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Some Bullen-Simpson type inequalities for differentiable s-convex functions

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Abstract. Convexity is one of the fundamental principles of analysis. Over the past few decades, many important inequalities have been established for different classes of convex functions. In this paper, some Bullen-Simpson type integral inequalities for functions whose first derivatives are s-convex in the second sense are established. The cases where the first derivatives are bounded as well as Hölderian are also provided. Some applications to numerical integration and inequalities involving means are given.

1. Introduction

Let I be an interval of real numbers.

A function $\mathcal{B} : I \to \mathbb{R}$ is said to be convex, if for all $\mu_1, \mu_2 \in I$ and all $\mathfrak{h} \in [0, 1]$ (see [35]), we have

$$\mathcal{B}(\mathfrak{h}\mu_1 + (1-\mathfrak{h})\mu_2) \leq \mathfrak{h}\mathcal{B}(\mu_1) + (1-\mathfrak{h})\mathcal{B}(\mu_2).$$

One of the famous inequalities for the class of convex functions is the so-called Hermite-Hadamard inequality (see [35]), which can be stated as follows:

Theorem 1. *Let f be a convex function on the interval $[\mu_1, \mu_2]$ with $\mu_1 < \mu_2$, then we have*

$$\mathcal{B}\left(\tfrac{\mu_1+\mu_2}{2}\right) \leq \tfrac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx \leq \tfrac{\mathcal{B}(\mu_1)+\mathcal{B}(\mu_2)}{2}. \tag{1}$$

Since its discovery, several articles related to inequality (1) have been published [2, 6, 7, 12, 14, 16, 21, 24, 30–34, 37, 41, 42].

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The concept of convexity has been also generalized in diverse manners. One of them is the so-called s-convex function or Breckner convex function defined as follows:

A nonnegative function $\mathcal{B}: I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if

$$\mathcal{B}\left(\mathfrak{h}\mu_1 + (1-\mathfrak{h})\,\mu_2\right) \leq \mathfrak{h}^s \mathcal{B}\left(\mu_1\right) + \left(1-\mathfrak{h}\right)^s \mathcal{B}(\mu_2)$$

holds for all $\mu_1, \mu_2 \in I$ and $\mathfrak{h} \in [0,1]$ (see [8]).

In [11], Dragomir and Fitzpatrick, proved the following variant of inequality (1) which holds for s-convex functions in the second sense.

Theorem 2. *Let $\mathcal{B}: [\mu_1, \mu_2] \to [0,\infty)$ be an integrable and s-convex function in the second sense, where $0 \leq \mu_1 < \mu_2$, then we have*

$$2^{s-1}\mathcal{B}\left(\tfrac{\mu_1+\mu_2}{2}\right) \leq \tfrac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \mathcal{B}\left(x\right)dx \leq \tfrac{\mathcal{B}(\mu_1)+\mathcal{B}(\mu_2)}{s+1}. \tag{2}$$

In [15], Hwang et al. established the following Bullen-type inequality

$$\left|\tfrac{1}{4}\left(\mathcal{B}\left(\mu_1\right) + 2\mathcal{B}\left(\tfrac{\mu_1+\mu_2}{2}\right) + \mathcal{B}\left(\mu_2\right)\right) - \tfrac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \mathcal{B}\left(x\right)dx\right|$$
$$\leq \tfrac{\mu_2-\mu_1}{16}\left(\left|\mathcal{B}'\left(\mu_1\right)\right| + \left|\mathcal{B}'\left(\mu_2\right)\right|\right).$$

In [38], Sarikaya et al. gave the following Simpson-type inequality

$$\left|\tfrac{1}{4}\left(\mathcal{B}\left(\mu_1\right) + 2\mathcal{B}\left(\tfrac{\mu_1+\mu_2}{2}\right) + \mathcal{B}\left(\mu_2\right)\right) - \tfrac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \mathcal{B}\left(x\right)dx\right|$$
$$\leq \tfrac{\mu_2-\mu_1}{16}\left(\left|\mathcal{B}'\left(\mu_1\right)\right| + \left|\mathcal{B}'\left(\mu_2\right)\right|\right).$$

Over the last two decades, error estimation of quadrature rules via different types of convexity has become an attractive and fascinating area of research and has gained popularity. Consequently, several papers treating integral inequalities under the principle of convexity have been widely studied by mathematicians and researchers. Regarding Newton-Cotes type inequalities involving one point, see [3,4,18,23,27,28], two-point Newton-Cotes type inequalities see [10, 25, 26, 40], for three-point Newton-Cotes type inequalities: see [1, 9, 17, 19, 29, 39], and Newton-Cotes type inequalities involving four points: see [5, 13, 20, 22, 36].

In this paper, we propose to study the so-called Bullen-Simpson type inequalities, which is a five-point Newton-Cotes Rule and can be represented as follows:

$$\int_{\mu_1}^{\mu_2}\mathcal{B}(x)dx\simeq\tfrac{\mu_2-\mu_1}{12}\Big(\mathcal{B}(\mu_1)+4\mathcal{B}\left(\tfrac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\tfrac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\tfrac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)\Big).$$

To do this, we first prove a new identity. Using this identity, we establish some Bullen-Simpson type inequalities for functions whose first derivatives are s-convex, we also discuss the cases where the first derivatives are bounded as well as Hölderian functions. At the end, we propose some applications to numerical quadratures and inequalities involving means.

2. Main results

Lemma 1. *Let $\mathcal{B}:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $\mu_1,\mu_2\in I^\circ$ with $\mu_1<\mu_2$, and $\mathcal{B}'\in L^1[\mu_1,\mu_2]$, then the following equality holds*

$$\begin{aligned}&\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\\
&=\tfrac{\mu_2-\mu_1}{16}\Bigg(\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\mathcal{B}'\left((1-\mathfrak{h})\mu_1+\mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right)d\mathfrak{h}\\
&+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\mathcal{B}'\left((1-\mathfrak{h})\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)d\mathfrak{h}\\
&+\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+\mu_2}{2}+\mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right)d\mathfrak{h}\\
&+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+3\mu_2}{4}+\mathfrak{h}\mu_2\right)d\mathfrak{h}\Bigg).\end{aligned}$$

Proof. Let

$$I_1=\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\mathcal{B}'\left((1-\mathfrak{h})\mu_1+\mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right)d\mathfrak{h},$$

$$I_2=\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\mathcal{B}'\left((1-\mathfrak{h})\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)d\mathfrak{h},$$

$$I_3=\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+\mu_2}{2}+\mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right)d\mathfrak{h}$$

and

$$I_4 = \int_0^1 \left(\mathfrak{h} - \tfrac{2}{3}\right) \mathcal{B}'\left(\left(1-\mathfrak{h}\right)\tfrac{\mu_1+3\mu_2}{4} + \mathfrak{h}\mu_2\right) d\mathfrak{h}.$$

Integrating by parts I_1, we get

$$\begin{aligned}(3)\quad I_1 =& \tfrac{4}{\mu_2-\mu_1}\left(\mathfrak{h} - \tfrac{1}{3}\right)\mathcal{B}\left(\left(1-\mathfrak{h}\right)\mu_1 + \mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right)\Big|_{\mathfrak{h}=0}^{\mathfrak{h}=1} \\ &- \tfrac{4}{\mu_2-\mu_1}\int_0^1 \mathcal{B}\left(\left(1-\mathfrak{h}\right)\mu_1 + \mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right) d\mathfrak{h} \\ =& \tfrac{8}{3(\mu_2-\mu_1)}\mathcal{B}\left(\tfrac{3\mu_1+\mu_2}{4}\right) + \tfrac{4}{3(\mu_2-\mu_1)}\mathcal{B}\left(\mu_1\right) \\ &- \tfrac{4}{\mu_2-\mu_1}\int_0^1 \mathcal{B}\left(\left(1-\mathfrak{h}\right)\mu_1 + \mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right) d\mathfrak{h} \\ =& \tfrac{8}{3(\mu_2-\mu_1)}\mathcal{B}\left(\tfrac{3\mu_1+\mu_2}{4}\right) + \tfrac{4}{3(\mu_2-\mu_1)}\mathcal{B}\left(\mu_1\right) - \tfrac{16}{(\mu_2-\mu_1)^2}\int_{\mu_1}^{\frac{3\mu_1+\mu_2}{4}} \mathcal{B}\left(u\right) du.\end{aligned}$$

Similarly, we get

$$(4)\quad I_2 = \tfrac{4}{3(\mu_2-\mu_1)}\mathcal{B}\left(\tfrac{\mu_1+\mu_2}{2}\right) + \tfrac{8}{3(\mu_2-\mu_1)}\mathcal{B}\left(\tfrac{3\mu_1+\mu_2}{4}\right) - \tfrac{16}{(\mu_2-\mu_1)^2}\int_{\frac{3\mu_1+\mu_2}{4}}^{\frac{\mu_1+\mu_2}{2}} \mathcal{B}\left(u\right) du\ ,$$

$$(5)\quad I_3 = \tfrac{8}{3(\mu_2-\mu_1)}\mathcal{B}\left(\tfrac{\mu_1+3\mu_2}{4}\right) + \tfrac{4}{3(\mu_2-\mu_1)}\mathcal{B}\left(\tfrac{\mu_1+\mu_2}{2}\right) - \tfrac{16}{(\mu_2-\mu_1)^2}\int_{\frac{\mu_1+\mu_2}{2}}^{\frac{\mu_1+3\mu_2}{4}} \mathcal{B}\left(u\right) du$$

and

$$(6)\quad I_4 = \tfrac{4}{3(\mu_2-\mu_1)}\mathcal{B}\left(\mu_2\right) + \tfrac{8}{3(\mu_2-\mu_1)}\mathcal{B}\left(\tfrac{\mu_1+3\mu_2}{4}\right) - \tfrac{16}{(\mu_2-\mu_1)^2}\int_{\frac{\mu_1+3\mu_2}{4}}^{\mu_2} \mathcal{B}\left(u\right) du.$$

Summing (3)-(6) and then multiplying the resulting equality by $\frac{\mu_2-\mu_1}{16}$, we get the desired result. □

Theorem 3. *Let $\mathcal{B}$ be as in Lemma 1. If $|\mathcal{B}'|$ is s-convex in the second sense for some fixed $s \in (0,1]$, then we have*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$
$$\leq \frac{\mu_2-\mu_1}{16(s+1)(s+2)}\left(\left(\frac{s-1}{3}+\frac{8}{9}\left(\frac{2}{3}\right)^s\right)\left(\left|\mathcal{B}'(\mu_1)\right|+2\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|+\left|\mathcal{B}'(\mu_2)\right|\right)\right.$$
$$\left.+\,2\left(\frac{2s+1}{3}+\frac{2}{9}\left(\frac{1}{3}\right)^s\right)\left(\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|+\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|\right)\right).$$

Proof. From Lemma 1, properties of modulus and s-convexity of $|\mathcal{B}'|$, we have

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$
$$\leq \frac{\mu_2-\mu_1}{16}\left(\int\limits_0^1\left|\mathfrak{h}-\frac{1}{3}\right|\left|\mathcal{B}'\left((1-\mathfrak{h})\,\mu_1+\mathfrak{h}\frac{3\mu_1+\mu_2}{4}\right)\right|d\mathfrak{h}\right.$$
$$+\int\limits_0^1\left|\mathfrak{h}-\frac{2}{3}\right|\left|\mathcal{B}'\left((1-\mathfrak{h})\,\frac{3\mu_1+\mu_2}{4}+\mathfrak{h}\frac{\mu_1+\mu_2}{2}\right)\right|d\mathfrak{h}$$
$$+\int\limits_0^1\left|\mathfrak{h}-\frac{1}{3}\right|\left|\mathcal{B}\left((1-\mathfrak{h})\,\frac{\mu_1+\mu_2}{2}+\mathfrak{h}\frac{\mu_1+3\mu_2}{4}\right)\right|d\mathfrak{h}$$
$$\left.+\int\limits_0^1\left|\mathfrak{h}-\frac{2}{3}\right|\left|\mathcal{B}\left((1-\mathfrak{h})\,\frac{\mu_1+3\mu_2}{4}+\mathfrak{h}\mu_2\right)\right|d\mathfrak{h}\right)$$
$$\leq \frac{\mu_2-\mu_1}{16}\left(\int\limits_0^1\left|\mathfrak{h}-\frac{1}{3}\right|\left((1-\mathfrak{h})^s\left|\mathcal{B}'(\mu_1)\right|+\mathfrak{h}^s\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|\right)d\mathfrak{h}\right.$$
$$+\int\limits_0^1\left|\mathfrak{h}-\frac{2}{3}\right|\left((1-\mathfrak{h})^s\left|\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)\right|+\mathfrak{h}^s\left|\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)\right|\right)d\mathfrak{h}$$
$$+\int\limits_0^1\left|\mathfrak{h}-\frac{2}{3}\right|\left((1-\mathfrak{h})^s\left|\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)\right|+\mathfrak{h}^s\left|\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)\right|\right)d\mathfrak{h}$$
$$+\int\limits_0^1\left|\mathfrak{h}-\frac{1}{3}\right|\left((1-\mathfrak{h})^s\left|\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)\right|+\mathfrak{h}^s\left|\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)\right|\right)d\mathfrak{h}$$

$$
\begin{aligned}
&+\int_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right|\left((1-\mathfrak{h})^s\left|\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|+\mathfrak{h}^s\left|\mathcal{B}\left(\mu_2\right)\right|\right)d\mathfrak{h}\Bigg)\\
=&\tfrac{\mu_2-\mu_1}{16}\Bigg(\left|\mathcal{B}\left(\mu_1\right)\right|\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|(1-\mathfrak{h})^s\,d\mathfrak{h}\\
&+\left|\mathcal{B}\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right|\left(\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|\mathfrak{h}^s d\mathfrak{h}+\int_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|(1-\mathfrak{h})^s\,d\mathfrak{h}\right)\\
&+\left|\mathcal{B}\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|\left(\int_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|\mathfrak{h}^s d\mathfrak{h}+\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|(1-\mathfrak{h})^s\,d\mathfrak{h}\right)\\
&+\left|\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|\left(\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|\mathfrak{h}^s d\mathfrak{h}+\int_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|(1-\mathfrak{h})^s\,d\mathfrak{h}\right)\\
&+\left|\mathcal{B}\left(\mu_2\right)\right|\int_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|\mathfrak{h}^s d\mathfrak{h}\Bigg)\\
=&\tfrac{\mu_2-\mu_1}{16(s+1)(s+2)}\Bigg(\left(\tfrac{s-1}{3}+\tfrac{8}{9}\left(\tfrac{2}{3}\right)^s\right)\left(\left|\mathcal{B}'\left(\mu_1\right)\right|+2\left|\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|+\left|\mathcal{B}'\left(\mu_2\right)\right|\right)\\
&+2\left(\tfrac{2s+1}{3}+\tfrac{2}{9}\left(\tfrac{1}{3}\right)^s\right)\left(\left|\mathcal{B}\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right|+\left|\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|\right)\Bigg),
\end{aligned}
$$

where we have used the facts that

$$
\begin{aligned}
\Upsilon_1(s)=&\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|(1-\mathfrak{h})^s\,d\mathfrak{h}=\int_0^1\left|\tfrac{2}{3}-\mathfrak{h}\right|\mathfrak{h}^s d\mathfrak{h}\\
=&\tfrac{s-1}{3(s+1)(s+2)}+\tfrac{2}{(s+1)(s+2)}\left(\tfrac{2}{3}\right)^{s+2}
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
\Upsilon_2(s)=&\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|\mathfrak{h}^s d\mathfrak{h}=\int_0^1\left|\tfrac{2}{3}-\mathfrak{h}\right|(1-\mathfrak{h})^s\,d\mathfrak{h}\\
=&\tfrac{2s+1}{3(s+1)(s+2)}+\tfrac{2}{(s+1)(s+2)}\left(\tfrac{1}{3}\right)^{s+2}.
\end{aligned}
\tag{8}
$$

The proof is completed. □

Corollary 1. *In Theorem 3 using the s-convexity of $|\mathcal{B}'|$, we get*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\begin{aligned}&\leq\frac{\mu_2-\mu_1}{48(s+1)^2(s+2)}\left(\frac{2^s s^2+8s+4-2^s}{2^s}+\frac{2^{s+3}s+2^{s+3}+2^{3-s}}{3^{s+1}}\right)\\ &\quad\times\left(\left|\mathcal{B}'(\mu_1)\right|+2\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|+\left|\mathcal{B}'(\mu_2)\right|\right) \qquad (9)\\ &\leq\frac{\left(s+1+2^{2-s}\right)(\mu_2-\mu_1)}{48(s+1)^3(s+2)}\left(\frac{2^s s^2+8s+4-2^s}{2^s}+\frac{2^{s+3}s+2^{s+3}+2^{3-s}}{3^{s+1}}\right)\\ &\quad\times\left(\left|\mathcal{B}'(\mu_1)\right|+\left|\mathcal{B}'(\mu_2)\right|\right).\end{aligned}$$

Corollary 2. *For $s=1$, Theorem 3 becomes*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\leq\frac{5(\mu_2-\mu_1)}{72}\left(\frac{8|\mathcal{B}'(\mu_1)|+29\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|+16\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|+29\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|+8|\mathcal{B}'(\mu_2)|}{90}\right).$$

Corollary 3. *In Corollary 1, if we take $s=1$, we obtain*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\begin{aligned}&\leq\frac{5(\mu_2-\mu_1)}{288}\left(\left|\mathcal{B}'(\mu_1)\right|+2\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|+\left|\mathcal{B}'(\mu_2)\right|\right)\\ &\leq\frac{5(\mu_2-\mu_1)}{144}\left(\left|\mathcal{B}'(\mu_1)\right|+\left|\mathcal{B}'(\mu_2)\right|\right). \qquad (10)\end{aligned}$$

Theorem 4. *Let $\mathcal{B}$ be as in Lemma 1. If $|\mathcal{B}'|^q$ is s-convex in the second sense for some fixed $s\in(0,1]$ and $q>1$ with $\frac{1}{q}+\frac{1}{p}=1$, then we have*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\begin{aligned}&\leq\frac{\mu_2-\mu_1}{144}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{B}'(\mu_1)|^q+\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.\\ &\quad+\left(\frac{\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|^q+\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\\ &\quad\left.+\left(\frac{\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q+\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|^q+|\mathcal{B}'(\mu_2)|^q}{s+1}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Proof. From Lemma 1, properties of modulus, Hölder's inequality and s-convexity of $|\mathcal{B}'|^q$, we have

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\leq \frac{\mu_2-\mu_1}{16}\left(\left(\int\limits_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|^p d\mathfrak{h}\right)^{\frac{1}{p}}\left(\int\limits_0^1\left|\mathcal{B}'\left((1-\mathfrak{h})\mu_1+\mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right)\right|^q d\mathfrak{h}\right)^{\frac{1}{q}}\right.$$

$$+\left(\int\limits_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|^p d\mathfrak{h}\right)^{\frac{1}{p}}\left(\int\limits_0^1\left|\mathcal{B}'\left((1-\mathfrak{h})\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)\right|^q d\mathfrak{h}\right)^{\frac{1}{q}}$$

$$+\left(\int\limits_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|^p d\mathfrak{h}\right)^{\frac{1}{p}}\left(\int\limits_0^1\left|\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+\mu_2}{2}+\mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right)\right|^q d\mathfrak{h}\right)^{\frac{1}{q}}$$

$$\left.+\left(\int\limits_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|^p d\mathfrak{h}\right)^{\frac{1}{p}}\left(\int\limits_0^1\left|\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+3\mu_2}{4}+\mathfrak{h}\mu_2\right)\right|^q d\mathfrak{h}\right)^{\frac{1}{q}}\right)$$

$$\leq \frac{\mu_2-\mu_1}{16}\left(\int\limits_0^{\frac{1}{3}}\left(\tfrac{1}{3}-\mathfrak{h}\right)^p d\mathfrak{h}+\int\limits_{\frac{1}{3}}^1\left(\mathfrak{h}-\tfrac{1}{3}\right)^p d\mathfrak{h}\right)^{\frac{1}{p}}$$

$$\times\left(\left(\int\limits_0^1\left((1-\mathfrak{h})^s\left|\mathcal{B}'(\mu_1)\right|^q+\mathfrak{h}^s\left|\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right|^q\right)d\mathfrak{h}\right)^{\frac{1}{q}}\right.$$

$$+\left(\int\limits_0^1\left((1-\mathfrak{h})^s\left|\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right|^q+\mathfrak{h}^s\left|\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^q\right)d\mathfrak{h}\right)^{\frac{1}{q}}$$

$$+\left(\int\limits_0^1\left((1-\mathfrak{h})^s\left|\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^q+\mathfrak{h}^s\left|\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|^q\right)d\mathfrak{h}\right)^{\frac{1}{q}}$$

$$\left.+\left(\int\limits_0^1\left((1-\mathfrak{h})^s\left|\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|^q+\mathfrak{h}^s\left|\mathcal{B}'(\mu_2)\right|^q\right)d\mathfrak{h}\right)^{\frac{1}{q}}\right)$$

$$= \frac{\mu_2-\mu_1}{144}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{B}'(\mu_1)|^q+\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|^q+\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q+\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|^q+|\mathcal{B}'(\mu_2)|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

The proof is completed. □

Corollary 4. *In Theorem 4, using the s-convexity of $|\mathcal{B}'|^q$, we get*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\leq \frac{\mu_2-\mu_1}{144}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{(4^s+3^s)|\mathcal{B}'(\mu_1)|^q+|\mathcal{B}'(\mu_2)|^q}{4^s(s+1)}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{|\mathcal{B}'(\mu_1)|^q+(4^s+3^s)|\mathcal{B}'(\mu_2)|^q}{4^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(3^s+2^s)|\mathcal{B}'(\mu_1)|^q+(1+2^s)|\mathcal{B}'(\mu_2)|^q}{4^s(s+1)}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{(1+2^s)|\mathcal{B}'(\mu_1)|^q+(3^s+2^s)|\mathcal{B}'(\mu_2)|^q}{4^s(s+1)}\right)^{\frac{1}{q}}\right).$$

Corollary 5. *In Corollary 4, using the discrete power mean inequality, we get*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\leq \frac{\mu_2-\mu_1}{72}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{1+4^s+3^s}{4^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{1+3^s+2^{s+1}}{4^s(s+1)}\right)^{\frac{1}{q}}\right)\left(\frac{|\mathcal{B}'(\mu_1)|^q+|\mathcal{B}'(\mu_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 6. *Taking $s=1$ in Theorem 4, we get*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\leq \frac{\mu_2-\mu_1}{144}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{B}'(\mu_1)|^q+\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|^q+\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q+\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|^q+|\mathcal{B}'(\mu_2)|^q}{2}\right)^{\frac{1}{q}}\right).$$

Corollary 7. *For $s=1$, Corollary 4, gives*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$
$$\leq\frac{\mu_2-\mu_1}{144}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\left(\frac{7|\mathcal{B}'(\mu_1)|^q+|\mathcal{B}'(\mu_2)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{B}'(\mu_1)|^q+7|\mathcal{B}'(\mu_2)|^q}{8}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{5|\mathcal{B}'(\mu_1)|^q+3|\mathcal{B}'(\mu_2)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{B}'(\mu_1)|^q+5|\mathcal{B}'(\mu_2)|^q}{8}\right)^{\frac{1}{q}}\right).$$

Corollary 8. *Taking $s=1$ in Corollary 5, we get*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$
$$\leq\frac{\mu_2-\mu_1}{36}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{B}'(\mu_1)|^q+|\mathcal{B}'(\mu_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 5. *Let $\mathcal{B}$ be as in Lemma 1. If $|\mathcal{B}'|^q$ is s-convex in the second sense for some fixed $s\in(0,1]$ and $q\geq1$, then we have*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$
$$\leq\frac{5(\mu_2-\mu_1)}{288}\left(\left(\frac{2\times3^{s+1}s-2\times3^{s+1}+2^{s+4}}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'(\mu_1)\right|^q+\frac{4\times3^{s+1}s+2\times3^{s+1}+4}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{4\times3^{s+1}s+2\times3^{s+1}+4}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|^q+\frac{2\times3^{s+1}s-2\times3^{s+1}+2^{s+4}}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{2\times3^{s+1}s-2\times3^{s+1}+2^{s+4}}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q+\frac{4\times3^{s+1}s+2\times3^{s+1}+4}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|^q\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{4\times3^{s+1}s+2\times3^{s+1}+4}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|^q+\frac{2\times3^{s+1}s-2\times3^{s+1}+2^{s+4}}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'(\mu_2)\right|^q\right)^{\frac{1}{q}}\right).$$

Proof. From Lemma 1, properties of modulus, power mean inequality and s-convexity of $|f'|^q$, we have

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$
$$\leq\frac{\mu_2-\mu_1}{16}\left(\left(\int\limits_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|d\mathfrak{h}\right)^{1-\frac{1}{q}}\left(\int\limits_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|\left|\mathcal{B}'\left((1-\mathfrak{h})\,\mu_1+\mathfrak{h}\frac{3\mu_1+\mu_2}{4}\right)\right|^q d\mathfrak{h}\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| d\mathfrak{h}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| \left|\mathcal{B}'\left((1-\mathfrak{h})\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)\right|^q d\mathfrak{h}\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|\mathfrak{h}-\tfrac{1}{3}\right| d\mathfrak{h}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|\mathfrak{h}-\tfrac{1}{3}\right| \left|\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+\mu_2}{2}+\mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right)\right|^q d\mathfrak{h}\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| d\mathfrak{h}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| \left|\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+3\mu_2}{4}+\mathfrak{h}\mu_2\right)\right|^q d\mathfrak{h}\right)^{\frac{1}{q}}\Bigg)$$

$$\leq \tfrac{\mu_2-\mu_1}{16}\Bigg(\left(\int_0^1 \left|\mathfrak{h}-\tfrac{1}{3}\right| d\mathfrak{h}\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left|\mathfrak{h}-\tfrac{1}{3}\right| \left((1-\mathfrak{h})^s\left|\mathcal{B}'(\mu_1)\right|^q+\mathfrak{h}^s\left|\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right|^q\right) d\mathfrak{h}\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| d\mathfrak{h}\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| \left((1-\mathfrak{h})^s\left|\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right|^q+\mathfrak{h}^s\left|\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^q\right) d\mathfrak{h}\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|\mathfrak{h}-\tfrac{1}{3}\right| d\mathfrak{h}\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left|\mathfrak{h}-\tfrac{1}{3}\right| \left((1-\mathfrak{h})^s\left|\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^q+\mathfrak{h}^s\left|\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|^q\right) d\mathfrak{h}\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| d\mathfrak{h}\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| \left((1-\mathfrak{h})^s\left|\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|^q+\mathfrak{h}^s\left|\mathcal{B}'(\mu_2)\right|^q\right) d\mathfrak{h}\right)^{\frac{1}{q}}\Bigg)$$

$$=\tfrac{\mu_2-\mu_1}{16}\left(\tfrac{5}{18}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left(\left|\mathcal{B}'(\mu_1)\right|^q\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|(1-\mathfrak{h})^s\,d\mathfrak{h}+\left|\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right|^q\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|\mathfrak{h}^s d\mathfrak{h}\right)^{\frac{1}{q}}\right.$$

$$+\left(\left|\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right|^q\int_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|(1-\mathfrak{h})^s\,d\mathfrak{h}+\left|\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^q\int_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|\mathfrak{h}^s d\mathfrak{h}\right)^{\frac{1}{q}}$$

$$+\left(\left|\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^q\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|(1-\mathfrak{h})^s\,d\mathfrak{h}+\left|\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|^q\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|\mathfrak{h}^s d\mathfrak{h}\right)^{\frac{1}{q}}$$

$$\left.+\left(\left|\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|^q\int_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|(1-\mathfrak{h})^s\,d\mathfrak{h}+\left|\mathcal{B}'(\mu_2)\right|^q\int_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|\mathfrak{h}^s d\mathfrak{h}\right)^{\frac{1}{q}}\right)$$

$$=\tfrac{5(\mu_2-\mu_1)}{288}$$

$$\times\left(\left(\tfrac{2\times3^{s+1}s-2\times3^{s+1}+2^{s+4}}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'(\mu_1)\right|^q+\tfrac{4\times3^{s+1}s+2\times3^{s+1}+4}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\tfrac{4\times3^{s+1}s+2\times3^{s+1}+4}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right|^q+\tfrac{2\times3^{s+1}s-2\times3^{s+1}+2^{s+4}}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{2\times3^{s+1}s-2\times3^{s+1}+2^{s+4}}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^q+\tfrac{4\times3^{s+1}s+2\times3^{s+1}+4}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|^q\right)^{\frac{1}{q}}$$

$$\left.+\left(\tfrac{4\times3^{s+1}s+2\times3^{s+1}+4}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|^q+\tfrac{2\times3^{s+1}s-2\times3^{s+1}+2^{s+4}}{5\times3^s(s+1)(s+2)}\left|\mathcal{B}'(\mu_2)\right|^q\right)^{\frac{1}{q}}\right),$$

where we have used (7) and (8). The proof is completed. □

Corollary 9. *For $s=1$, Theorem 5 becomes*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\leq\tfrac{5(\mu_2-\mu_1)}{288}\left(\left(\frac{16|\mathcal{B}'(\mu_1)|^q+29\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|^q}{45}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{29\left|\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|^q+16\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q}{45}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{16\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q+29\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|^q}{45}\right)^{\frac{1}{q}}+\left(\frac{29\left|\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|^q+16|\mathcal{B}'(\mu_2)|^q}{45}\right)^{\frac{1}{q}}\right).$$

Corollary 10. *In Corollary 9, using the convexity of* $|\mathcal{B}'|^q$*, we get*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$
$$\leq\frac{5(\mu_2-\mu_1)}{288}\left(\left(\frac{151|\mathcal{B}'(\mu_1)|^q+29|\mathcal{B}'(\mu_2)|^q}{180}\right)^{\frac{1}{q}}+\left(\frac{119|\mathcal{B}'(\mu_1)|^q+61|\mathcal{B}'(\mu_2)|^q}{180}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{61|\mathcal{B}'(\mu_1)|^q+119|\mathcal{B}'(\mu_2)|^q}{180}\right)^{\frac{1}{q}}+\left(\frac{29|\mathcal{B}'(\mu_1)|^q+151|\mathcal{B}'(\mu_2)|^q}{180}\right)^{\frac{1}{q}}\right).$$

Corollary 11. *In Corollary 10, using the discrete power mean inequality we get*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$
$$\leq\frac{5(\mu_2-\mu_1)}{72}\left(\frac{|\mathcal{B}'(\mu_1)|^q+|\mathcal{B}'(\mu_2)|^q}{2}\right)^{\frac{1}{q}}.$$

3. Further results

Theorem 6. *Let* $\mathcal{B}$ *be as in Lemma 1. If there exist constants* $-\infty<m<M<+\infty$ *such that* $m\leq\mathcal{B}'(x)\leq M$ *for all* $x\in[\mu_1,\mu_2]$*, then we have*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$
$$\leq\frac{5(\mu_2-\mu_1)(M-m)}{144}.$$

Proof. From Lemma 1, we have

$$\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx$$
$$=\frac{\mu_2-\mu_1}{16}\left(\int\limits_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\,\mu_1+\mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)d\mathfrak{h}\right.$$
$$+\int\limits_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\,\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)d\mathfrak{h}$$
$$+\int\limits_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\,\tfrac{\mu_1+\mu_2}{2}+\mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)d\mathfrak{h}$$

$$+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+3\mu_2}{4}+\mathfrak{h}\mu_2\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)d\mathfrak{h}\Bigg)$$

$$=\tfrac{\mu_2-\mu_1}{16}\Bigg(\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\mu_1+\mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)d\mathfrak{h}$$

$$+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)d\mathfrak{h}$$

$$+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)d\mathfrak{h}$$

$$+\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+\mu_2}{2}+\mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)d\mathfrak{h}$$

$$+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+3\mu_2}{4}+\mathfrak{h}\mu_2\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)d\mathfrak{h}\Bigg)$$

$$=\tfrac{\mu_2-\mu_1}{16}\Bigg(\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\mu_1+\mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right)-\tfrac{m+M}{2}\right)d\mathfrak{h}$$

$$+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)-\tfrac{m+M}{2}\right)d\mathfrak{h}$$

$$+\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+\mu_2}{2}+\mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right)-\tfrac{m+M}{2}\right)d\mathfrak{h}$$

$$+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+3\mu_2}{4}+\mathfrak{h}\mu_2\right)-\tfrac{m+M}{2}\right)d\mathfrak{h}\Bigg), \tag{11}$$

where we have taken into consideration that

$$\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)d\mathfrak{h}+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)d\mathfrak{h}+\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)d\mathfrak{h}+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)d\mathfrak{h}=0.$$

Using absolute value on both sides of (11), we get

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\leq \tfrac{\mu_2-\mu_1}{16}\left(\int\limits_0^1 \left|\mathfrak{h}-\tfrac{1}{3}\right| \left|\mathcal{B}'\left((1-\mathfrak{h})\,\mu_1+\mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right)-\tfrac{m+M}{2}\right| d\mathfrak{h}\right.$$

$$+\int\limits_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| \left|\mathcal{B}'\left((1-\mathfrak{h})\,\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)-\tfrac{m+M}{2}\right| d\mathfrak{h}$$

$$+\int\limits_0^1 \left|\mathfrak{h}-\tfrac{1}{3}\right| \left|\mathcal{B}'\left((1-\mathfrak{h})\,\tfrac{\mu_1+\mu_2}{2}+\mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right)-\tfrac{m+M}{2}\right| d\mathfrak{h}$$

$$\left.+\int\limits_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| \left|\mathcal{B}'\left((1-\mathfrak{h})\,\tfrac{\mu_1+3\mu_2}{4}+\mathfrak{h}\mu_2\right)-\tfrac{m+M}{2}\right| d\mathfrak{h}\right). \tag{12}$$

Since $m \leq \mathcal{B}'(x) \leq M$ for all $x \in [\mu_1, \mu_2]$, then we have

$$\left|\mathcal{B}'\left((1-\mathfrak{h})\,\mu_1+\mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right)-\tfrac{m+M}{2}\right| \leq \tfrac{M-m}{2}, \tag{13}$$

$$\left|\mathcal{B}'\left((1-\mathfrak{h})\,\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)-\tfrac{m+M}{2}\right| \leq \tfrac{M-m}{2}, \tag{14}$$

$$\left|\mathcal{B}'\left((1-\mathfrak{h})\,\tfrac{\mu_1+\mu_2}{2}+\mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right)-\tfrac{m+M}{2}\right| \leq \tfrac{M-m}{2} \tag{15}$$

and

$$\left|\mathcal{B}'\left((1-\mathfrak{h})\,\tfrac{\mu_1+3\mu_2}{4}+\mathfrak{h}\mu_2\right)-\tfrac{m+M}{2}\right| \leq \tfrac{M-m}{2}. \tag{16}$$

Using (13)-(16) in (12), we get

$$\left|\tfrac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\tfrac{1}{\mu_2-\mu_1}\int\limits_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\leq \tfrac{(\mu_2-\mu_1)(M-m)}{16}\left(\int\limits_0^1 \left|\mathfrak{h}-\tfrac{1}{3}\right| d\mathfrak{h}+\int\limits_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| d\mathfrak{h}\right)$$

$$= \tfrac{5(\mu_2-\mu_1)(M-m)}{144},$$

where we have used

$$\int\limits_0^1 \left|\mathfrak{h}-\tfrac{1}{3}\right| d\mathfrak{h} = \int\limits_0^1 \left|\mathfrak{h}-\tfrac{2}{3}\right| d\mathfrak{h} = \tfrac{5}{18}.$$

The proof is completed. □

Theorem 7. *Let $\mathcal{B}$ be as in Lemma 1. If $\mathcal{B}'$ is r-L-Hölderian function on $[\mu_1, \mu_2]$ (i.e. there exist $L > 0$ and $0 < r \leq 1$ such that $|\mathcal{B}'(x) - \mathcal{B}'(y)| \leq L|x-y|^r$), then we have*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12} - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \mathcal{B}(x)\,dx\right|$$
$$\leq \frac{(\mu_2-\mu_1)^{r+1}}{4^{r+1}(r+1)(r+2)}\left(\frac{r^2+9r+2}{12} + \frac{1+2^{r+2}}{3^{r+2}}\right)L.$$

Proof. From Lemma 1, we have

$$\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12} - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \mathcal{B}(x)\,dx$$
$$=\frac{\mu_2-\mu_1}{16}\left(\int_0^1 \left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\mu_1 + \mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right) - \mathcal{B}'(\mu_1) + \mathcal{B}'(\mu_1)\right)d\mathfrak{h}\right.$$
$$+\int_0^1 \left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{3\mu_1+\mu_2}{4} + \mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right) - \mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right) + \mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right)d\mathfrak{h}$$
$$+\int_0^1 \left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+\mu_2}{2} + \mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right) - \mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right) + \mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right)d\mathfrak{h}$$
$$\left.+\int_0^1 \left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+3\mu_2}{4} + \mathfrak{h}\mu_2\right) - \mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right) + \mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right)d\mathfrak{h}\right)$$
$$=\frac{\mu_2-\mu_1}{16}\left(\int_0^1 \left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\mu_1 + \mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right) - \mathcal{B}'(\mu_1)\right)d\mathfrak{h}\right.$$
$$+\int_0^1 \left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{3\mu_1+\mu_2}{4} + \mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right) - \mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right)d\mathfrak{h}$$
$$+\int_0^1 \left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+\mu_2}{2} + \mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right) - \mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right)d\mathfrak{h}$$
$$+\int_0^1 \left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+3\mu_2}{4} + \mathfrak{h}\mu_2\right) - \mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right)d\mathfrak{h}$$

$$+\mathcal{B}'(\mu_1)\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)d\mathfrak{h}+\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)d\mathfrak{h}$$

$$\left.+\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)d\mathfrak{h}+\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)d\mathfrak{h}\right)$$

$$=\tfrac{\mu_2-\mu_1}{16}\left(\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\,a+\mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right)-\mathcal{B}'(\mu_1)\right)d\mathfrak{h}\right.$$

$$+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)-\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right)d\mathfrak{h}$$

$$+\int_0^1\left(\mathfrak{h}-\tfrac{1}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+\mu_2}{2}+\mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right)-\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right)d\mathfrak{h}$$

$$+\int_0^1\left(\mathfrak{h}-\tfrac{2}{3}\right)\left(\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+3\mu_2}{4}+\mathfrak{h}\mu_2\right)-\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right)d\mathfrak{h}$$

(17)

$$\left.+\tfrac{1}{6}\left(\mathcal{B}'(\mu_2)-\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right)+\tfrac{1}{6}\left(\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)-\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right)\right).$$

Using the absolute value on both sides of (17) and the fact that $\mathcal{B}'$ is r-L-Hölderian on $[\mu_1,\mu_2]$, we obtain

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$

$$\leq\tfrac{\mu_2-\mu_1}{16}\left(\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|\left|\mathcal{B}'\left((1-\mathfrak{h})\,\mu_1+\mathfrak{h}\tfrac{3\mu_1+\mu_2}{4}\right)-\mathcal{B}'(\mu_1)\right|d\mathfrak{h}\right.$$

$$+\int_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|\left|\mathcal{B}'\left((1-\mathfrak{h})\tfrac{3\mu_1+\mu_2}{4}+\mathfrak{h}\tfrac{\mu_1+\mu_2}{2}\right)-\mathcal{B}'\left(\tfrac{3\mu_1+\mu_2}{4}\right)\right|d\mathfrak{h}$$

$$+\int_0^1\left|\mathfrak{h}-\tfrac{1}{3}\right|\left|\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+\mu_2}{2}+\mathfrak{h}\tfrac{\mu_1+3\mu_2}{4}\right)-\mathcal{B}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|d\mathfrak{h}$$

$$+\int_0^1\left|\mathfrak{h}-\tfrac{2}{3}\right|\left|\mathcal{B}'\left((1-\mathfrak{h})\tfrac{\mu_1+3\mu_2}{4}+\mathfrak{h}\mu_2\right)-\mathcal{B}'\left(\tfrac{\mu_1+3\mu_2}{4}\right)\right|d\mathfrak{h}$$

$$+\frac{1}{6}\left|\mathcal{B}'(\mu_1)-\mathcal{B}'\left(\frac{3\mu_1+\mu_2}{4}\right)\right|+\frac{1}{6}\left|\mathcal{B}'\left(\frac{\mu_1+\mu_2}{2}\right)-\mathcal{B}'\left(\frac{\mu_1+3\mu_2}{4}\right)\right|\bigg)$$
$$\leq\frac{\mu_2-\mu_1}{8}\left(\frac{\mu_2-\mu_1}{4}\right)^r L\left(\int_0^1\left|\mathfrak{h}-\frac{1}{3}\right|\mathfrak{h}^r d\mathfrak{h}+\int_0^1\left|\mathfrak{h}-\frac{2}{3}\right|\mathfrak{h}^r d\mathfrak{h}+\frac{1}{6}\right)$$
$$=\frac{L}{2}\left(\frac{\mu_2-\mu_1}{4}\right)^{r+1}\left(\Upsilon_2(r)+\Upsilon_1(r)+\frac{1}{6}\right)$$
$$=\frac{(\mu_2-\mu_1)^{r+1}}{4^{r+1}(r+1)(r+2)}\left(\frac{r^2+9r+2}{12}+\frac{1+2^{r+2}}{3^{r+2}}\right)L,$$

where we have used (7) and (8). The proof is completed. □

Corollary 12. *For $\mathcal{B}'$ L-Lipschitzian function, we have*

$$\left|\frac{\mathcal{B}(\mu_1)+4\mathcal{B}\left(\frac{3\mu_1+\mu_2}{4}\right)+2\mathcal{B}\left(\frac{\mu_1+\mu_2}{2}\right)+4\mathcal{B}\left(\frac{\mu_1+3\mu_2}{4}\right)+\mathcal{B}(\mu_2)}{12}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{B}(x)\,dx\right|$$
$$\leq\frac{(\mu_2-\mu_1)^2}{72}L.$$

4. Applications

Let Υ be the partition of the points $\mu_1=\phi_0<\phi_1<\cdots<\phi_n=\mu_2$ of the interval $[\mu_1,\mu_2]$, and consider the quadrature formula

$$\int_{\mu_1}^{\mu_2}\mathcal{B}(u)\,du=\lambda(\mathcal{B},\Upsilon)+R(\mathcal{B},\Upsilon),$$

where

$$\lambda(\mathcal{B},\Upsilon)=\sum_{i=0}^{n-1}\frac{\phi_{i+1}-\phi_i}{12}\left(\mathcal{B}(\phi_i)+4\mathcal{B}\left(\frac{3\phi_i+\phi_{i+1}}{4}\right)+2\mathcal{B}\left(\frac{\phi_i+\phi_{i+1}}{2}\right)\right.$$
$$\left.+4\mathcal{B}\left(\frac{\phi_i+3\phi_{i+1}}{4}\right)+\mathcal{B}(\phi_{i+1})\right)$$

and $R(\mathcal{B},\Upsilon)$ denotes the associated approximation error.

Proposition 1. *Let $n\in\mathbb{N}$ and $\mathcal{B}$ be as in Theorem 1. Then we have*

$$|R(\mathcal{B},\Upsilon)|\leq\sum_{i=0}^{n-1}\frac{(\phi_{i+1}-\phi_i)^2}{48(s+1)^2(s+2)}\left(\frac{2^s s^2+8s+4-2^s}{2^s}+\frac{2^{s+3}s+2^{s+3}+2^{3-s}}{3^{s+1}}\right)$$
$$\times\left(\left|\mathcal{B}'(\phi_i)\right|+2\left|\mathcal{B}'\left(\frac{\phi_i+\phi_{i+1}}{2}\right)\right|+\left|\mathcal{B}'(\phi_{i+1})\right|\right).$$

Proof. Using the inequality (9) of Corollary 1 on $[\phi_i, \phi_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the partition Υ, we get

$$\left| \frac{\mathcal{B}(\phi_i)+4\mathcal{B}\left(\frac{3\phi_i+\phi_{i+1}}{4}\right)+2\mathcal{B}\left(\frac{\phi_i+\phi_{i+1}}{2}\right)+4\mathcal{B}\left(\frac{\phi_i+3\phi_{i+1}}{4}\right)+f(\phi_{i+1})}{12} - \frac{1}{\phi_{i+1}-\phi_i} \int\limits_{\phi_i}^{\phi_{i+1}} \mathcal{B}(x)\,dx \right|$$
$$\leq \frac{\phi_{i+1}-\phi_i}{48(s+1)^2(s+2)} \left(\frac{2^s s^2+8s+4-2^s}{2^s} + \frac{2^{s+3}s+2^{s+3}+2^{3-s}}{3^{s+1}} \right)$$
$$\times \left(\left|\mathcal{B}'(\phi_i)\right| + 2\left|\mathcal{B}'\left(\frac{\phi_i+\phi_{i+1}}{2}\right)\right| + \left|\mathcal{B}'(\phi_{i+1})\right| \right).$$

Multiplying both sides of above inequality by $(\phi_{i+1} - \phi_i)$, summing the obtained inequalities, for all $i = 0, 1, \ldots, n-1$, and using the triangular inequality, we get the desired result. □

Proposition 2. *Let $n \in \mathbb{N}$ and $\mathcal{B}$ be as in Theorem 2. If $|\mathcal{B}'|^q$ is a convex function, then we have*

$$|R(\mathcal{B}, \Upsilon)|$$
$$\leq \sum_{i=0}^{n-1} \frac{(\phi_{i+1}-\phi_i)^2}{144} \left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \left(\left(\frac{7|\mathcal{B}'(\phi_i)|^q+|\mathcal{B}'(\phi_{i+1})|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{B}'(\phi_i)|^q+7|\mathcal{B}'(\phi_{i+1})|^q}{8}\right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\frac{5|\mathcal{B}'(\phi_i)|^q+3|\mathcal{B}'(\phi_{i+1})|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{3|\mathcal{B}'(\phi_i)|^q+5|\mathcal{B}'(\phi_{i+1})|^q}{8}\right)^{\frac{1}{q}} \right).$$

Proof. Using Corollary 7 on $[\phi_i, \phi_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the partition Υ, we get

$$\left| \frac{\mathcal{B}(\phi_i)+4\mathcal{B}\left(\frac{3\phi_i+\phi_{i+1}}{4}\right)+2\mathcal{B}\left(\frac{\phi_i+\phi_{i+1}}{2}\right)+4\mathcal{B}\left(\frac{\phi_i+3\phi_{i+1}}{4}\right)+f(\phi_{i+1})}{12} - \frac{1}{\phi_{i+1}-\phi_i} \int\limits_{\phi_i}^{\phi_{i+1}} \mathcal{B}(x)\,dx \right|$$
$$\leq \frac{\phi_{i+1}-\phi_i}{144} \left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \left(\left(\frac{7|\mathcal{B}'(\phi_i)|^q+|\mathcal{B}'(\phi_{i+1})|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{|\mathcal{B}'(\phi_i)|^q+7|\mathcal{B}'(\phi_{i+1})|^q}{8}\right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\frac{5|\mathcal{B}'(\phi_i)|^q+3|\mathcal{B}'(\phi_{i+1})|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{3|\mathcal{B}'(\phi_i)|^q+5|\mathcal{B}'(\phi_{i+1})|^q}{8}\right)^{\frac{1}{q}} \right).$$

In a similar way, multiplying both sides of the above inequality by $(\phi_{i+1} - \phi_i)$, summing the obtained inequalities, for all $i = 0, 1, \ldots, n-1$, and using the triangular inequality, we get the desired result. □

Proposition 3. *Let $n \in \mathbb{N}$ and $\mathcal{B}$ be as in Theorem 3. If $|\mathcal{B}'|^q$ is a convex function, then we have*

$$|R(\mathcal{B}, \Upsilon)| \leq \sum_{i=0}^{n-1} \frac{5(\phi_{i+1}-\phi_i)^2}{72} \left(\frac{|\mathcal{B}'(\phi_i)|^q+|\mathcal{B}'(\phi_{i+1})|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. Using Corollary 11 on $[\phi_i, \phi_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the partition Υ, we get

$$\left|\frac{\mathcal{B}(\phi_i)+4\mathcal{B}\left(\frac{3\phi_i+\phi_{i+1}}{4}\right)+2\mathcal{B}\left(\frac{\phi_i+\phi_{i+1}}{2}\right)+4\mathcal{B}\left(\frac{\phi_i+3\phi_{i+1}}{4}\right)+f(\phi_{i+1})}{12} - \frac{1}{\phi_{i+1}-\phi_i}\int\limits_{\phi_i}^{\phi_{i+1}} \mathcal{B}(x)\,dx\right|$$

$$\leq \frac{5(\phi_{i+1}-\phi_i)}{72}\left(\frac{|\mathcal{B}'(\phi_i)|^q+|\mathcal{B}'(\phi_{i+1})|^q}{2}\right)^{\frac{1}{q}}.$$

And again, multiplying both sides of the above inequality by $(\phi_{i+1} - \phi_i)$, summing the obtained inequalities, for all $i = 0, 1, \ldots, n-1$, and using the triangular inequality, we get the desired result. □

Application to special means

For arbitrary real numbers $\mu_1, \mu_2, \mu_3, \mu_4$, we have:
The Arithmetic mean: $A(\mu_1, \mu_2) = \frac{\mu_1+\mu_2}{2}$ and $A(\mu_1, \mu_2, \mu_3, \mu_4) = \frac{\mu_1+\mu_2+\mu_3+\mu_4}{4}$.
The p-Logarithmic mean: $L_p(\mu_1, \mu_2) = \left(\frac{\mu_2^{p+1}-\mu_1^{p+1}}{(p+1)(\mu_2-\mu_1)}\right)^{\frac{1}{p}}$, $\mu_1, \mu_2 > 0, \mu_1 \neq \mu_2$ and $p \in \mathbb{R}\setminus\{-1, 0\}$.

Proposition 4. *Let $0 < \mu_1 < \mu_2$ be real numbers, then we have*

$$\left|A\left(\mu_1^2, \mu_2^2\right) + A^2(\mu_1, \mu_2) + 2A^2(\mu_1, \mu_1, \mu_1, \mu_2) + 2A^2(\mu_1, \mu_2, \mu_2, \mu_2) - 6L_2^2(\mu_1, \mu_2)\right| \leq \frac{5}{12}\left(\mu_2^2 - \mu_1^2\right).$$

Proof. It suffices to apply inequality (2.8) of Corollary 3, to the function $f(x) = x^2$. □

Proposition 5. *Let $0 < \mu_1 < \mu_2$ and $q \geq 1$, be real numbers, then we have*

$$\left|A\left(\mu_1^2, \mu_2^2\right) + A^2(\mu_1, \mu_2) + 2A^2(\mu_1, \mu_1, \mu_1, \mu_2) + 2A^2(\mu_1, \mu_2, \mu_2, \mu_2) - 6L_2^2(\mu_1, \mu_2)\right| \leq \frac{5(\mu_2-\mu_1)\left(\mu_2^2-\mu_1^2\right)}{8}.$$

Proof. It suffices to apply Theorem 6, to the function $\mathcal{B}(x) = x^3$. □

5. Conclusion

In this study, we have considered the Bullen-Simpson type integral inequalities. We have proved a new integral identity. Based on this identity, we have established some new Bullen-Simpson type inequalities for functions whose first derivatives are s-convex. We have also discussed the cases when the first derivatives are bounded as well as Hölderian, and we derived some special cases. Some applications to numerical quadratures and inequalities involving means are provided at the end.

References

[1] T. Abdeljawad, S. Rashid, Z. Hammouch, İ. İşcan and Y.-M. Chu, *Some new Simpson-type inequalities for generalized p-convex function on fractal sets with applications*, Advances in Difference Equations, 2020 (2020), Article ID: 496, 1–26.

[2] M. A. Ali, C. S. Goodrich and H. Budak, *Some new parameterized Newton-type inequalities for differentiable functions via fractional integrals*, Journal of Inequalities and Applications, 2023 (2023), Article ID: 49, 1–17.

[3] M. Alomari, M. Darus, S. S. Dragomir and P. Cerone, *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*, Applied Mathematics Letters, 23 (9) (2010), 1071–1076.

[4] M. W. Alomari, M. Darus and U. S. Kirmaci, *Some inequalities of Hermite-Hadamard type for s-convex functions*, Acta Mathematica Scientia. Series B. English Edition, 31 (4) (2011), 1643–1652.

[5] M. W. Alomari and S. S. Dragomir, *Various error estimations for several Newton-Cotes quadrature formulae in terms of at most first derivative and applications in numerical integration*, Jordan Journal of Mathematics and Statistics, 7 (2) (2014), 89–108.

[6] M. U. Awan, M. A. Noor, T. Du and K. I. Noor, *New refinements of fractional Hermite-Hadamard inequality*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemàticas (RACSAM), 113 (1) (2019), 21–29.

[7] M. U. Awan, N. Akhtar, A. Kashuri, M. A. Noor and Y.-M. Chu, *2D approximately reciprocal ρ-convex functions and associated integral inequalities*, AIMS Mathematics, 5 (5) (2020), 4662–4680.

[8] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen* (in German), Publications de l'Institut Mathématique (Beograd) (N.S.), 23 (37) (1978), 13–20.

[9] M. Djenaou and B. Meftah, *Milne type inequalities for differentiable s-convex functions*, Honam Mathematical Journal, 44 (3) (2022), 325–338.

[10] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Applied Mathematics Letters, 11 (5) (1998), 91–95.

[11] S. S. Dragomir and S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense, Demonstratio Mathematica, 32 (4) (1999), 687–696.

[12] T. Du, C. Luo and Z. Cao, *On the Bullen-type inequalities via generalized fractional integrals and their applications*, Fractals, 29 (07) (2021), Article ID: 2150188.

[13] I. Franjić and J. Pečarić, *On corrected Bullen-Simpson's 3/8 inequality*, Tamkang Journal of Mathematics, 37 (2) (2006), 135–148.

[14] F. Hezenci and H. Budak, *Novel results on trapezoid-type inequalities for conformable fractional integrals*, Turkish Journal of Mathematics, 47 (2) (2023), 425–438.

[15] S.-R. Hwang, K.-L. Tseng and K.-C. Hsu, *New inequalities for fractional integrals and their applications*, Turkish Journal of Mathematics, 40 (3) (2016), 471–486.

[16] H. Kara, H. Budak and A. O. Akdemir, *Some new parameterized inequalities based on Riemann-Liouville fractional integrals*, Filomat 37 (23) (2023), 7867–7880.

[17] A. Kashuri, B. Meftah and P.O. Mohammed, Some weighted Simpson type inequalities for differentiable s-convex functions and their applications, Journal of Fractional Calculus and Nonlinear Systems, 1 (1) (2020), 75–94.

[18] A. Kashuri, B. Meftah, P. O. Mohammed, A. A. Lupa, B. Abdalla, Y. S. Hamed and T. Abdeljawad, *Fractional weighted Ostrowski type inequalities and their applications*, Symmetry, 13 (6) (2021), ArticleID: 968, 1–20.

[19] A. Lakhdari, W. Saleh, B. Meftah and A. Iqbal, *Corrected Dual-Simpson-Type Inequalities for Differentiable Generalized Convex Functions on Fractal Set*, Fractal and Fractional, 6 (12) (2022), Article ID: 710, 1–17.

[20] N. Laribi and B. Meftah, 3/8-*Simpson type inequalities for functions whose modulus of first derivatives and its q-th powers are s-convex in the second sense*, Jordan Journal of Mathematics and Statistics, 16 (1) (2023), 79–98.

[21] C. Luo and T. Du, *Generalized Simpson type inequalities involving Riemann-Liouville fractional integrals and their applications*, Filomat, 34 (3) (2020), n751–760.

[22] L. Mahmoudi and B. Meftah, *Parameterized Simpson-like inequalities for differential s-convex functions*, Analysis (Berlin), 43 (1) (2023), 59–70.

[23] B. Meftah, *Ostrowski inequalities for functions whose first derivatives are logarithmically preinvex*, Chinese Journal of Mathematics, 2016 (2016), Article ID: 5292603, 1–10.

[24] B. Meftah, *Ostrowski's inequalities for functions whose first derivatives are s-logarithmically preinvex in the second sense*, Mathematica Moravica, 22 (2) (2018), 11–28.

[25] B. Meftah, M. Merad, N. Ouanas, A. Souahi, *Some new Hermite-Hadamard type inequalities for functions whose nth derivatives are convex*, Acta et Commentationes Universitatis Tartuensis de Mathematica, 23 (2) (2019), 163–178.

[26] B. Meftah, *Fractional Hermite-Hadamard type integral inequalities for functions whose modulus of derivatives are co-ordinated* log-*preinvex*, Punjab University Journal of Mathematics, 51 (2) (2019), 21–37.

[27] B. Meftah and K. Mekalfa, Some weighted trapezoidal inequalities for differentiable log-convex functions, Journal of Interdisciplinary Mathematics, 23 (2020), 1–13.

[28] B. Meftah and A. Souahi, *Some weighted Ostrowski-type inequalities for differentiable preinvex functions*, Mathematical Methods in the Applied Sciences, 44 (18) (2021), 14892–14914.

[29] B. Meftah, A. Souahi and M. Merad, *Some local fractional Maclaurin type inequalities for generalized convex functions and their applications*, Chaos, Solitons and Fractals, 162 (2022), Article ID: 112504, 1–7.

[30] B. Meftah, H. Boulares, R. Shafqat, A. Ben Makhlouf and R. Benaicha, *Some new fractional weighted Simpson type inequalities for functions whose first derivatives are convex*, Mathematical Problems in Engineering, 2023 (2023), 1–19.

[31] B. Meftah and A. Lakhdari, Dual Simpson type inequalities for multiplicatively convex functions, Filomat, 37 (22) (2023), 7673–7683.

[32] J. Nasir, S. Qaisar, S. I. Butt, K. A. Khan and R. M. Mabela, *Some Simpson's Riemann-Liouville fractional integral inequalities with applications to special functions*, Journal of Function Spaces, 2022 (2022), Article ID: 2113742, 1–12.

[33] M. A. Noor, K. I. Noor and M. U. Awan, Fractional Ostrowski inequalities for s-Godunova-Levin functions, International Journal of Analysis and Applications, 5 (2) (2014), 167–173.

[34] M. A. Noor, K. I. Noor, S. Iftikhar and M. U. Awan, *Strongly generalized harmonic convex functions and integral inequalities*, Journal of Mathematical Analysis and Applications, 7 (3) (2016), 66–77.

[35] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, 187, Academic Press, Inc., Boston, MA, 1992.

[36] M. Rostamian Delavar, A. Kashuri and M. De La Sen, *On Weighted Simpson's* 3/8 *Rule*, Symmetry, 13 (10) (2021), Article ID: 1933.

[37] W. Saleh, A. Lakhdari, T. Abdeljawad and B. Meftah, *On fractional biparameterized Newton-type inequalities*, Journal of Inequalities and Applications, 2023 (2023), Article ID: 122, 1–18.

[38] M. Z. Sarikaya, E. Set and M. E. Özdemir, On new inequalities of Simpson's type for convex functions, RGMIA Research Report Collection, 13 (2) (2010), Article ID: 2, 1–8.

[39] M. Z. Sarikaya, E. Set and M. E. Ozdemir, *On new inequalities of Simpson's type for s-convex functions*, Computers & Mathematics with Applications, 60 (8) (2010), 2191–2199.

[40] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Başak, *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, 57 (9-10) (2013), 2403-2407.

[41] H. Xu, A. Lakhdari, W. Saleh and B. Meftah, *Some new parametrized inequalities on fractal set*, Fractals, 32 (3) (2024), Article ID: 2450063.

[42] W. S. Zhu, B. Meftah, H. Xu, F. Jarad and A. Lakhdari, *On parameterized inequalities for fractional multiplicative integrals*, Demonstratio Mathematica, 57 (1) (2024), Article ID: 20230155, 1–17.

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