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On a generalization of the Gadovan numbers

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Abstract. In this paper, we define $(k,l)-$Gadovan numbers. We give the Binet-like formula, the generating functions, the exponential generating function of the $(k,l)-$ Gadovan numbers. Also, we derive Cassini-like identity, Catalan-like identity, Vajda-like identity, Honsberger-like identity and D'ocagne-like identity for the $(k,l)-$Gadovan numbers.

1. Introduction

First, Fibonacci numbers were studied and it was seen that Fibonacci numbers can find application in nature, many fields of mathematics and other sciences [8–10, 12, 14, 20].

Then many generalizations of Fibonacci numbers have been given. Thus, new number sequences were defined [1, 14, 16]. The relation of these number with Fibonacci sequence has been given. Thus, the newly defined number sequences can indirectly find applications in nature, in almost every branch of mathematics and in other sciences, which has led to an increased interest in number sequences [17–19].

One of these generalized numbers is the Gadovan numbers [4].
The Gadovan numbers are defined by

$$GP_{n+3} = GP_{n+1} + GP_n, \quad n \geq 0,$$

with $GP_0 = a$, $GP_1 = b$ and $GP_2 = c$, where a, b, c.

Diskaya and Menken defined the Gadovan numbers, which generalizes a new class of Padovan numbers [4].

Now, before describing the generalization of Gadavon numbers, let us give the sequences of numbers that we can derive using this generalization.

The Padovan numbers are defined by

$$P_{n+3} = P_{n+1} + P_n, \quad n \geq 0,$$

with $P_0 = 1$, $P_1 = 0$ and $P_2 = 1$.

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The Padovan polynomials are defined by

$$P_{n+3}(x) = xP_{n+1}(x) + P_n(x), \quad n \geq 0,$$

with $P_0(x) = 1$, $P_1(x) = 0$ and $P_2(x) = x$ [5].

The bivariate Padovan polynomials are defined by

$$P_{n+3}(x, y) = xP_{n+1}(x, y) + yP_n(x, y), \quad n \geq 0,$$

with $P_0(x, y) = 1$, $P_1(x, y) = 0$ and $P_2(x, y) = x$ [6].

Padovan numbers and their generalizations have been studied by many researchers [2, 3, 7, 8, 12, 21–25].

The Fibonacci-Pell numbers are defined as follows

$$F_{n+3} = F_{n+1} + F_n, \quad n \geq 0,$$

with $F_0 = 1$, $F_1 = 0$ and $F_2 = 2$ [22].

The Lucas-Pell numbers are defined as follows

$$B_{n+3} = B_{n+1} + B_n, \quad n \geq 0,$$

with $B_0 = 3$, $B_1 = 0$ and $B_2 = 4$ [22].

The Gaussian Padovan numbers are defined by

$$GP_n = GP_{n-2} + GP_{n-3}, \quad n \geq 3,$$

with $GP_0 = 1$, $GP_1 = 1 + i$ and $GP_2 = 1 + i$ [24].

The Perrin numbers are as follows

$$R_{n+3} = R_{n+1} + R_n, \quad n \geq 0,$$

with $R_0 = 3$, $R_1 = 0$ and $R_2 = 2$ [21].

In this paper, we present the generalization of Gadovan numbers named $(k, l)-$Gadovan numbers. We give the Binet-like formula, the generating functions, the exponential generating function of the $(k, l)-$Gadovan numbers. In addition, we derive the Cassini-like identity, Catalan-like identity, Vajda-like identity and D'ocagne identity-like for these numbers.

2. $(k, l)-$ Gadovan numbers

Definition 1. We define the $(k, l)-$Gadovan numbers $\left\{GP_{(k,l),n}\right\}$ by the third order recurrence relation that

$$GP_{(k,l),n+3} = kGP_{(k,l),n+1} + lGP_{(k,l),n}, \quad n \geq 0,$$

with initial conditions $GP_{(k,l),0} = a$, $GP_{(k,l),1} = b$ and $GP_{(k,l),2} = c$.

We show some terms of the $(k, l)-$Gadovan numbers in Table 1.

TABLE 1. Some terms of the $(k, l)-$Gadovan numbers.

n	0	1	2	3	4	5	6
$GP_{(k,l),n}$	a	b	c	$kb + la$	$kc + lb$	$k^2b + kla + lc$	$k^2c + 2klb + l^2a$

Note that, the following exceptions occur:

1. For $k=1,\ l=1$, we obtain Gadovan numbers.
2. For $k=1,\ l=1,\ a=1,\ b=0,\ c=1$, we obtain the Padovan numbers.
3. For $k=x,\ l=1,\ a=1,\ b=0,\ c=x$, we obtain the Padovan polynomials.
4. For $k=x,\ l=y,\ a=1,\ b=0,\ c=x$, we obtain the Bivariate Padovan polynomials.
5. For $k=1$, $l=1$, $a=1$, $b=1+i$, $c=1+i$, we obtain the Gaussian Padovan numbers.
6. For $k=1$, $l=1$, $a=3$, $b=0$, $c=2$, we obtain the Perrin numbers.
7. For $k=1$, $l=1$, $a=0$, $b=0$, $c=1$, we obtain the Perrin-Padovan numbers.
8. For $k=1$, $l=1$, $a=1$, $b=0$, $c=2$, we obtain the Fibonacci-Pell numbers.
9. For $k=1,\ l=1,\ a=3,\ b=0,\ c=4$, we obtain the Lucas-Pell numbers.

The characteristic equation of the $(k,l)-$Gadovan numbers is

$$x^3-kx-l=0.$$

The characteristic equation has the following roots

$$x_1=\frac{\sqrt[3]{\frac{2}{3}}k}{r}+\frac{r}{\sqrt[3]{18}},$$

$$x_2=-\frac{\left(1+i\sqrt{3}\right)k}{\sqrt[3]{12}r}-\frac{\left(1-i\sqrt{3}\right)r}{2\sqrt[3]{18}r}$$

and

$$x_3=-\frac{\left(-1-i\sqrt{3}\right)k}{2\sqrt[3]{18}}-\frac{\left(1+i\sqrt{3}\right)r}{2\sqrt[3]{18}r},$$

where

$$r=\sqrt[3]{\sqrt{3}\sqrt{27l^2-4k^3}+9l}.$$

It can be easily seen that the following equations are satisfied:

$$x_1+x_2+\ x_3=0,$$

$$x_1x_2+x_2x_3+\ x_1x_3=-k,$$

$$x_1x_2x_3=l,$$

$$x_1^3=kx_1+l,\quad x_2^3=kx_2+l,\quad x_3^3=kx_3+l.$$

Theorem 1. *The Binet-like Formula for $(k,l)-$Gadovan numbers$\left\{GP_{(k,l),n}\right\}$ is*

$$GP_{(k,l),n} = Ax_1^n + Bx_2^n + Cx_3^n,$$

where

$$A = \frac{c - bx_2 - bx_3 + ax_2x_3}{x_1^3 - x_1^2x_2 - x_1^2x_3 + x_1x_2x_3},$$

$$B = \frac{c - bx_1 - bx_3 - ax_1x_3}{x_2^3 - x_2^2x_3 - x_2^2x_1 + x_1x_2x_3},$$

$$C = \frac{c - bx_1 - bx_2 - ax_1x_2}{x_3^3 - x_1x_3^2 - x_3^2x_2 + x_1x_2x_3}.$$

Proof. For $n = 0$, we have

$$GP_{(k,l),0} = a = A + B + C.$$

For $n = 1$, we have

$$GP_{(k,l),1} = b = Ax_1 + Bx_2 + Cx_3.$$

For $n = 2$, we have

$$GP_{(k,l),1} = c = Ax_1^2 + Bx_2^2 + Cx_3^2.$$

If this system of equations is solved, then the coefficients A, B and C are found.

Thus, the desired result is achieved. $\square$

Theorem 2. *The generating function of* $\left\{GP_{(k,l),n}\right\}$ *is*

$$G(x) = \frac{a + bx + (c-a)\,x^2}{1 - kx^2 - l\ x^3}.$$

Proof.

$$G(x) = \sum_{n=1}^{\infty} GP_{(k,l),n}x^n = GP_{k,0} + GP_{(k,l),1}x + GP_{(k,l),2}x^2 + GP_{(k,l),3}x^3 + \cdots + GP_{(k,l),n}x^n + \cdots \tag{1}$$

respectively multiplying both sides of this identity by kx^2 and lx^3.

$$kx^2G(x) = GP_{(k,l),0}kx^2 + GP_{(k,l),1}kx^3 + GP_{(k,l),2}kx^4 + GP_{(k,l),3}kx^5 + \cdots + GP_{(k,l),n}kx^{n+2} + \cdots \tag{2}$$

$$lx^3G(x) = GP_{(k,l),0}lx^3 + GP_{(k,l),1}lx^4 + GP_{(k,l),2}lx^5 + GP_{(k,l),3}lx^6 + \cdots + GP_{(k,l),n}lx^{n+3} + \cdots \tag{3}$$

From (1), (2) and (3), we get

$$G(x)\left(1-kx^2-l\,x^3\right)=GP_{(k,l),0}+GP_{(k,l),1}x+GP_{(k,l),2}x^2-GP_{(k,l),0}kx^2$$
$$+\left(GP_{(k,l),3}-kGP_{(k,l),1}-lGP_{(k,l),0}\right)x^3$$
$$+\left(GP_{(k,l),4}-kGP_{(k,l),2}-lGP_{(k,l),1}\right)x^4+\cdots$$

After necessary calculations and using the recurrence relation, we obtain

$$G(x)=\frac{GP_{(k,l),0}+GP_{(k,l),1}x+GP_{(k,l),2}x^2-GP_{(k,l),0}kx^2}{1-kx^2-l\,x^3},$$

$$G(x)=\frac{a+bx+(c-a)\,x^2}{1-kx^2-l\,x^3}.$$ □

Theorem 3. *The exponential generating function for* $(k,l)-$*Gadovan numbers*

$$\sum_{n=0}^{\infty}\frac{GP_{(k,l),n}x^n}{n!}=Ae^{x_1x}+Be^{x_2x}+Ce^{x_3x},$$

where $GP_{(k,l),n}=Ax_1^n+Bx_2^n+Cx_3^n$.

Proof. For the proof, we use Binet formula.

$$\sum_{n=0}^{\infty}\frac{GP_{(k,l),n}x^n}{n!}=\sum_{n=0}^{\infty}\frac{(Ax_1^n+Bx_2^n+Cx_3^n)x^n}{n!}$$
$$=A\sum_{n=0}^{\infty}\frac{(x_1x)^n}{n!}+B\sum_{n=0}^{\infty}\frac{(x_2x)^n}{n!}+C\sum_{n=0}^{\infty}\frac{(x_3x)^n}{n!}.$$

We know that

$$e^{x_1x}=\sum_{n=1}^{\infty}\frac{(x_1x)^n}{n!},$$
$$e^{x_2x}=\sum_{n=1}^{\infty}\frac{(x_2x)^n}{n!},$$
$$e^{x_3x}=\sum_{n=1}^{\infty}\frac{(x_3x)^n}{n!}.$$

So,

$$\sum_{n=0}^{\infty}\frac{GP_{(k,l),n}x^n}{n!}=Ae^{x_1x}+Be^{x_2x}Ce^{x_3x}.$$

Thus, the proof is completed. □

Theorem 4. *For* $m,n\in\mathbb{Z}^+$, *there is the following equation:*

$$\sum_{n=1}^{m=1}\binom{m}{n}k^nl^{m-n}GP_{(k,l),n}=GP_{(k,l),3m}.$$

Proof.

$$\begin{aligned}
&\sum_{n=1}^{m}\binom{m}{n}k^{n}l^{m-n}GP_{(k,l),n}\\
&=\sum_{n=1}^{m}\binom{m}{n}\left(A(kx_1)^n+B(kx_2)^n+C(kx_3)^n\right)l^{m-n}\\
&=A\sum_{n=1}^{m}\binom{m}{n}(kx_1)^n l^{m-n}+B\sum_{n=1}^{m}\binom{m}{n}(kx_2)^n l^{m-n}+C\sum_{n=1}^{m}\binom{m}{n}(kx_3)^n l^{m-n}\\
&=A(kx_1+l)^m+B(kx_2+l)^m+C(kx_3+l)^m\\
&=A{x_1}^{3m}+B{x_2}^{3m}+C{x_3}^{3m}=GP_{(k,l),3m}.
\end{aligned}$$

□

Theorem 5. *For $m,n \in \mathbb{Z}^+$, there is the following equation.*

$$\sum_{r=1}^{m}\binom{m}{r}k^{n-r}l^{m}GP_{(k,l),n-r} = GP_{(k,l),n+2m}.$$

Proof.

$$\begin{aligned}
&\sum_{r=1}^{m}\binom{m}{r}k^{n-r}l^{m}GP_{(k,l),n-r}\\
&=\sum_{r=1}^{m}\binom{m}{r}k^{n-r}\left(Ax_1^{n-r}+Bx_2^{n-r}+Cx_3^{n-r}\right)l^m\\
&=A\left[\sum_{r=1}^{m}\binom{m}{r}k^{m-r}x_1^{m-r}l^m\right]x_1^{n-m}+B\left[\sum_{r=1}^{m}\binom{m}{r}k^{m-r}x_2^{m-r}l^m\right]x_2^{n-m}\\
&\quad+C\left[\sum_{r=1}^{m=1}\binom{m}{r}k^{m-r}x_3^{m-r}l^m\right]x_3^{n-m}\\
&=A\left[\sum_{r=1}^{m}\binom{m}{r}(kx_1)^{m-r}l^m\right]x_1^{n-m}+B\left[\sum_{r=1}^{m}\binom{m}{r}(kx_2)^{m-r}l^m\right]x_2^{n-m}\\
&\quad+C\left[\sum_{r=1}^{m=1}\binom{m}{r}(kx_3)^{m-r}l^m\right]x_3^{n-m}\\
&=A(kx_1+l)^m x_1^{n-m}+B(kx_2+l)^m x_2^{n-m}+C(kx_3+l)^m x_3^{n-m}\\
&=A_1{x_1}^{n+2m}+B_1{x_2}^{n+2m}+C_1{x_3}^{n+2m}=GP_{(k,l),n+2m}.
\end{aligned}$$

Thus, the proof is obtained. □

Theorem 6 (Cassini-like Identity)**.** *For $n \geq 1$, we have,*

$$\begin{aligned}
&GP_{(k,l),n+1}GP_{(k,l),n-1}-{GP_{(k,l),n}}^2\\
&=k^{n-1}\Big(ABx_3{}^{n-1}(x_1-x_2)^2+BCx_2{}^{n-1}(x_1-x_3)^2+BCx_1{}^{n-1}(x_2-x_3)^2\Big).
\end{aligned}$$

Proof. For the proof, we use the Binet-like formula.

$$
\begin{aligned}
& GP_{(k,l),n+1}GP_{(k,l),n-1} - {GP_{(k,l),n}}^2 \\
&= \left(Ax_1^{n+1} + Bx_2^{n+1} + Cx_3^{n+1}\right)\left(Ax_1^{n-1} + Bx_2^{n-1} + Cx_3^{n-1}\right) \\
&\quad - \left(Ax_1^{n} + Bx_2^{n} + Cx_3^{n}\right)^2 \\
&= A^{\mathbf{2}}x_1^{2n} + ABx_1^{n+1}x_2^{n-1} + ACx_1^{n+1}x_3^{n-1} + BAx_2^{n+1}x_1^{n-1} + B^2x_2^{2n} \\
&\quad + BCx_2^{n+1}x_3^{n-1} + CAx_3^{n+1}x_1^{n-1} + CBx_3^{n+1}x_2^{n-1} + C^2x_3^{2n} \\
&\quad - A^{\mathbf{2}}x_1^{2n} - ABx_1^nx_2^n - ACx_3^nx_1^n - BAx_1^nx_2^n - B{B_1}^2x_2^{2n} \\
&\quad - BCx_3^nx_2^n - CAx_1^nx_3^n - CBx_2^nx_3^n - C^2x_3^{2n} \\
&= ABx_1^nx_2^n\left(\frac{x_1}{x_2} + \frac{x_2}{x_1} - 2\right) + ACx_1^nx_3^n\left(\frac{x_1}{x_3} + \frac{x_3}{x_1} - 2\right) \\
&\quad + BCx_2^nx_3^n\left(\frac{x_2}{x_3} + \frac{x_3}{x_2} - 2\right) + AB(x_1x_2)^{n-1}(x_1 - x_2)^2 \\
&\quad + AC(x_1x_3)^{n-1}(x_1 - x_3)^2 + BC(x_2x_3)^{n-1}(x_2 - x_3)^2 \\
&\quad + ABk^{n-1}{x_3}^{n-1}(x_1 - x_2)^2 + ACk^{n-1}{x_2}^{n-1}(x_1 - x_3)^2 \\
&\quad + BCk^{n-1}{x_1}^{n-1}(x_2 - x_3)^2 \\
&= k^{n-1}\left(ABx_3{}^{n-1}(x_1 - x_2)^2 + ACx_2{}^{n-1}(x_1 - x_3)^2 + BCx_1{}^{n-1}(x_2 - x_3)^2\right).
\end{aligned}
$$

So, the desired is obtained. □

Theorem 7. *(Catalan-like Identity) For $n \geq t$, we have,*

$$
\begin{aligned}
GP_{(k,l),n+t}GP_{(k,l),n-t} - {GP_{(k,l),n}}^2 = k^{n-t}\Big(&ABx_3{}^{n-t}\big({x_1}^t - {x_2}^t\big)^2 \\
&+ ACx_2{}^{n-t}\big({x_1}^t - {x_3}^t\big)^2 + BCx_1{}^{n-t}\big({x_2}^t - {x_3}^t\big)^2\Big).
\end{aligned}
$$

Proof. For the proof, we use the Binet-like formula.

$$
\begin{aligned}
& GP_{(k,l),n+t}GP_{(k,l),n-t} - {GP_{(k,l),n}}^2 \\
&= \left(Ax_1^{n+t} + Bx_2^{n+t} + Cx_3^{n+t}\right)\left(Ax_1^{n-t} + Bx_2^{n-t} + Cx_3^{n-t}\right) \\
&\quad - \left(Ax_1^{n} + Bx_2^{n} + Cx_3^{n}\right)^2 \\
&= A^{\mathbf{2}}x_1^{2n} + ABx_1^{n+t}x_2^{n-t} + ACx_1^{n+t}x_3^{n-t} + BAx_2^{n+t}x_1^{n-t} + B^2x_2^{2n} \\
&\quad + BCx_2^{n+t}x_3^{n-t} + CAx_3^{n+t}x_1^{n-t} + CBx_3^{n+t}x_2^{n-t} + C^2x_3^{2n} \\
&\quad - A^{\mathbf{2}}x_1^{2n} - ABx_1^nx_2^n - ACx_3^nx_1^n - BAx_1^nx_2^n - B^2x_2^{2n} \\
&\quad - BCx_3^nx_2^n - CAx_1^nx_3^n - CBx_2^nx_3^n - C^2x_3^{2n} \\
&= ABx_1^nx_2^n\left(\frac{{x_1}^t}{{x_2}^t} + \frac{{x_2}^t}{{x_1}^t} - 2\right) + ACx_1^nx_3^n\left(\frac{{x_1}^t}{{x_3}^t} + \frac{{x_3}^t}{{x_1}^t} - 2\right) \\
&\quad + BCx_2^nx_3^n\left(\frac{{x_2}^t}{{x_3}^t} + \frac{{x_3}^t}{{x_2}^t} - 2\right) + AB(x_1x_2)^{n-t}\big({x_1}^t - {x_2}^t\big)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ AC(x_1x_3)^{n-t}\left({x_1}^t - {x_3}^t\right)^2 + BC(x_2x_3)^{n-t}\left({x_2}^t - {x_3}^t\right)^2 \\
&+ ABk^{n-t}{x_3}^{n-t}\left({x_1}^t - {x_2}^t\right)^2 + ACk^{n-t}{x_2}^{n-1}\left({x_1}^t - {x_3}^t\right)^2 \\
&+ BCk^{n-t}{x_1}^{n-t}\left({x_2}^t - {x_3}^t\right)^2 \\
= {}& k^{n-t}\Big(AB{x_3}^{n-t}\left({x_1}^t - {x_2}^t\right)^2 + AC{x_2}^{n-t}\left({x_1}^t - {x_3}^t\right)^2 \\
&+ BC{x_1}^{n-t}\left({x_2}^t - {x_3}^t\right)^2\Big).
\end{aligned}
$$

So, the desired is obtained.

If we take $k = 1$, we get the Cassini-like identity. □

Theorem 8 (D'ocagne-like Identity)**.** *Let n and m be any integers. Then the following identity is true.*

$$
\begin{aligned}
GP_{(k,l),m+1}GP_{(k,l),n} - GP_{(k,l),m}GP_{(k,l),n+1} &= AB\,(x_1 - x_2)\,(x_1^m x_2^n - x_2^m x_1^n) \\
+ AC\,(x_1 - x_3)\,(x_1^m x_3^n - x_3^m x_1^n) &+ BC\,(x_2 - x_3)\,(x_2^m x_3^n - x_3^m x_2^n)\,.
\end{aligned}
$$

Proof. For the proof, we use the Binet-like formula.

$$
\begin{aligned}
&GP_{(k,l),m+1}GP_{(k,l),n} - GP_{(k,l),m}GP_{(k,l),n+1} \\
={}& \left(Ax_1^{m+1} + Bx_2^{m+1} + Cx_3^{m+1}\right)\left(Ax_1^n + Bx_2^n + Cx_3^n\right) \\
&- \left(Ax_1^m + Bx_2^m + Cx_3^m\right)\left(Ax_1^{n+1} + Bx_2^{n+1} + Cx_3^{n+1}\right) \\
={}& A^{\mathbf{2}}x_1^{m+n+1} + ABx_1^{m+1}x_2^n + ACx_1^{m+1}x_3^n + BAx_2^{m+1}x_1^n + B^2x_2^{m+n+1} \\
&+ BCx_2^{m+1}x_3^n + CAx_3^{m+1}x_1^n + CBx_3^{m+1}x_2^n + C^2x_3^{m+n+1} \\
&- A^{\mathbf{2}}x_1^{m+n+1} - ABx_1^m x_2^{n+1} - ACx_1^m x_3^{n+1} - BAx_2^m x_1^{n+1} - B^2x_2^{m+n+1} \\
&- BCx_2^m x_3^{n+1} - CAx_3^m x_1^{n+1} - CBx_3^m x_2^{n+1} - C^2x_3^{m+n+1} \\
={}& ABx_1^m x_2^n\,(x_1 - x_2) + BAx_2^m x_1^n\,(x_2 - x_1) + ACx_1^m x_3^n\,(x_1 - x_3) \\
&+ CAx_3^m x_1^n\,(x_3 - x_1) + BCx_2^m x_3^n\,(x_2 - x_3) + CBx_3^m x_2^n\,(x_3 - x_2) \\
={}& AB\,(x_1 - x_2)\,(x_1^m x_2^n - x_2^m x_1^n) + AC\,(x_1 - x_3)\,(x_1^m x_3^n - x_3^m x_1^n) \\
&+ BC\,(x_2 - x_3)\,(x_2^m x_3^n - x_3^m x_2^n)\,.
\end{aligned}
$$

So, the proof is complete. □

Theorem 9 (Honsberger-like Identity)**.** *Let n and m be any integers. Then the following identity is true.*

$$
\begin{aligned}
&GP_{(k,l),m}GP_{(k,l),n} + GP_{(k,l),m*1}GP_{(k,l),n+1} \\
={}& A^2x_1^{m+n}\left(1 + x_1^2\right) + B^2x_2^{m+n}\left(1 + x_2^2\right) + C^2x_3^{m+n}\left(1 + x_3^2\right) \\
&+ AB\,(1 + x_1x_2)\,(x_1^n x_2^m + x_2^n x_1^m) \\
&+ AC\,(1 + x_1x_3)\,(x_1^m x_3^n + x_3^m x_1^n) \\
&+ BC\,(1 + x_2x_3)\,(x_2^m x_3^n + x_3^m x_2^n)\,.
\end{aligned}
$$

Proof. For the proof, we use the Binet-like formula.

$$
\begin{aligned}
&GP_{(k,l),m}GP_{(k,l),n} + GP_{(k,l),m*1}GP_{(k,l),n+1} \\
&= \left(Ax_1^m + Bx_2^m + Cx_3^m\right)\left(Ax_1^n + Bx_2^n + Cx_3^n\right) \\
&\quad + \left(Ax_1^{m+1} + Bx_2^{m+1} + Cx_3^{m+1}\right)\left(Ax_1^{n+1} + Bx_2^{n+1} + Cx_3^{n+1}\right) \\
&= \mathbf{A^2}x_1^{m+n} + ABx_1^m x_2^n + ACx_1^m x_3^n + BAx_2^m x_1^n + B^2 x_2^{m+n} \\
&\quad + BCx_2^m x_3^n + CAx_3^m x_1^n + CBx_3^m x_2^n + C^2 x_3^{m+n} \\
&\quad + \mathbf{A^2}x_1^{m+n+2} + ABx_1^{m+1}x_2^{n+1} + ACx_1^{m+1}x_3^{n+1} + BAx_2^{m+1}x_1^{n+1} + B^2 x_2^{m+n+2} \\
&\quad + BCx_2^{m+1}x_3^{n+1} + CAx_3^{m+1}x_1^{n+1} + CBx_3^{m+1}x_2^{n+1} + C^2 x_3^{m+n+2} \\
&= \mathbf{A^2}x_1^{m+n}\left(1 + x_1^2\right) + B^2 x_2^{m+n}\left(1 + x_2^2\right) + C^2 x_3^{m+n}\left(1 + x_3^2\right) \\
&\quad + AB\left(1 + x_1x_2\right)\left(x_1^n x_2^m + x_2^n x_1^m\right) + AC\left(1 + x_1x_3\right)\left(x_1^m x_3^n + x_3^m x_1^n\right) \\
&\quad + BC\left(1 + x_2x_3\right)\left(x_2^m x_3^n + x_3^m x_2^n\right). \qquad \square
\end{aligned}
$$

Theorem 10 (Vajda-like Identity)**.** *Let n and m be any integers. Then the following identity is true.*

$$
\begin{aligned}
&GP_{(k,l),n+m}GP_{(k,l),n+r} - GP_{(k,l),n}GP_{(k,l),n+m+r} \\
&= k^n\Big[ABx_3{}^{-n}\left(x_1{}^m - x_2{}^m\right)\left(x_2^r - x_1^r\right) + ACx_2^{-n}\left(x_1^m - x_3{}^m\right)\left(x_3^r - x_1^r\right) \\
&\qquad + BCx_1^{-n}\left(x_2{}^m - x_3{}^m\right)\left(x_3^r - x_2^r\right)\Big].
\end{aligned}
$$

Proof. For the proof, we use the Binet-like formula.

$$
\begin{aligned}
&GP_{(k,l),n+m}GP_{(k,l),n+r} - GP_{(k,l),n}GP_{(k,l),n+m+r} \\
&= \left(Ax_1^{n+m} + Bx_2^{n+m} + Cx_3^{n+m}\right)\left(Ax_1^{n+r} + Bx_2^{n+r} + Cx_3^{n+r}\right) \\
&\quad - \left(Ax_1^n + Bx_2^n + Cx_3^n\right)\left(Ax_1^{n+m+r} + Bx_2^{n+m+r} + Cx_3^{n+m+r}\right) \\
&= \mathbf{A^2}x_1^{2n+m+r} + ABx_1^{n+m}x_2^{n+r} + ACx_1^{n+m}x_3^{n+r} \\
&\quad + BAx_2^{n+m}x_1^{n+r} + B^2 x_2^{2n+m+r} + BCx_2^{n+m}x_3^{n+r} \\
&\quad + CAx_3^{n+m}x_1^{n+r} + CBx_3^{n+m}x_2^{n+r} + C^2 x_3^{2n+m+r} \\
&\quad - \mathbf{A^2}x_1^{2n+m+r} - ABx_1^n x_2^{n+m+r} - ACx_1^n x_3^{n+m+r} \\
&\quad - BAx_2^n x_1^{n+m+r} - B^2 x_2^{2n+m+r} - BCx_2^n x_3^{n+m+r} \\
&\quad - CAx_3^n x_1^{n+m+r} - CBx_3^n x_2^{n+m+r} - C^2 x_3^{2n+m+r} \\
&= ABx_1^n x_2^{n+r}\left(x_1^m - x_2^m\right) + BAx_2^n x_1^{n+r}\left(x_2^m - x_1^m\right) \\
&\quad + ACx_1^n x_3^{n+r}\left(x_1^m - x_3^m\right) + CAx_3^n x_1^{n+r}\left(x_3^m - x_1^m\right) \\
&\quad + BCx_2^n x_3^{n+r}\left(x_2^m - x_3^m\right) + CBx_3^n x_2^{n+r}\left(x_3^m - x_2^m\right) \\
&= AB\left(x_1{}^m - x_2{}^m\right)x_1^n x_2^n\left(x_2^r - x_1^r\right) + AC\left(x_1^m - x_3{}^m\right)x_1^n x_3^n\left(x_3^r - x_1^r\right) \\
&\quad + BC\left(x_2{}^m - x_3{}^m\right)x_2^n x_3^n\left(x_3^r - x_2^r\right).
\end{aligned}
$$

$$= k^n \left[ABx_3{}^{-n} \left(x_1{}^m - x_2{}^m\right) \left(x_2^r - x_1^r\right) + ACx_2^{-n} \left(x_1^m - x_3{}^m\right) \left(x_3^r - x_1^r\right) \right. \\ \left. + BCx_1^{-n} \left(x_2{}^m - x_3{}^m\right) \left(x_3^r - x_2^r\right) \right].$$

So, the proof is complete. □

3. CONCLUSION

We introduced the $(k, l)-$Gadovan numbers and some identities of the $(k, l)-$Gadovan numbers. We obtained relation to some numbers, binomial sums, the generating functions. We obtained the Cassini-like, Catalan-like, Vajda-like, Honsberger-like and D'ocagne-like identities, which are important identities related to number sequences.

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