

Operator upper bounds for Davis-Choi-Jensen's difference in Hilbert spaces

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ABSTRACT. In this paper we obtain several operator inequalities providing upper bounds for the Davis-Choi-Jensen's Difference

$$\Phi(f(A)) - f(\Phi(A))$$

for any convex function $f : I \rightarrow \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\text{Sp}(A) \subset I$ and any linear, positive and normalized map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, where H and K are Hilbert spaces. Some examples of convex and operator convex functions are also provided.

1. INTRODUCTION

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H . We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H .

Let H, K be complex Hilbert spaces. Following [1] (see also [9, p. 18]) we can introduce the following definition.

Definition 1. A map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely

$$\Phi(\lambda A + \mu B) = \lambda\Phi(A) + \mu\Phi(B)$$

for any $\lambda, \mu \in \mathbb{C}$ and $A, B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is normalized if it preserves the identity operator, i.e., $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

2020 *Mathematics Subject Classification.* Primary: 47A63; Secondary: 47A99.

Key words and phrases. Selfadjoint bounded linear operators, Functions of operators, Operator convex functions, Jensen's operator inequality, Linear, positive and normalized map.

Full paper. Received 29 May 2023, accepted 7 March 2024, available online 19 March 2024.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$.

If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \leq A \leq \beta 1_H$, then $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by putting $\Phi = \Psi^{-1/2}(1_H) \Psi \Psi^{-1/2}(1_H)$ we get $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalized.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda)f(A) + \lambda f(B)$$

for all $\lambda \in [0, 1]$ and for every selfadjoint operators $A, B \in \mathcal{B}(H)$ whose spectra are contained in I .

The following Jensen's type result is well known [9, p. 22]:

Theorem 1 (Davis-Choi-Jensen's Inequality). *Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$(1) \quad f(\Phi(A)) \leq \Phi(f(A)).$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H) \Psi \Psi^{-1/2}(1_H)$ in (1) we get

$$f\left(\Psi^{-1/2}(1_H) \Psi(A) \Psi^{-1/2}(1_H)\right) \leq \Psi^{-1/2}(1_H) \Psi(f(A)) \Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps*

$$(2) \quad \Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H) \Psi(A) \Psi^{-1/2}(1_H)\right) \Psi^{1/2}(1_H) \leq \Psi(f(A)).$$

Let $C_j \in \mathcal{B}(H)$, $j = 1, \dots, k$ be contractions with

$$(3) \quad \sum_{j=1}^k C_j^* C_j = 1_H.$$

The map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ defined by [9, p. 19]

$$\Phi(A) := \sum_{j=1}^k C_j^* A C_j$$

is a normalized positive linear map on $\mathcal{B}(H)$.

For more results on inequalities for selfadjoint operators in Hilbert spaces, see [2, 3, 6–8] and the references therein.

In this paper we obtain several operator inequalities providing upper bounds for the Davis-Choi-Jensen's Difference

$$\Phi(f(A)) - f(\Phi(A))$$

for any convex function $f : I \rightarrow \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\text{Sp}(A) \subset I$ and any linear, positive and normalized map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, where H and K are Hilbert spaces. Some examples of convex and operator convex functions are also provided.

2. MAIN RESULTS

We use the following result that was obtained in [4].

Lemma 1. *If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function on $[a, b]$, then*

$$(4) \quad 0 \leq \frac{(b-t)f(a) + (t-a)f(b)}{b-a} - f(t) \\ \leq (b-t)(t-a) \frac{f'_-(b) - f'_+(a)}{b-a} \leq \frac{1}{4} (b-a) [f'_-(b) - f'_+(a)]$$

for any $t \in [a, b]$.

If the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $1/4$ are sharp.

We have:

Theorem 2. *Let $f : [m, M] \rightarrow \mathbb{R}$ be a convex function on $[m, M]$ and A a selfadjoint operator with the spectrum $\text{Sp}(A) \subset [m, M]$.*

If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$(5) \quad \Phi(f(A)) - f(\Phi(A)) \\ \leq \frac{f'_-(M) - f'_+(m)}{M-m} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\ \leq \frac{1}{4} (M-m) [f'_-(M) - f'_+(m)] 1_K.$$

Proof. Utilizing the continuous functional calculus for a selfadjoint operator T with $0 \leq T \leq 1_H$ and the convexity of f on $[m, M]$, we have

$$(6) \quad f(m(1_H - T) + MT) \leq f(m)(1_H - T) + f(M)T$$

in the operator order.

If we take in (6)

$$0 \leq T = \frac{A - m1_H}{M - m} \leq 1_H,$$

then we get

$$(7) \quad f\left(m\left(1_H - \frac{A - m1_H}{M - m}\right) + M\frac{A - m1_H}{M - m}\right) \\ \leq f(m)\left(1_H - \frac{A - m1_H}{M - m}\right) + f(M)\frac{A - m1_H}{M - m}.$$

Observe that

$$\begin{aligned} & m \left(1_H - \frac{A - m1_H}{M - m} \right) + M \frac{A - m1_H}{M - m} \\ &= \frac{m(M1_H - A) + M(A - m1_H)}{M - m} = A \end{aligned}$$

and

$$\begin{aligned} & f(m) \left(1_H - \frac{A - m1_H}{M - m} \right) + f(M) \frac{A - m1_H}{M - m} \\ &= \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \end{aligned}$$

and by (7) we get the following inequality of interest

$$(8) \quad f(A) \leq \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}.$$

If we take the map Φ in (8), then we get

$$\begin{aligned} \Phi(f(A)) &\leq \Phi \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] \\ &= \frac{f(m)\Phi(M1_H - A) + f(M)\Phi(A - m1_H)}{M - m} \\ &= \frac{f(m)(M\Phi(1_H) - \Phi(A)) + f(M)(\Phi(A) - m\Phi(1_H))}{M - m} \\ &= \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m}, \end{aligned}$$

which implies that

$$(9) \quad \begin{aligned} & \Phi(f(A)) - f(\Phi(A)) \\ &\leq \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m} - f(\Phi(A)). \end{aligned}$$

Since $m1_K \leq \Phi(A) \leq M1_K$, then by using (4) for $a = m$, $b = M$ and the continuous functional calculus, we have

$$(10) \quad \begin{aligned} & \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m} - f(\Phi(A)) \\ &\leq \frac{f'_-(M) - f'_+(m)}{M - m} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\ &\leq \frac{1}{4} (M - m) [f'_-(M) - f'_+(m)] 1_K. \end{aligned}$$

By making use of (9) and (10) we get the desired result (5). \square

Corollary 1. Let $f : [m, M] \rightarrow \mathbb{R}$ be an operator convex function on $[m, M]$ and A a selfadjoint operator with the spectrum $\text{Sp}(A) \subset [m, M]$.

If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$(11) \quad \begin{aligned} 0 &\leq \Phi(f(A)) - f(\Phi(A)) \\ &\leq \frac{f'_-(M) - f'_+(m)}{M - m} (M1_K - \Phi(A))(\Phi(A) - m1_K) \\ &\leq \frac{1}{4} (M - m) [f'_-(M) - f'_+(m)] 1_K. \end{aligned}$$

We also have the following scalar inequality of interest:

Lemma 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$ and $t \in [0, 1]$, then

$$(12) \quad \begin{aligned} 2 \min\{t, 1-t\} &\left[\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \right] \\ &\leq (1-t)f(a) + tf(b) - f((1-t)a + tb) \\ &\leq 2 \max\{t, 1-t\} \left[\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \right]. \end{aligned}$$

The proof follows, for instance, by Corollary 1 from [5] for $n = 2$, $p_1 = 1 - t$, $p_2 = t$, $t \in [0, 1]$ and $x_1 = a$, $x_2 = b$.

Theorem 3. Let $f : [m, M] \rightarrow \mathbb{R}$ be a convex function on $[m, M]$ and A a selfadjoint operator with the spectrum $\text{Sp}(A) \subset [m, M]$.

If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$(13) \quad \begin{aligned} 2 &\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\ &\times \left(\frac{1}{2} (M - m) 1_K - \left| \Phi(A) - \frac{1}{2} (m + M) 1_K \right| \right) \\ &\leq \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m} - f(\Phi(A)) \\ &\leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\ &\times \left(\frac{1}{2} (M - m) 1_K + \left| \Phi(A) - \frac{1}{2} (m + M) 1_K \right| \right) \end{aligned}$$

and

$$(14) \quad \begin{aligned} 2 &\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\ &\times \left(\frac{1}{2} (M - m) 1_K - \Phi \left(\left| A - \frac{1}{2} (m + M) 1_K \right| \right) \right) \\ &\leq \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m} - \Phi(f(A)) \end{aligned}$$

$$\begin{aligned} &\leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\ &\quad \times \left(\frac{1}{2}(M-m)1_K + \Phi \left(\left| A - \frac{1}{2}(m+M)1_H \right| \right) \right). \end{aligned}$$

Proof. We have from (12) that

$$\begin{aligned} (15) \quad &2 \left(\frac{1}{2} - \left| t - \frac{1}{2} \right| \right) \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\ &\leq (1-t)f(m) + tf(M) - f((1-t)m + tM) \\ &\leq 2 \left(\frac{1}{2} + \left| t - \frac{1}{2} \right| \right) \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right], \end{aligned}$$

for all $t \in [0, 1]$.

Utilizing the continuous functional calculus for a selfadjoint operator T with $0 \leq T \leq 1_H$ we get from (15) that

$$\begin{aligned} (16) \quad &2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \left(\frac{1}{2}1_H - \left| T - \frac{1}{2}1_H \right| \right) \\ &\leq (1-T)f(m) + Tf(M) - f((1-T)m + TM) \\ &\leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \left(\frac{1}{2}1_H + \left| T - \frac{1}{2}1_H \right| \right), \end{aligned}$$

in the operator order.

If we take in (16)

$$0 \leq T = \frac{A - m1_H}{M - m} \leq 1_H,$$

then, like in the proof of Theorem 2, we get

$$\begin{aligned} (17) \quad &2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\ &\quad \times \left(\frac{1}{2}(M-m)1_H - \left| A - \frac{1}{2}(m+M)1_H \right| \right) \\ &\leq \frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} - f(A) \\ &\leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\ &\quad \times \left(\frac{1}{2}(M-m)1_H + \left| A - \frac{1}{2}(m+M)1_H \right| \right). \end{aligned}$$

Since $m1_K \leq \Phi(A) \leq M1_K$, then by writing the inequality (17) for $\Phi(A)$ instead of A we get (13).

If we take Φ in (17), then we get

$$\begin{aligned}
& 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\
& \quad \times \Phi \left(\frac{1}{2} (M-m) 1_H - \left| A - \frac{1}{2} (m+M) 1_H \right| \right) \\
& \leq \Phi \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} \right] - \Phi f(A) \\
& \leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\
& \quad \times \Phi \left(\frac{1}{2} (M-m) 1_H + \left| A - \frac{1}{2} (m+M) 1_H \right| \right),
\end{aligned}$$

which is equivalent to (14). \square

Corollary 2. *Let $f : [m, M] \rightarrow \mathbb{R}$ be an operator convex function on $[m, M]$ and A a selfadjoint operator with the spectrum $\text{Sp}(A) \subset [m, M]$.*

If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$\begin{aligned}
(18) \quad & 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\
& \quad \times \left(\frac{1}{2} (M-m) 1_K - \Phi \left(\left| A - \frac{1}{2} (m+M) 1_H \right| \right) \right) \\
& \leq \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M-m} - \Phi(f(A)) \\
& \leq \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M-m} - f(\Phi(A)) \\
& \leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\
& \quad \times \left(\frac{1}{2} (M-m) 1_K + \left| \Phi(A) - \frac{1}{2} (m+M) 1_K \right| \right).
\end{aligned}$$

We also have:

Corollary 3. *Let $f : [m, M] \rightarrow \mathbb{R}$ be a convex function on $[m, M]$ and A a selfadjoint operator with the spectrum $\text{Sp}(A) \subset [m, M]$.*

If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$\begin{aligned}
(19) \quad & \Phi(f(A)) - f(\Phi(A)) \leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] \\
& \quad \times \left(\frac{1}{2} (M-m) 1_K + \left| \Phi(A) - \frac{1}{2} (m+M) 1_K \right| \right) \\
& \leq 2(M-m) \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right) \right] 1_K.
\end{aligned}$$

Proof. From (9) we have

$$\begin{aligned} & \Phi(f(A)) - f(\Phi(A)) \\ & \leq \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m} - f(\Phi(A)) \end{aligned}$$

and from (14) we have

$$\begin{aligned} & \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m} - f(\Phi(A)) \\ & \leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m + M}{2}\right) \right] \\ & \quad \times \left(\frac{1}{2}(M - m)1_K + \left| \Phi(A) - \frac{1}{2}(m + M)1_K \right| \right), \end{aligned}$$

which produce the desired result (19). \square

Remark 1. If $f : [m, M] \rightarrow \mathbb{R}$ is an operator convex function on $[m, M]$, A a selfadjoint operator with the spectrum $\text{Sp}(A) \subset [m, M]$ and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$\begin{aligned} (20) \quad & 0 \leq \Phi(f(A)) - f(\Phi(A)) \\ & \leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m + M}{2}\right) \right] \\ & \quad \times \left(\frac{1}{2}(M - m)1_K + \left| \Phi(A) - \frac{1}{2}(m + M)1_K \right| \right) \\ & \leq 2(M - m) \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m + M}{2}\right) \right] 1_K. \end{aligned}$$

We also have [4]:

Lemma 3. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$. If f' is K -Lipschitzian on $[a, b]$, then

$$\begin{aligned} (21) \quad & |(1 - t)f(a) + tf(b) - f((1 - t)a + tb)| \\ & \leq \frac{1}{2}K(b - t)(t - a) \leq \frac{1}{8}K(b - a)^2 \end{aligned}$$

for all $t \in [0, 1]$.

The constants $1/2$ and $1/8$ are the best possible in (21).

Remark 2. If $f : [a, b] \rightarrow \mathbb{R}$ is twice differentiable and $f'' \in L_\infty[a, b]$, then

$$\begin{aligned} (22) \quad & |(1 - t)f(a) + tf(b) - f((1 - t)a + tb)| \\ & \leq \frac{1}{2} \|f''\|_{[a, b], \infty} (b - t)(t - a) \leq \frac{1}{8} \|f''\|_{[a, b], \infty} (b - a)^2, \end{aligned}$$

where $\|f''\|_{[a, b], \infty} := \text{esssup}_{t \in [a, b]} |f''(t)| < \infty$. The constants $1/2$ and $1/8$ are the best possible in (22).

We have:

Theorem 4. *Let $f : [m, M] \rightarrow \mathbb{R}$ be a twice differentiable convex function on $[m, M]$ with $\|f''\|_{[m, M], \infty} := \operatorname{ess\,sup}_{t \in [m, M]} f''(t) < \infty$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A) \subset [m, M]$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then*

$$(23) \quad \begin{aligned} & \Phi(f(A)) - f(\Phi(A)) \\ & \leq \frac{1}{2} \|f''\|_{[m, M], \infty} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\ & \leq \frac{1}{8} \|f''\|_{[m, M], \infty} (M - m)^2 1_K. \end{aligned}$$

Proof. From (22) and the continuous functional calculus, we get

$$(24) \quad \begin{aligned} 0 & \leq \frac{f(m)(M1_H - B) + f(M)(B - m1_H)}{M - m} - f(B) \\ & \leq \frac{1}{2} \|f''\|_{[m, M], \infty} (M1_H - B)(B - m1_H) \\ & \leq \frac{1}{8} \|f''\|_{[m, M], \infty} (M - m)^2 1_H \end{aligned}$$

where B is a selfadjoint operator with the spectrum $\operatorname{Sp}(B) \subset [m, M]$.

If we use (24) for $\Phi(A)$ we get

$$(25) \quad \begin{aligned} 0 & \leq \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m} - f(\Phi(A)) \\ & \leq \frac{1}{2} \|f''\|_{[m, M], \infty} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\ & \leq \frac{1}{8} \|f''\|_{[m, M], \infty} (M - m)^2 1_K. \end{aligned}$$

Since

$$\begin{aligned} & \Phi(f(A)) - f(\Phi(A)) \\ & \leq \frac{f(m)(M1_K - \Phi(A)) + f(M)(\Phi(A) - m1_K)}{M - m} - f(\Phi(A)), \end{aligned}$$

hence by (25) we get (23). \square

Corollary 4. *Let $f : [m, M] \rightarrow \mathbb{R}$ be an operator convex function on $[m, M]$ and A a selfadjoint operator with the spectrum $\operatorname{Sp}(A) \subset [m, M]$.*

If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$(26) \quad \begin{aligned} 0 & \leq \Phi(f(A)) - f(\Phi(A)) \\ & \leq \frac{1}{2} \|f''\|_{[m, M], \infty} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\ & \leq \frac{1}{8} \|f''\|_{[m, M], \infty} (M - m)^2 1_K. \end{aligned}$$

3. SOME EXAMPLES

We consider the exponential function $f(x) = \exp(\alpha x)$ with $\alpha \in \mathbb{R} \setminus \{0\}$. This function is convex but not operator convex on \mathbb{R} . If A is selfadjoint with $\text{Sp}(A) \subset [m, M]$ for some $m < M$ and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by (5), (19) and (23) we have

$$\begin{aligned}
 (27) \quad & \Phi(\exp(\alpha A)) - \exp(\alpha \Phi(A)) \\
 & \leq \alpha \frac{\exp(\alpha M) - \exp(\alpha m)}{M - m} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\
 & \leq \frac{1}{4} \alpha (M - m) [\exp(\alpha M) - \exp(\alpha m)] 1_K,
 \end{aligned}$$

$$\begin{aligned}
 (28) \quad & \Phi(\exp(\alpha A)) - \exp(\alpha \Phi(A)) \\
 & \leq 2 \left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha \frac{m + M}{2}\right) \right] \\
 & \quad \times \left(\frac{1}{2} (M - m) 1_K + \left| \Phi(A) - \frac{1}{2} (m + M) 1_K \right| \right) \\
 & \leq 2(M - m) \left[\frac{\exp(\alpha m) + f(\alpha M)}{2} - \exp\left(\alpha \frac{m + M}{2}\right) \right] 1_K
 \end{aligned}$$

and

$$\begin{aligned}
 (29) \quad & \Phi(\exp(\alpha A)) - \exp(\alpha \Phi(A)) \\
 & \leq \frac{1}{2} \alpha^2 \begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \times (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\
 & \leq \frac{1}{8} \alpha^2 (M - m)^2 \begin{cases} \exp(\alpha M) & \text{if } \alpha > 0 \\ \exp(\alpha m) & \text{if } \alpha < 0 \end{cases} \times 1_K.
 \end{aligned}$$

The function $f(x) = -\ln x$, $x > 0$ is operator convex on $(0, \infty)$. If A is selfadjoint with $\text{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by (11), (20) and (26) we have

$$\begin{aligned}
 (30) \quad & 0 \leq \ln(\Phi(A)) - \Phi(\ln(A)) \\
 & \leq \frac{1}{mM} (M1_V - \Phi(A)) (\Phi(A) - m1_K) \leq \frac{1}{4mM} (M - m)^2 1_K,
 \end{aligned}$$

$$\begin{aligned}
 (31) \quad & 0 \leq \ln(\Phi(A)) - \Phi(\ln(A)) \\
 & \leq 2 \ln\left(\frac{m + M}{2\sqrt{mM}}\right) \left(\frac{1}{2} (M - m) 1_K + \left| \Phi(A) - \frac{1}{2} (m + M) 1_K \right| \right) \\
 & \leq 2(M - m) \ln\left(\frac{m + M}{2\sqrt{mM}}\right) 1_K
 \end{aligned}$$

and

$$\begin{aligned}
 (32) \quad 0 &\leq \ln(\Phi(A)) - \Phi(\ln(A)) \\
 &\leq \frac{1}{2m^2} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\
 &\leq \frac{1}{8m^2} (M - m)^2 1_K.
 \end{aligned}$$

We observe that if $M > 2m$ then the bound in (30) is better than the one from (32). If $M < 2m$, then the conclusion is the other way around.

The function $f(x) = x \ln x$, $x > 0$ is operator convex on $(0, \infty)$. If A is selfadjoint with $\text{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $\Phi \in \mathfrak{F}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by (11), (20) and (26) we have

$$\begin{aligned}
 (33) \quad 0 &\leq \Phi(A \ln(A)) - \Phi(A) \ln(\Phi(A)) \\
 &\leq \frac{\ln(M) - \ln(m)}{M - m} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\
 &\leq \frac{1}{4} (M - m) [\ln(M) - \ln(m)] 1_K,
 \end{aligned}$$

(34)

$$\begin{aligned}
 0 &\leq \Phi(A \ln(A)) - \Phi(A) \ln(\Phi(A)) \\
 &\leq 2 \left[\frac{m \ln(m) + M \ln(M)}{2} - \left(\frac{m + M}{2} \right) \ln \left(\frac{m + M}{2} \right) \right] \\
 &\quad \times \left(\frac{1}{2} (M - m) 1_K + \left| \Phi(A) - \frac{1}{2} (m + M) 1_K \right| \right) \\
 &\leq 2 (M - m) \left[\frac{m \ln(m) + M \ln(M)}{2} - \left(\frac{m + M}{2} \right) \ln \left(\frac{m + M}{2} \right) \right] 1_K
 \end{aligned}$$

and

$$\begin{aligned}
 (35) \quad 0 &\leq \Phi(A \ln(A)) - \Phi(A) \ln(\Phi(A)) \\
 &\leq \frac{1}{2m} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \leq \frac{1}{8m} (M - m)^2 1_K.
 \end{aligned}$$

Consider the power function $f(x) = x^r$, $x \in (0, \infty)$ and r a real number. If $r \in (-\infty, 0] \cup [1, \infty)$, then f is convex and for $r \in [-1, 0] \cup [1, 2]$ is operator convex. If we use the inequalities (5), (19) and (23) we have for $r \in (-\infty, 0] \cup [1, \infty)$ that

$$\begin{aligned}
 (36) \quad &\Phi(A^r) - (\Phi(A))^r \\
 &\leq r \frac{M^{r-1} - m^{r-1}}{M - m} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\
 &\leq \frac{1}{4} r (M - m) (M^{r-1} - m^{r-1}) 1_K,
 \end{aligned}$$

$$\begin{aligned}
(37) \quad & \Phi(A^r) - (\Phi(A))^r \\
& \leq 2 \left[\frac{m^r + M^r}{2} - \left(\frac{m + M}{2} \right)^r \right] \\
& \quad \times \left(\frac{1}{2} (M - m) 1_K + \left| \Phi(A) - \frac{1}{2} (m + M) 1_K \right| \right) \\
& \leq 2(M - m) \left[\frac{m^r + M^r}{2} - \left(\frac{m + M}{2} \right)^r \right] 1_K
\end{aligned}$$

and

$$\begin{aligned}
(38) \quad & \Phi(A^r) - (\Phi(A))^r \\
& \leq \frac{1}{2} r(r-1) \begin{cases} M^{r-2} & \text{for } r \geq 2 \\ m^{r-2} & \text{for } r \in (-\infty, 0] \cup [1, 2] \end{cases} \\
& \quad \times (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\
& \leq \frac{1}{8} r(r-1) (M-m)^2 \begin{cases} M^{r-2} & \text{for } r \geq 2 \\ m^{r-2} & \text{for } r \in (-\infty, 0] \cup [1, 2] \end{cases} \times 1_K,
\end{aligned}$$

where A is selfadjoint with $\text{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

If $r \in [-1, 0] \cup [1, 2]$, then we also have $0 \leq \Phi(A^r) - (\Phi(A))^r$ in the inequalities (36)-(38).

For $r = -1$ we have the inequalities

$$\begin{aligned}
(39) \quad & 0 \leq \Phi(A^{-1}) - (\Phi(A))^{-1} \\
& \leq \frac{M+m}{M^2 m^2} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \\
& \leq \frac{1}{4} (M-m)^2 \frac{M+m}{M^2 m^2} 1_K,
\end{aligned}$$

$$\begin{aligned}
(40) \quad & 0 \leq \Phi(A^{-1}) - (\Phi(A))^{-1} \\
& \leq \frac{(M-m)^2}{mM(m+M)} \left(\frac{1}{2} (M-m) 1_K + \left| \Phi(A) - \frac{1}{2} (m+M) 1_K \right| \right) \\
& \leq \frac{(M-m)^3}{mM(m+M)} 1_K
\end{aligned}$$

and

$$\begin{aligned}
(41) \quad & 0 \leq \Phi(A^{-1}) - (\Phi(A))^{-1} \\
& \leq \frac{1}{m^3} (M1_K - \Phi(A)) (\Phi(A) - m1_K) \leq \frac{1}{4m^3} (M-m)^2 1_K,
\end{aligned}$$

where A is selfadjoint with $\text{Sp}(A) \subset [m, M]$ for some $0 < m < M$ and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

4. CONCLUSION

In this paper we obtained several operator inequalities providing upper bounds for the celebrated Davis-Choi-Jensen's Difference for any convex function $f : I \rightarrow \mathbb{R}$, any selfadjoint operator A in H with the spectrum $\text{Sp}(A) \subset I$ and any linear, positive and normalized map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, where H and K are Hilbert spaces. Some examples for fundamental convex and operator convex functions of interest, to illustrate the main results, were also provided.

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