 2 ([29])**.** The Hadamard fractional derivative of order $r > 0$ applied to the function $h \in AC^n_\delta([1,+\infty),\mathbb{R})$ is defined as

$$(^H D^q_1 h)(t) = \delta^n(^H I^{n-r}_1 h)(t),$$

where $n-1 < r < n$, $n = [r]+1$, and $[r]$ is the integer part of r.

Definition 3 ([26])**.** For a given function $h \in AC^n_\delta([a,b],\mathbb{R})$, such that $0 < a < b$, the Caputo-Hadamard fractional derivative of order $r > 0$ is defined as follows:

$$^{Hc}D^r y(t) =^H D^r \left[y(s) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{s}{a}\right)^k \right](t),$$

where $Re(\alpha) \geq 0$ and $n = [Re(\alpha)] + 1$.

Lemma 1. *[[26]] Let $y \in AC^n_\delta([a,b],\mathbb{R})$ or $C^n_\delta([a,b],\mathbb{R})$ and $\alpha \in \mathbb{C}$. Then*

$$^H I^r(^{Hc}D^r y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{t}{a}\right)^k.$$

Theorem 1 ([14]). *Let X be a Banach space, and A, B two operators satisfying*

(i) *A is a contraction*
(ii) *B is completely continuous*

Then either

(a) *The operator equation $y = A(y) + B(y)$ has a solution, or*
(b) *The set $\mathcal{E} = \{u \in X, \lambda A\left(\frac{u}{\lambda}\right) + \lambda B(u)\}$ is unbounded for $\lambda \in (0,1)$*

Definition 4. A function $y \in AC^1_\delta([-r,1],\mathbb{R})$ is said to be a solution of (1)-(2) if y satisfies the equation ${}^C_H D^\alpha y(t) = f(t, y_t) + g(t, y_t)$ on J, and the condition $y(t) = \phi(t)$ on $[-r, 1]$.

For the existence of solutions for the problem (3)-(4), we need the following auxiliary lemma.

Lemma 2. *Let h belong to $AC^1_\delta([1,T],\mathbb{R})$. For $r \in (0,1]$, a function y is a solution of the fractional integral equation*

$$y(t) = y_1 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} h(s) \frac{d}{ds}, \tag{5}$$

if and only if y is a solution of the fractional IVP

$${}^C_H D^\alpha y(t) = h(t), \qquad 0 < \alpha < 1, \tag{6}$$

$$y(1) = y_1. \tag{7}$$

Proof. Applying the Hadamard fractional integral of order α to both sides of (6), and by using Lemma 1, we find

$$y(t) = c_1 + {}^H I^r h(t). \tag{8}$$

Then by using the conditions (7), we get

$$c_1 = y_1,$$

Finally, we obtain the solution (5). Conversely, it is clear that if y satisfies equation (5), then equations (6) and (7) hold. □

Theorem 2. *Assume the following hypotheses hold:*

(H1) *The function $f : [1,T] \times C([-r,T],\mathbb{R}) \to \mathbb{R}$ is continuous*
(H2) *There exist $\phi_f \in L^1(J,\mathbb{R}_+)$, and $\psi : [0,\infty) \longrightarrow (0,\infty)$ continuous and nondecreasing such that*

$$|f(t,u)| \le \phi_f(t)\psi(|u|) \quad \textit{for a.e. } t \in J \textit{ and each } u \in \mathbb{R}.$$

(H3) *There exist a constant $k > 0$ such that*

$$|g(t,x) - g(t,y)| \le k\|x-y\|_C, \quad \textit{for a.e. } t \in J \textit{ and each } x, y \in C([-r,T],\mathbb{R}).$$

(H4) *there exists an number $M > 0$ such that*

$$\frac{M}{\|\phi\|_C + \frac{\|\phi_f\|_{L^1}(\log T)^\alpha \psi(M)}{\Gamma(\alpha+1)} + \frac{kM(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{g^* k(\log T)^\alpha}{\Gamma(\alpha+1)}} < 1 \tag{9}$$

where

$$g^* = \sup_{s\in J} \|g(s,0)\|.$$

If

$$\frac{k(\log T)^\alpha}{\Gamma(\alpha+1)} < 1, \tag{10}$$

then the IVP (1)*eq*-(2) *has at least one solution on* $[-r, T]$.

Proof. Consider the operators:

$$\tilde{F}, G : AC^1_\delta([-r,1],\mathbb{R}) \to AC^1_\delta([-r,1],\mathbb{R})$$

defined by

$$\tilde{F}(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,1], \\ \phi(1) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, y_s) \frac{d}{ds}, & \text{if } t \in [1,T], \end{cases}$$

and

$$G(y)(t) = \begin{cases} 0, & \text{if } t \in [-r,1], \\ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s, y_s) \frac{d}{ds}, & \text{if } t \in [0,b]. \end{cases}$$

Then the problem of finding the solutions of the IVP (3)-(4) is reduced to finding the solutions of the operator equation $\tilde{F}(y)(t) + G(y)(t) = y(t), t \in J$. We shall show that the operators $\tilde{F}$ and G satisfy all the conditions of Theorem 1.

Step 1: $\tilde{F}$ *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y \in AC^1_\delta([-r,T],\mathbb{R})$. Then, for each $t \in J$,

$$\begin{aligned} |(\tilde{F}y_n)(t) - (\tilde{F}y)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, y_{ns})) - f(s, y(s))| \frac{d}{ds} \\ &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \sup_{s\in[1,T]} |f(s, y_{ns})) - f(s, y(s))| \frac{d}{ds} \\ &\le \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \|f(., y_n(.)) - f(., y(.))\|_\infty. \end{aligned}$$

Since f is continuous, we have $\|\tilde{F}(y_n) - \tilde{F}(y)\|_\infty \to 0$ as $n \to \infty$.

Step 2: *$\tilde{F}$ maps bounded sets into bonded sets in $AC^1_\delta([-r,1],\mathbb{R})$.*

Indeed, it is enough to show that for any $\mu^* > 0$, there exists a positive constant ℓ such that for each $y \in B_{\mu^*} = \{y \in AC^1_\delta([-r,T],\mathbb{R}) : \|y\|_\infty \leq \mu^*\}$, we have for each $t \in [1,T]$

$$\begin{aligned}
|\tilde{F}(y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s,y_s)| \frac{d}{ds} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_f(t)\psi(|u|) \frac{d}{ds} \\
&\leq \frac{\|\phi_f\|_{L^1} (\log T)^\alpha \psi(\mu^*)}{\Gamma(\alpha+1)},
\end{aligned}$$

Thus

$$\|\tilde{F}\|(y)\|_\infty \leq \frac{\|\phi_f\|_{L^1} (\log T)^\alpha \psi(\mu^*)}{\Gamma(\alpha+1)} = \ell.$$

Step 3: *$\tilde{F}$ maps bounded sets into equicontinuous sets of $AC^1_\delta([-r,T],\mathbb{R})$.*

Let $t_1, t_2 \in J$, $t_1 < t_2$, and let B_{μ^*} be bounded set of $C(J,\mathbb{R})$ as in Step 2. Let $y \in B_{\mu^*}$ and $h \in (\tilde{F}y)$. Then

$$\begin{aligned}
|\tilde{F}(t_2) - \tilde{F}(t_1)| = &|\frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1}\right] f(s,y_s) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} f(s,y_s) \frac{ds}{s}| \\
\leq &\frac{\phi_f(s)\psi(|y(s)|)}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1}\right] \frac{ds}{s} \\
&+ \frac{\phi_f(s)\psi(|y(s)|)}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s}.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that $\tilde{F} : AC^1_\delta([-r,1],\mathbb{R}) \to AC^1_\delta([-r,1],\mathbb{R})$ is completely continuous.

Step 4: *G is a contraction*

Let $x, y \in AC^1_\delta([-r,T],\mathbb{R})$, then for each $t \in J$ we have

$$\begin{aligned}
|(Gx)(t) - (Gy)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |g(s,x_s) - g(s,y_s)| \frac{d}{ds} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} k\|x-y\|_\infty \frac{d}{ds} \\
&\leq \frac{k(\log T)^\alpha}{\Gamma(\alpha+1)} \|x-y\|_\infty.
\end{aligned}$$

Thus

$$\|G(x)-G(y)\|_\infty \le \|x-y\|_\infty.$$

and consequently G is a contraction, since $8\,\dfrac{k(\log T)^\alpha}{\Gamma(\alpha+1)}<1$.

Step 5: *A priori bounds*

Now it remains to show that the set
$\mathcal{E} = \{y \in AC^1_\delta([-r,T],\mathbb{R}), \lambda\tilde{F}(y)+\lambda G\left(\frac{y}{\lambda}\right)$ for some , $,0<\lambda<1\}$ is bounded.

Let $y\in\mathcal{E}$, then $y=\lambda\tilde{F}(y)+\lambda G\left(\frac{y}{\lambda}\right)$ for some $0<\lambda<1$. Thus, for each $t\in J$ we have

$$\begin{aligned} y(t) = \lambda\Bigg[\phi(0) &+ \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} f(s,y_s)\frac{d}{ds}\\ &+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} g\left(s,\frac{y_s}{\lambda}\right)ds\Bigg].\end{aligned}$$

This implies by (H2) and (H3) that for each $t\in J$, we have

$$\begin{aligned} \|y(t)\| &\le \|\phi(0)\| + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} |f(s,y_s)|\frac{d}{ds}\\ &\quad+\frac{\lambda}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left\|g\left(s,\frac{y_s}{\lambda}\right)-g(s,0)\right\|\frac{d}{ds}\\ &\quad+\frac{\lambda}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\|g(s,0)\|\frac{d}{ds}\\ &\le \|\phi\|_C+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_f(t)\psi(|u|)\frac{d}{ds}\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}k\|y_s\|_C\frac{d}{ds}+\frac{g^*k(\log T)^\alpha}{\Gamma(\alpha+1)}\\ &\le \|\phi\|_C+\frac{\|\phi_f\|_{L^1}(\log T)^\alpha\psi(\|y\|_C)}{\Gamma(\alpha+1)}\\ &\quad+\frac{k\|y_s\|_C(\log T)^\alpha}{\Gamma(\alpha+1)}+\frac{g^*k(\log T)^\alpha}{\Gamma(\alpha+1)},\end{aligned}$$

where

$$g^*=\sup_{s\in J}\|g(s,0)\|.$$

Thus

$$\frac{\|y\|_C}{\|\phi\|_C+\dfrac{\|\phi_f\|_{L^1}(\log T)^\alpha\psi(\|y\|_C)}{\Gamma(\alpha+1)}+\dfrac{k\|y_s\|_C(\log T)^\alpha}{\Gamma(\alpha+1)}+\dfrac{g^*k(\log T)^\alpha}{\Gamma(\alpha+1)}}<1,$$

then by condition (H5), there exists M such that $\|y\|_C \leq M$. This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 1, we deduce that $\tilde{F}(y)+G(y)$ has a fixed point which is a solution of the problem (3)-(4). $\square$

2.1. **Existence of extremal solutions.** In this section we shall prove the existence of minimal and maximal solutions for the IVP (3)-(4) under suitable monotonicity conditions on the functions involved in it.

Definition 5. A nonempty closed subset C of a Banach space X is said to be a cone if

(i) $C + C \subset C$,
(ii) $\lambda C \subset C$, and
(iii) $\{-C\} \cap \{C\} = \{0\}$.

A cone C is called normal if the norm $\|\cdot\|$ is semi-monotone on C, i.e., there exists a constant $N > 0$ such that $\|x\| \leq N\|y\|$, whenever $x \leq y$. We equip the space $X = C(J, E)$ with the order relation $\leq$ induced by a regular cone in E, that is for all $y, \overline{y} \in X : y \leq \overline{y}$ if and only if $\overline{y}(t) - y(t) \geq 0,\ \forall t \in J$. Cones and their properties are detailed in [27]. Let $a, b \in X$ be such that $a \leq b$. Then, by an order interval $[a, b]$ we mean a set of points in X given by

$$[a, b] = \{x \in X \mid a \leq x \leq b\}.$$

Definition 6. Let X be an ordered Banach space. A mapping $T : X \to X$ is called isotone increasing if $T(x) \leq T(y)$ for any $x, y \in X$ with $x < y$. Similarly, T is called isotone decreasing if $T(x) \geq T(y)$ whenever $x < y$.

Definition 7 ([27])**.** We say that $x \in X$ is the least fixed point of G in X if $x = Gx$ and $x \leq y$ whenever $y \in X$ and $y = Gy$. The greatest fixed point of G in X is defined similarly by reversing the inequality. If both least and greatest fixed points of G in X exist, we call them extremal fixed point of G in X.

We need the following fixed point theorem in the sequel.

Theorem 3 ([19])**.** *Let $[a, b]$ be an order interval in a Banach space and let $B_1, B_2 : [a, b] \to X$ be two functions satisfying:*

(a) *B_1 is a contraction,*
(b) *B_2 is completely continuous,*
(c) *B_1 and B_2 are strictly monotone increasing, and*
(d) *$B_1(x) + B_2(x) \in [a, b],\ \forall\, x \in [a, b]$.*

Further if the cone K in X is normal, then the equation $x = B_1(x) + B_2(x)$ has a least fixed point x_* and a greatest fixed point $x^* \in [a, b]$. Moreover $x_* = \lim\limits_{n\to\infty} x_n$ and $x^* = \lim\limits_{n\to\infty} y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[a, b]$ defined by

$$x_{n+1} = B_1(x_n) + B_2(x_n),\ x_0 = a \ \text{ and } \ y_{n+1} = B_1(y_n) + B_2(y_n),\ y_0 = b.$$

We adopt the following definitions.

Definition 8. A function $AC^1_\delta([-r,T],\mathbb{R})$ is called a lower solution of the IVP (3)-(4) if ${}^C_H D^\alpha v(t) \leq f(t,v_t)+g(t,v_t)$, for each $t\in J$ and $v(t)\leq \phi(t)$ if $t\in[-r,1]$.

Similarly, an upper solution w of IVP (3)-(4) is defined by reversing the order of the above inequalities.

Definition 9. A solution x_M of the IVP (3)-(4) is said to be maximal if for any other solution x of the IVP (3)-(4) on $[-r,T]$ we have $x(t)\leq x_M(t)$ for each $t\in[-r,T]$.

Similarly a minimal solution of IVP (3)-(4) is defined by reversing the order of the inequalities.

Definition 10. A function $f(t,x)$ is called strictly monotone increasing in x almost everywhere for $t\in J$, if $f(t,x)\leq f(t,y)$ for each $t\in J$ and all $x,y\in E$ with $x<y$. Similarly $f(t,x)$ is called strictly monotone decreasing in x almost everywhere for $t\in J$, if $f(t,x)\geq f(t,y)$ a.e. $t\in J$ for all $x,y\in E$ with $x<y$.

We need the following assumptions in the sequel.

- (B4) The functions $f(t,y)$ and $g(t,y)$ are strictly monotone nondecreasing in y for each $t\in J$.
- (B5) The IVP (3)-(4) has a lower solution v and an upper solution w with $v\leq w$.

Theorem 4. *Assume that assumptions (B1)-(B5) hold. Then the IVP* (3)-(4) *has minimal and maximal solutions on* $[-r,T]$.

Proof. It can be shown, as in the proof of Theorem 2 that $\tilde F$ is completely continuous and G is a contraction on $[v,w]$. We shall show that $\tilde F$ and G are isotone increasing on $[v,w]$. Let $y,\overline{y}\in[v,w]$ be such that $y\leq\overline{y}$, $y\neq\overline{y}$. Then by (B 4), we have for each $t\in J$

$$\begin{aligned}\tilde F(y)(t) &= \phi(1)+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} f(s,y_s)\frac{d}{ds}\\ &\leq \phi(1)+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} f(s,\overline{y}_s)ds\\ &= \tilde F(\overline{y})(t).\end{aligned}$$

Similarly, $G(y)\leq G(\overline{y})$. Therefore $\tilde F$ and G are isotone increasing on $[v,w]$. Finally, let $x\in[v,w]$ be any element. By (B5), we deduce that

$$v\leq \tilde F(v)+G(v)\leq \tilde F(x)+G(x)\leq \tilde F(w)+G(w)\leq w,$$

which shows that $\tilde F(x)+G(x)\in[v,w]$ for all $x\in[v,w]$. Thus, the functions $\tilde F$ and G satisfy all the conditions of Theorem 3, hence the IVP eq3-(4) has minimal and maximal solutions on $[-r,T]$.

This completes the proof. $\square$

3. Conclusions

In this paper, we have presented the existence solutions for an initial-value problem (IVP for short)of perturbed functional fractional differential equations with Caputo-Hadamard derivative. The result is obtained by applying the fixed point theorem of Burton and Kirk. Further, we have investigated the existence of extremal solutions by using the concept of upper and lower solutions combined with a fixed point theorem in ordered Banach space established by Dhage and Henderson.

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