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Trigonometric approximation of periodic functions in Morrey spaces using matrix means of their Fourier series

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Abstract. We examine the generalized methods of summability of Fourier series of functions belonging to Morrey spaces $L^{p,\lambda}$, $0<\lambda\leq 2$, $1<p<\infty$. In this study, the approximation of functions by matrix means in terms of the continuity modulus in Morrey spaces $L^{p,\lambda}$, $0<\lambda\leq 2$, $1<p<\infty$, is investigated.

1. Introduction and main results

Let $\mathbb{T}$ denote the interval $[0,2\pi]$. Let $L^p(\mathbb{T})$, $1\leq p<\infty$ be the Lebesgue space of all measurable $2\pi-$periodic functions defined on $\mathbb{T}$ such that

$$\|f\|_p := \left(\int_{\mathbb{T}} |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty.$$

The *Morrey spaces* $L_0^{p,\lambda}(\mathbb{T})$ for a given $0\leq\lambda\leq 2$ and $p\geq 1$, we define as the set of functions $f\in L_{loc}^p(\mathbb{T})$ such that

$$\|f\|_{L_0^{p,\lambda}(\mathbb{T})} := \left\{\sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}}\int_I |f(t)|^p\,dt\right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all intervals $I\subset[0,2\pi]$. Note that $L_0^{p,\lambda}(\mathbb{T})$ becomes a Banach spaces, where $\lambda=2$ coincides with $L^p(\mathbb{T})$ and for $\lambda=0$ with $L^\infty(\mathbb{T})$. If $0\leq\lambda_1\leq\lambda_2\leq 2$, then $L_0^{p,\lambda_1}(\mathbb{T})\subset L_0^{p,\lambda_2}(\mathbb{T})$. Also, if $f\in L_0^{p,\lambda}(\mathbb{T})$, then $f\in L^p(\mathbb{T})$ and hence $f\in L^1(\mathbb{T})$. The Morrey spaces, were introduced by C. B. Morrey in 1938 [25]. The properties of the these spaces have been investigated intensively by several authors and

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together with weighted Lebesgue spaces L^p_ω play an important role in the theory of partial equations, in the study of the local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces L^p. The detailed information about properties of the Morrey spaces can be found in references [7, 8, 9, 13, 15, 26, 27].

Denote by $C^\infty(\mathbb{T})$ the set of all functions that are realized as the restriction to $\mathbb{T}$ of elements in $C^\infty(\mathbb{T})$. Also we define $L^{p,\lambda}(\mathbb{T})$ to be closure of $C^\infty(\mathbb{T})$ in $L_0^{p,\lambda}(\mathbb{T})$. $L^{p,\lambda}(\mathbb{T})$ is modified Morrey spaces which contains the set of trigonometric polynomials as a dense subset.

We define Steklov means f_h by

$$f_h(x) := \frac{1}{2h}\int\limits_{-h}^{h} f(x+t)\,dt, \quad 0 < h < \pi,\ x \in \mathbb{T}.$$

According to [9] the inequality

$$\|f_h\|_{L^{p,\lambda}(\mathbb{T})} \le c\,\|f\|_{L^{p,\lambda}(\mathbb{T})}$$

holds. Hence the operator f_h is bounded in the space $L^{p,\lambda}(\mathbb{T})$, $0 \le \lambda \le 2$ and $p \ge 1$.

The function

$$\Omega_{p,\lambda}(\delta, f) := \sup_{|h|\le\delta} \|f - f_h\|_{L^{p,\lambda}(\mathbb{T})}, \quad \delta > 0$$

is called the *modulus of continuity* of $f \in L^{p,\lambda}(\mathbb{T})$, $0 \le \lambda \le 2$ and $p \ge 1$.

The modulus of continuity $\Omega_{p,\lambda}(\delta, f)$ is a nondecreasing, nonnegative, continuous function and

$$\Omega_{p,\lambda}(\delta, f+g) \le \Omega_{p,\lambda}(\delta, f) + \Omega_{p,\lambda}(\delta, g)$$

for $f, g \in L^{p,\lambda}(\mathbb{T})$, $0 \le \lambda \le 2$ and $p \ge 1$.

We define the following class of functions

$$Lip_{p,\lambda}(\omega, M) = \left\{ f \in L^{p,\lambda}(\mathbb{T}) : \Omega_{p,\lambda}(\delta, f) \le M\omega(\delta),\ \delta > 0 \right\},$$

where ω is a function of modulus of continuity type on the interval $[0, 2\pi]$, i.e., a nondecreasing continious function having the following properties $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2)$ for any $0 \le \delta_1 \le \delta_2 \le \delta_1 + \delta_2 \le 2\pi$ and M is some positive constant.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x, f) \tag{1}$$

be Fourier series of the function $f \in L_1(\mathbb{T})$, where

$$A_k(x, f) := (a_k(f)\cos kx + b_k(f)\sin kx),$$

$a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f \in L_1(\mathbb{T})$.

The n-th *partial sums* of the series (1) is defined as

$$S_n(x,f)=\frac{a_0}{2}+\sum_{\nu=1}^{n}A_\nu(x,f),$$

The best approximation to $f\in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$, $1<p<\infty$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{L^{p,\lambda}(\mathbb{T})}:=\inf\left\{\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}:T_n\in\prod\nolimits_n\right\}.$$

Let $A:=(a_{n,k})_{0\leq k,n<\infty}$ be an infinite matrix of the real numbers such that

$$a_{n,k}\geq 0 \text{ when } k,n=0,1,2,\ldots,\ \lim_{n\to\infty}a_{n,k}=0 \text{ and } \sum_{k=0}^{\infty}a_{n,k}=1$$

or $A_0:=(a_{n,k})_{0<k,\leq n<\infty}$, where

$$a_{n,k}=0 \text{ when } k>n,$$

then

$$T_{n,A}(x,f):=\sum_{k=0}^{\infty}a_{n,k}S_k(x,f),\quad n=0,1,2,\ldots$$

or

$$T_{n,A_0}(x,f):=\sum_{k=0}^{n}a_{n,k}S_k(x,f),\quad n=0,1,2,\ldots$$

respectively.

We will use the relation $f=O(g)$ which, means that $f\leq cg$ for a constant c independent of f and g.

The approximation of the functions by trigonometric polynomials in non-weighted and weighted Morrey spaces has been studied by several authors (see, for example, [6, 10,18, 20, 23, 30]). In this study, we investigate the approximation of functions using matrix means in terms of continuity modulus in Morrey spaces $L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ and $1<p<\infty$. These approximations were first obtained by W. Lenski and B. Szal [24] in Lebesgue spaces with variable exponents $L_{2\pi}^{p(x)}$ with $p(x)\geq 1$. Similar results in different spaces were studied by several authors (see, for example, [1-5, 11,12, 16, 19, 21, 22,29, 31]).

Note that in this study, to prove main results the methods used in [17], [24] and [28] has been followed.

Our main results are the following.

Theorem 1. *Let* $f \in L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leq 2$ *and* $1 < p < \infty$. *If the conditions*

$$\sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| = O\left(\frac{1}{n+1}\right) \tag{2}$$

for some $\beta \geq 0$ *and*

$$\sum_{k=0}^{\infty} (k+1)\, a_{n,k} = O(n+1) \tag{3}$$

hold, then the inequality

$$\|T_{n,A}(\cdot, f) - f\|_{L^{p,\lambda}(\mathbb{T})} = O\left(\Omega_{p,\lambda}(f, \frac{1}{n+1}) + \sum_{k=0}^{n} a_{n,k}\Omega_{p,\lambda}(, f, \frac{1}{k+1})\right)$$

holds.

Theorem 2. *Let* $f \in L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leq 2$ *and* $1 < p < \infty$. *If the conditions*

$$\sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k}+1}{(k+2)^{\beta}} \right| = O(a_{n,n}) \tag{4}$$

for some $\beta \geq 0$ *and*

$$(n+1)\, a_{n,n} = O(1) \tag{5}$$

hold, then the inequality

$$\|T_{n,A_0}(\cdot, f) - f\|_{L^{p,\lambda}(\mathbb{T})} = O\left(\sum_{k=0}^{n} a_{n,k}\Omega_{p,\lambda}(, f, \frac{1}{k+1})\right) \tag{6}$$

holds.

The following theorems hold specifically for the class of functions $Lip_{p,\lambda}(\omega, M)$.

Theorem 3. *Let* $f \in Lip_{p,\lambda}(\omega, M)$. *If the conditions* (2) *for some* $\beta > 0$ *and* (3) *hold, then the estimation*

$$\|T_{n,A}(\cdot, f) - f\|_{L^{p,\lambda}(\mathbb{T})} = O\left(\omega\Big(\frac{1}{n+1}\Big)\right) \tag{7}$$

holds.

Theorem 4. *Let* $f \in Lip_{p,\lambda}(\omega, M)$. *If the conditions* (4) *for some* $\beta > 0$ *and* (5) *hold, then the estimation*

$$\|T_{n,A_0}(\cdot, f) - f\|_{L^{p,\lambda}(\mathbb{T})} = O\left(\omega\Big(\frac{1}{n+1}\Big)\right)$$

holds.

Note that $a_{n,k} = e^{-n}\sum_{j=k}^{\infty}\frac{n^j}{(j+1)!}$ where $n,k = 0,1,2,\ldots$, satisfies the condition (2) for any $\beta \geq 0$ and (3). Also $a_{n,k} = \frac{(k+1)^{\beta}-k^{\beta}}{(n+1)^{\beta}}$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$, where $n,k = 0,1,2,\ldots$, satisfies the conditions (4) for any $\beta > 1$ and (5). (see, [24, Example 1 and Example 2]).

2. Auxiliary results

In the proof of the main results we use the following auxiliary results.

Lemma 1 ([32]). *Let $L^{p,\lambda}(T)$ be a Morrey space with $0 < \lambda \leq 2$ and $1 < p < \infty$, $f(x,y)$ is measurable on $\mathbb{R}^2$ and 2π periodic in each variable. Then*

$$\left\|\int_{\mathbb{T}} |f(\cdot,y)|\, dy\right\|_{L^{p,\lambda}(\mathbb{T})} \leq \int_{\mathbb{T}} \|f(\cdot,y)\|_{L^{p,\lambda}(\mathbb{T})}\, dy.$$

Lemma 2 ([24]). *If (2) for some $\beta \geq 0$ and (3) hold, then*

$$\frac{1}{2\pi}\int_{\mathbb{T}}\left|\sum_{k=0}^{\infty} a_{n,k}\sum_{m=0}^{m} l_m \cos mt\right| dt = O(1),$$

where

$$l_m = \begin{cases} 1, & \text{when } m = 0; \\ \frac{\pi}{4\sin\frac{\pi}{8}}, & \text{when } m > 0. \end{cases}$$

Lemma 3. *Let $L^{p,\lambda}(T)$ be a Morrey space with $0 < \lambda \leq 2$ and $1 < p < \infty$. Then*

$$\left\|f_{\frac{1}{2s}}(\cdot+\tau)\right\|_{L^{p,\lambda}(\mathbb{T})} = O(1)\,\|f(\cdot)\|_{L^{p,\lambda}(\mathbb{T})}$$

for every real τ, where $f_{\frac{1}{2s}}(+\tau) = s\int_{-\frac{1}{2s}+\tau}^{\frac{1}{2s}+\tau} f(t)\,dt$ with $s > 1$.

Proof. Let $I = [a,b] \subset [0,2\pi]$ and $b-a \leq 2\pi$. Using the generalized Minkowskii inequality we have

$$\left\{\frac{1}{|I|^{1-\frac{\lambda}{2}}}\int_I \left|f_{\frac{1}{2s}}(x+\tau)\right|^p dx\right\}^{\frac{1}{p}} = \left\{\frac{1}{|I|^{1-\frac{\lambda}{2}}}\int_I \left|s\int_{-\frac{1}{2s}+\tau}^{\frac{1}{2s}+\tau} f(t)\right|^p dt\right\}^{\frac{1}{p}}$$

$$\leq \frac{1}{|I|^{\left(1-\frac{\lambda}{2}\right)/p}}\, s\int_{-\frac{1}{2s}+\tau}^{\frac{1}{2s}+\tau}\left(\int_I |f(t)|^p\, dt\right)^{\frac{1}{p}} dx. \tag{8}$$

Taking the supremum in the left-hand side of (8) over I we obtain the inequality of Lemma 3. □

Lemma 4. *Let $L^{p,\lambda}(T)$ be a Morrey space with $0<\lambda\leq 2$, $1<p<\infty$ and T_n be a trigonometric polynomial of the degree at most n, such that $\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}=O(1)\,\Omega_{p,\lambda}(,f,\frac{1}{n+1})$. If (2) for some $\beta\geq 0$ and (3) hold, then the estimation*

$$\left\|\sum_{k=0}^{\infty}a_{n,k}S_k(\cdot,f-T_n)\right\|_{L^{p,\lambda}(\mathbb{T})}=O\left(\Omega_{p,\lambda}(,f,\frac{1}{n+1})\right)$$

holds.

Proof. We define $f_h(t)=\frac{1}{2h}\int_{-h}^{h}f(y+t)\,dy$ and $T_{n,h}(t)=\frac{1}{2h}\int_{-h}^{h}T_n(y+t)\,dy$. If the calculations in study [24] are taken into account we have

$$\begin{aligned}&\sum_{k=0}^{\infty}a_{n,k}S_k(x,f-T_n)\\ &=\frac{1}{\pi}\int_{\mathbb{T}}(f_h(t+x)-T_{n,h}(t+x))\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{m=0}^{k}\frac{mh}{\sin mh}\cos mt\right).\end{aligned}\tag{9}$$

Let $0<h<\frac{1}{2}$ and $|t|\leq\pi$ be given. Using (9), Lemma 1 and Lemma 3 we find that

$$\begin{aligned}&\left\|\sum_{k=0}^{\infty}a_{n,k}S_k(\cdot,f-T_n)\right\|_{L^{p,\lambda}(\mathbb{T})}\\ &\leq\frac{2}{\pi}\int_{\mathbb{T}}\|f_{h_h}(t+\cdot)-T_{n,h}(t+\cdot)\|_{L^{p,\lambda}(\mathbb{T})}\left|\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{m=0}^{k}\frac{mh}{\sin mh}\cos mt\right)\right|dt\\ &=O(1)\,\frac{1}{\pi}\int_{\mathbb{T}}\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}\left|\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{m=0}^{k}\frac{mh}{\sin mh}\cos mt\right)\right|dt.\end{aligned}$$

In the last estimation if $h=\frac{\pi}{8m}<\frac{1}{2}$ is taken for $m=1,2,\ldots,$ we conclude that

$$\begin{aligned}&\left\|\sum_{k=0}^{\infty}a_{n,k}S_k(\cdot,f-T_n)\right\|_{L^{p,\lambda}(\mathbb{T})}\\ &\leq O(1)\,\frac{1}{2\pi}\int_{\mathbb{T}}\left|\sum_{k=0}^{\infty}a_{n,k}\left(1+\frac{\pi}{4\sin\frac{\pi}{8}}\sum_{m=1}^{k}\cos mt\right)dt\right|\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}.\end{aligned}\tag{10}$$

By considering of Lemma 2

$$\frac{1}{2\pi}\int_{\mathbb{T}}\left|\sum_{k=0}^{\infty}a_{n,k}\left(1+\frac{\pi}{4\sin\frac{\pi}{8}}\sum_{m=1}^{k}\cos mt\right)dt\right|=O(1).\tag{11}$$

is obtained.

The consideration of (10) and (11) gives us

$$\left\| \sum_{k=0}^{\infty} a_{n,k} S_k\left(\cdot, f - T_n\right) \right\|_{L^{p,\lambda}(\mathbb{T})} = O\left(1\right) \left\| f - T_n \right\|_{L^{p,\lambda}(\mathbb{T})}$$

$$= O\left(1\right) \Omega_{p,\lambda}\big(, f, \frac{1}{n+1}\big).$$

The proof of Lemma 4 is completed. □

3. Proofs of the main results

Proof of Theorem 1. Let $f \in L^{p,\lambda}\left(\mathbb{T}\right)$, $0 < \lambda \leq 2$, $1 < p < \infty$ and T_n be the polynomial satisfying the condition $\left\| f - T_n \right\|_{L^{p,\lambda}(\mathbb{T})} = \Omega_{p,\lambda}(, f, \frac{1}{n+1})$. It is clear that

$$S_k\left(x, f - T_n\right) = \begin{cases} S_k\left(x, f\right) - T_k\left(x\right), & \text{for } k \leq n \\ S_k\left(x, f\right) - T_n\left(x\right), & \text{for } k \geq n \end{cases} \tag{12}$$

Then taking account of (12) we have

$$\begin{aligned} &\left\| T_{n,A}\left(\cdot, f\right) - f \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &= \Big\| T_{n,A}\left(\cdot, f\right) - \sum_{k=0}^{n} a_{n,k} T_k - \sum_{k=n+1}^{\infty} a_{n,k} T_n \\ &\quad + \sum_{k=0}^{n} a_{n,k} T_k + \sum_{k=n+1}^{\infty} a_{n,k} T_n - f \Big\|_{L^{p,\lambda}(\mathbb{T})} \\ &\leq \left\| T_{n,A}\left(\cdot, f\right) - \sum_{k=0}^{n} a_{n,k} T_k - \sum_{k=n+1}^{\infty} a_{n,k} T_n \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &\quad + \left\| \sum_{k=0}^{n} a_{n,k} T_k - \sum_{k=n+1}^{\infty} a_{n,k} T_n - f \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &= \left\| \sum_{k=0}^{n} a_{n,k} \left\{ S_k\left(., f\right) - T_k \right\} + \sum_{k=n+1}^{\infty} a_{n,k} \left\{ S_k\left(., f\right) - T_n \right\} \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &\quad + \left\| \sum_{k=0}^{n} a_{n,k} \left(f - T_k\right) + \sum_{k=n+1}^{\infty} a_{n,k} \left(f - T_n\right) \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &\leq \left\| \sum_{k=0}^{\infty} a_{n,k} S_k\left(., f - T_n\right) \right\|_{L^{p,\lambda}(\mathbb{T})} + \left\| \sum_{k=0}^{n} a_{n,k} \left(f - T_k\right) \right\|_{L^{p,\lambda}(\mathbb{T})} \\ &\quad + \left\| \sum_{k=n+1}^{\infty} a_{n,k} \left(f - T_n\right) \right\|_{L^{p,\lambda}(\mathbb{T})} \end{aligned} \tag{13}$$

$$= \left\|\sum_{k=0}^{\infty} a_{n,k} S_k\left(.,f-T_n\right)\right\|_{L^{p,\lambda}(\mathbb{T})} + O\left(1\right)\sum_{k=0}^{n} a_{n,k}\Omega_{p,\lambda}\Big(,f,\frac{1}{k+1}\Big)$$
$$+\sum_{k=n+1}^{\infty} a_{n,k}\Omega_{p,\lambda}\Big(,f,\frac{1}{n+1}\Big).$$

By using of Lemma 4 we have

$$\left\|\sum_{k=0}^{\infty} a_{n,k} S_k\left(.,f-T_n\right)\right\|_{L^{p,\lambda}(\mathbb{T})} = O\left(\Omega_{p,\lambda}\Big(,f,\frac{1}{n+1}\Big)\right) \tag{14}$$

By combininig (13) and (14) we obtain the inequality of Theorem 1. □

Proof of Theorem 2. If we take into account the definition of matrix A_0, we can write the following estimation [24]:

$$\sum_{k=0}^{\infty}(k+1)^{\beta}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|$$
$$=\sum_{k=0}^{n-1}(k+1)^{\beta}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right| + a_{n,n}$$
$$= O\left(a_{n,n}\right) + a_{n,n} = O\left(a_{n,n}\right) = O\left(\frac{1}{n+1}\right)$$

and

$$\sum_{k=0}^{\infty}(k+1)\,a_{n,k} = \sum_{k=0}^{n}(k+1)\,a_{n,k} \le (n+1)\sum_{k=0}^{n} a_{n,k} = n+1.$$

Using the monotonicity of $\Omega_{p,\lambda}(f,\delta)$ for $\delta > 0$ and $\sum_{k=0}^{n} a_{n,k} = 1$ we obtain

$$\sum_{k=0}^{n} a_{n,k}\Omega_{p,\lambda}(,f,\frac{1}{k+1}) \ge \Omega_{p,\lambda}\Big(,f,\frac{1}{n+1}\Big)\sum_{k=0}^{n} a_{n,k}$$
$$= \Omega_{p,\lambda}\Big(,f,\frac{1}{n+1}\Big).$$

The last inequality and Theorem 1 imply that (6).

The proof of Theorem 2 is completed. □

Proof of Theorem 3. For the function ω of modulus of continuity type the inequality $\omega(n\delta) \leq n\omega(\delta)$ holds because of the inequality $\omega(s\delta) \leq (s+1)\omega(\delta)$, where $n \in \mathbb{N}$ and $s \geq 0$. Then

$$\omega(\delta_2) = \omega\left(\frac{\delta_1}{\delta_1}\delta_2\right) \leq \omega\left(\frac{\delta_2}{\delta_1}+1\right)\omega(\delta_1)$$
$$= \omega\left(\frac{\delta_2}{\delta_1}+\frac{\delta_1}{\delta_1}\right)\omega(\delta_1) \leq \omega\left(\frac{\delta_2}{\delta_1}+\frac{\delta_2}{\delta_1}\right)\omega(\delta_1) = 2\frac{\delta_2}{\delta_1}\omega(\delta_1)$$

holds, where $0 < \delta_1 \leq \delta_2$. Then, from the last inequality we conclude that [24]

$$\frac{\omega(\delta_2)}{\delta_2} \leq \frac{2\omega(\delta_1)}{\delta_1}, \quad 0 < \delta_1 \leq \delta_2$$

If we use the last inequality and (2) for $\beta > 0$, we get

$$\sum_{k=0}^{n} a_{n,k}\omega\left(\frac{1}{k+1}\right) = \sum_{k=0}^{n}\frac{a_{n,k}}{k+1}\frac{\omega\left(\frac{1}{k+1}\right)}{\frac{1}{k+1}}$$
$$\leq 2(n+1)\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{n}\frac{a_{n,k}}{k+1}$$
$$= 2(n+1)\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{\infty}(k+1)^{\beta-1}\frac{a_{n,k}}{(k+1)^{\beta}}$$
$$\leq 2(n+1)\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{\infty}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|\sum_{m=0}^{k}(m+1)^{\beta}$$
$$= O(n+1)\omega\left(\frac{1}{n+1}\right)\sum_{k=0}^{\infty}(k+1)^{\beta}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|$$
$$= O(n+1)\omega\left(\frac{1}{n+1}\right)O\left(\frac{1}{n+1}\right) = O\left(\omega\left(\frac{1}{n+1}\right)\right).$$

This relation and Theorem 1 immediately yield (7).

The proof of Theorem 3 is completed. □

Using Theorem 2, the proof of Theorem 4 is similar to the proof of Theorem 3.

4. Conclusion

Morrey spaces were introduced by Morrey [25] in 1938 in connection with certain problems in elliptic partial defferential equuations and calculus of variations. Later, Morrey spaces have played an important role in applications related to the Navier Stokes and Schrödinger equations, as well as elliptic problems with discontinuous coefficients and potential theory. For this reason, it is important to study the problems of approximation theory of functions within Morrey spaces.We investigate the approximation of functions by matrix means in terms of continuity modulus in Morrey spaces $L^{p,\lambda}(\mathbb{T})$, with $0 < \lambda \leq 2$ and $1 < p < \infty$. We consider the general methods of summablity of Fourier series of functions from Morrey spaces $L^{p,\lambda}(T)$ with $0 < \lambda \leq 2$ and $1 < p < \infty$. Note that for the estimating of the error of approximation of functions by the matrix means we use a modulus of continuity constructed by the Steklov functions.

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