

Clairaut slant submersion from almost Hermitian manifolds

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ABSTRACT. Our main aim is to introduce Clairaut slant submersions in complex geometry. We give the notion of Clairaut slant submersions from almost Hermitian manifolds onto Riemannian manifold in this article. We obtain some basic results on discussed submersions. Furthermore, we provide some examples to explore the geometry of Clairaut slant submersions.

1. INTRODUCTION

Let M be a Riemannian manifold endowed with a Riemannian metric g . An almost Hermitian manifold is a subclass of almost complex manifold. Since the Riemannian Submersions have many applications in science and technology, especially in the theory of relativity and cosmology, therefore it attracts many researchers to do the research in this area.

In 1966, the theory of Riemannian submersion was initiated by O' Neill [15] and it has been further studied by Gray [8], in 1967. Later, Watson [30] defined almost Hermitian submersions and showed that horizontal and vertical distributions are invariant with respect to the almost complex structure. The Riemannian submersions play a vital role not only in the differential geometry but also in science and technology. It is noticed that the theory of Riemannian submersions are capable to handle many issues of the singularity theory, Yang-Mills theory, quantum theory, Kaluza-Klein theory, relativity, superstring theories, etc. (see, [2, 6, 9]). For more details, we cite the books ([7, 23]) and the references therein. The Riemannian submersions motivate the researchers to define the anti-invariant submersion [24], semi-invariant submersion [26], invariant submersions [23], slant submersions [25], semi-slant submersions [16], conformal anti-invariant submersions ([11, 17]), conformal semi-slant submersions [20], quasi-bi-slant submersions [18], (for details see [10, 19, 21, 22]) etc.

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In 1972, Bishop [4] introduced and studied a new and interesting class of Riemannian submersion as : if there is a function $r : M \rightarrow R^+$ such that for every geodesic, making an angle θ with the horizontal subspaces, $r \sin \theta$ is constant, then submersion $\pi : M \rightarrow N$ is said to be a Clairaut submersion. Afterwards, this notion has been widely studied in Lorentzian spaces [1], timelike and spacelike spaces [14], static spacetimes ([28, 29]). In 1991, Aso et al. [3] generalized Clairaut submersions and the new conditions for anti-invariant Riemannian submersions to be Clairaut were described in [14]. In 2017, Sahin introduced Clairaut Riemannian map [27] and studied it's geometric properties and S. Kumar et al. [13] studied pointwise slant submersions from Kenmotsu manifolds. Recently, Yadav and Meena [31] defined Clairaut anti-invariant Riemannian maps from Kahler manifolds and Kumar et al. studied Clairaut semi-invariant Riemannian maps in [12].

The above studies inspire us to introduce the notion of Clairaut slant submersions from Hermitian manifolds onto Riemannian manifolds. We exhibit our work as follows: section 2, contains some basic concepts which are needed in throughout the paper. In section 3, we define the Clairaut slant submersions from Kähler manifolds onto Riemannian manifolds and discuss the differential geometric properties of such submersions. Curvature Relations of Clairaut slant submersions are discussed in section 4 and the last section contains some explicit examples of discussed submersions.

2. PRELIMINARIES

Let J be a $(1,1)$ tensor field on an even-dimensional differentiable manifold N_1 and I is identity operator in such a manner that

$$(1) \quad J^2 = -I.$$

Then J is called an almost complex structure on N_1 . The manifold N_1 with an almost complex structure J is called an almost complex manifold. Nijenhuis tensor N of an almost complex structure is defined as:

$$(2) \quad N(V_1, W_1) = [JV_1, JW_1] - [V_1, W_1] - J[JV_1, W_1] - J[V_1, JW_1],$$

for all $V_1, W_1 \in \Gamma(TN_1)$.

The almost complex manifold N_1 is called a complex manifold, If N vanishes on an almost complex manifold N_1 .

Let g_1 be a Riemannian metric on N_1 , then g_1 is called a Hermitian metric on N_1 if

$$(3) \quad g_1(JZ_1, JW_1) = g_1(Z_1, W_1), \text{ for all } Z_1, W_1 \in \Gamma(TN_1).$$

Now, manifold N_1 with Hermitian metric g_1 is called an almost Hermitian manifold. The Riemannian connection ∇ of the N_1 can be extended to the whole tensor algebra on N_1 . Tensor fields $(\nabla_{Y_1} J)$ is defined as

$$(4) \quad (\nabla_{Y_1} J)Z_1 = \nabla_{Y_1} JZ_1 - J\nabla_{Y_1} Z_1,$$

for all $Y_1, Z_1 \in \Gamma(TN_1)$.

An almost Hermitian manifold (N_1, g_1, J) is called a Kähler manifold [5] if

$$(5) \quad (\nabla_{Y_1} J)Z_1 = 0,$$

for all $Y_1, Z_1 \in \Gamma(TN_1)$.

For a Kähler manifold (N_1, g_1, J) , we have

$$(6) \quad R(X_1, X_2, X_3, X_4) = R(JX_1, JX_2, JX_3, JX_4),$$

$$(7) \quad R(X_1, X_2, X_3, X_4) = R(X_1, X_2, JX_3, JX_4),$$

$$(8) \quad R(X_1, X_2, JX_3, X_4) = -R(X_1, X_2, X_3, JX_4),$$

$$(9) \quad R(JX_1, X_2, JX_3, X_4) = R(X_1, JX_2, X_3, JX_4),$$

$$(10) \quad \begin{aligned} R(X_1, X_2, X_3, X_4) &= R(JX_1, JX_2, X_3, X_4) + R(JX_1, X_2, JX_3, X_4) \\ &\quad + R(JX_1, X_2, X_3, JX_4) \end{aligned}$$

for all $X_1, X_2, X_3, X_4 \in \Gamma(TN_1)$, where $R(X_1, X_2)X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1, X_2]}X_3$ denotes the Riemannian curvature tensor field of N_1 .

Define O'Neill's tensors [15] $\mathcal{T}$ and $\mathcal{A}$ by

$$(11) \quad \mathcal{A}_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}E_2,$$

$$(12) \quad \mathcal{T}_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{V}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}E_2,$$

for any vector fields E_1, E_2 on N_1 , where ∇ is the Levi-Civita connection of g_1 . It is easy to see that $\mathcal{T}_{E_1}$ and $\mathcal{A}_{E_1}$ are skew-symmetric operators on the tangent bundle of N_1 reversing the vertical and the horizontal distributions.

From equations (11) and (12), we have

$$(13) \quad \nabla_{Z_1}U_1 = \mathcal{T}_{Z_1}U_1 + \mathcal{V}\nabla_{Z_1}U_1,$$

$$(14) \quad \nabla_{Z_1}W_1 = \mathcal{T}_{Z_1}W_1 + \mathcal{H}\nabla_{Z_1}W_1,$$

$$(15) \quad \nabla_{W_1}Z_1 = \mathcal{A}_{W_1}Z_1 + \mathcal{V}\nabla_{W_1}Z_1,$$

$$(16) \quad \nabla_{V_1}W_1 = \mathcal{H}\nabla_{V_1}W_1 + \mathcal{A}_{V_1}W_1,$$

for all $Z_1, U_1 \in \Gamma(\ker F_*)$ and $V_1, W_1 \in \Gamma(\ker F_*)^\perp$, where $\mathcal{H}\nabla_{Z_1}W_1 = \mathcal{A}_{W_1}Z_1$, if W_1 is basic. It is clear that $\mathcal{T}$ performs on the fibers as the second fundamental form, while $\mathcal{A}$ performs on the horizontal distribution and measures the obstruction to the integrability of this distribution.

A differentiable map F between two Riemannian manifolds is totally geodesic if

$$(\nabla F_*)(Z_1, Z_2) = 0, \text{ for all } Z_1, Z_2 \in \Gamma(TN_1).$$

A totally geodesic map is that which maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths. A Riemannian

submersion is called a Riemannian submersion with totally umbilical fibers if [26]

$$(17) \quad \mathcal{T}_{X_1} Y_1 = g_1(X_1, Y_1)H,$$

for all $X_1, Y_1 \in \Gamma(\ker F_*)$, where H is the mean curvature vector field of fibers.

Let $F : (N_1, g_1) \rightarrow (N_2, g_2)$ is a smooth map. Then F_* of F can be observed a section of the bundle $Hom(TN_1, F^{-1}TN_2) \rightarrow N_1$, where $F^{-1}TN_2$ is the bundle which has fibers $(F^{-1}TN_2)_x = T_{F(x)}N_2$ has a connection ∇ induced from the Riemannian connection ∇^{N_1} and the pullback connection. Then the second fundamental form of F is given by

$$(18) \quad (\nabla F_*)(Z_1, V_1) = \nabla_{Z_1}^F F_*(V_1) - F_*(\nabla_{Z_1}^{N_1} V_1),$$

for vector field $Z_1, V_1 \in \Gamma(TN_1)$, where ∇^F is the pullback connection. We know that the second fundamental form is symmetric.

Now, we recall following definitions for further use:

Definition 1 ([23]). Let F be a Riemannian submersion from an almost Hermitian manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2) . Then, we say that F is an invariant submersion if the vertical distribution is invariant with respect to the complex structure J , i.e.,

$$J(\ker F_*) = \ker F_*.$$

Definition 2 ([24]). Let N_1 be an almost Hermitian manifold with Hermitian metric g_1 and almost complex structure J and N_2 be a Riemannian manifold with Riemannian metric g_2 . Suppose that there exists a submersion $F : (N_1, g_1, J) \rightarrow (N_2, g_2)$ such that $J(\ker F_*) \subseteq (\ker F_*)^\perp$. Then we say that F is an anti-invariant submersion.

Definition 3 ([25]). Let F be a Riemannian map from an almost Hermitian manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2) . If for any non-zero vector $Z \in (\ker F_*)$, the angle $\Theta(Z)$ between JZ and the space $(\ker F_*)$ is a constant, i.e. it is independent of the choice of the point $p \in N_1$ and choice of the tangent vector Z in $(\ker F_*)$, then we say that F is a slant submersion. In this case, the angle Θ is called the slant angle of the slant submersion.

3. CLAIRAUT SLANT SUBMERSIONS

Bishop [4] gave the notion of Clairaut Riemannian submersion. He defined that a Riemannian submersion $F : (N_1, g_1) \rightarrow (N_2, g_2)$ is called a Clairaut Riemannian submersion if there exists a positive function r on N_1 , such that for any geodesic α on N_1 , the function $(r \circ \alpha) \sin \theta$ is constant, where for any t , $\theta(t)$ is the angle between $\dot{\alpha}(t)$ and the horizontal space at $\alpha(t)$.

The necessary and sufficient condition for a Riemannian submersion to be a Clairaut Riemannian submersion was also given by Bishop as follows.

Theorem 1 ([4]). *Let $F : (N_1, g_1) \rightarrow (N_2, g_2)$ be a submersion with connected fibers. Then, F is a Clairaut submersion with $r = e^h$ if each fiber is totally umbilical and has the mean curvature vector field $H = -\nabla h$ with respect to g_1 .*

Now, we present the notion of Clairaut slant submersion as follows.

Definition 4. Let (N_1, g_1, J) be a Kähler manifold and (N_2, g_2) be a Riemannian manifold. Any slant submersion from (N_1, g_1, J) onto (N_2, g_2) is called Clairaut slant submersion if it satisfies the condition of Clairaut submersion.

We denote the complementary distribution to $\omega(\ker F_*)$ in $(\ker F_*)^\perp$ by μ . Then for $X_1 \in (\ker F_*)$, we get

$$(19) \quad JX_1 = \phi X_1 + \omega X_1,$$

where ϕX_1 and ωX_1 are vertical and horizontal parts of JX_1 . Also for $X_2 \in \Gamma(\ker F_*)^\perp$, we have

$$(20) \quad JX_2 = BX_2 + CX_2,$$

where BX_2 and CX_2 are vertical and horizontal components of JX_2 .

The proof of the following result is the same as given in [1], therefore, we omit its proof.

Lemma 1. *Let F be a slant submersion from an almost Hermitian manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2) . Then, we have*

- (i) $\phi_1^2 W_1 = -(\cos^2 \Theta_1) W_1$,
- (ii) $g_1(\phi W_1, \phi W_2) = \cos^2 \Theta_1 g_1(W_1, W_2)$,
- (iii) $g_1(\omega W_1, \omega W_2) = \sin^2 \Theta_1 g_1(W_1, W_2)$,

for all $W_1, W_2 \in \Gamma(\ker F_*)$.

Lemma 2. *Let F be a slant submersion from a Kähler manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2) . Then, we have*

$$(21) \quad \mathcal{V}\nabla_{Y_1} \phi Y_2 + \mathcal{T}_{Y_1} \omega Y_2 = B\mathcal{T}_{Y_1} Y_2 + \phi \mathcal{V}\nabla_{Y_1} Y_2,$$

$$(22) \quad \mathcal{T}_{Y_1} \phi Y_2 + \mathcal{H}\nabla_{Y_1} \omega Y_2 = C\mathcal{T}_{Y_1} Y_2 + \omega \mathcal{V}\nabla_{Y_1} Y_2,$$

$$(23) \quad \mathcal{V}\nabla_{Y_1} B W_1 + \mathcal{T}_{Y_1} C W_1 = \phi \mathcal{T}_{Y_1} W_1 + B\mathcal{H}\nabla_{Y_1} W_1,$$

$$(24) \quad \mathcal{T}_{Y_1} B W_1 + \mathcal{H}\nabla_{Y_1} C W_1 = \omega \mathcal{T}_{Y_1} W_1 + C\mathcal{H}\nabla_{Y_1} W_1,$$

$$(25) \quad \mathcal{V}\nabla_{W_1} \phi Y_1 + \mathcal{A}_{W_1} \omega Y_1 = B\mathcal{A}_{W_1} Y_1 + \phi \mathcal{V}\nabla_{W_1} Y_1,$$

$$(26) \quad \mathcal{A}_{W_1} \phi Y_1 + \mathcal{H}\nabla_{W_1} \omega Y_1 = \omega \mathcal{V}_{W_1} Y_1 + C\mathcal{A}_{W_1} Y_1,$$

$$(27) \quad \mathcal{V}\nabla_{W_1} B W_2 + \mathcal{A}_{W_1} C W_2 = B\mathcal{H}\nabla_{W_1} W_2 + \phi \mathcal{A}_{W_1} W_2,$$

$$(28) \quad \mathcal{A}_{W_1} B W_2 + \mathcal{H}\nabla_{W_1} C W_2 = \omega \mathcal{A}_{W_1} W_2 + C\mathcal{H}\nabla_{W_1} W_2,$$

for any $Y_1, Y_2 \in \Gamma(\ker F_*)$ and $W_1, W_2 \in \Gamma(\ker F_*)^\perp$.

Proof. Using equations (13)-(16), (19) and (20), we get the lemma completely. $\square$

Now, we define

$$(29) \quad (\nabla_{Y_1} \phi)Y_2 = \mathcal{V}\nabla_{Y_1} \phi Y_2 - \phi \mathcal{V}\nabla_{Y_1} Y_2,$$

$$(30) \quad (\nabla_{Y_1} \omega)Y_2 = \mathcal{H}\nabla_{Y_1} \omega Y_2 - \omega \mathcal{V}\nabla_{Y_1} Y_2,$$

$$(31) \quad (\nabla_{W_1} C)W_2 = \mathcal{H}\nabla_{W_1} C W_2 - C \mathcal{H}\nabla_{W_1} W_2,$$

$$(32) \quad (\nabla_{W_1} B)W_2 = \mathcal{V}\nabla_{W_1} B W_2 - B \mathcal{H}\nabla_{W_1} W_2,$$

for any $Y_1, Y_2 \in \Gamma(\ker F_*)$ and $W_1, W_2 \in \Gamma(\ker F_*)^\perp$.

Lemma 3. *Let F be a slant submersion from a Kähler manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2) . Then, we have*

$$(\nabla_{Y_1} \phi)Y_2 = B\mathcal{T}_{Y_1} Y_2 - \mathcal{T}_{Y_1} \omega Y_2,$$

$$(\nabla_{Y_1} \omega)Y_2 = C\mathcal{T}_{Y_1} Y_2 - \mathcal{T}_{Y_1} \phi Y_2,$$

$$(\nabla_{W_1} C)W_2 = \omega \mathcal{A}_{W_1} W_2 - \mathcal{A}_{W_1} B W_2,$$

$$(\nabla_{W_1} B)W_2 = \phi \mathcal{A}_{W_1} W_2 - \mathcal{A}_{W_1} C W_2,$$

for any vectors $Y_1, Y_2 \in \Gamma(\ker F_*)$ and $W_1, W_2 \in \Gamma(\ker F_*)^\perp$.

Proof. The proof of the above lemma is straightforward, so we omit its proof. $\square$

If the tensors ϕ and ω are parallel with respect to the linear connection ∇ on N_1 respectively, then

$$B\mathcal{T}_{Y_1} Y_2 = \mathcal{T}_{Y_1} \omega Y_2, C\mathcal{T}_{Y_1} Y_2 = \mathcal{T}_{Y_1} \phi Y_2$$

for any $Y_1, Y_2 \in \Gamma(TN_1)$.

Lemma 4. *Let F be a slant submersion from a Kähler manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2) . If $\alpha : I \subset \mathbb{R} \rightarrow M$ is a regular curve and $Y_1(t)$ and $Y_2(t)$ are the vertical and horizontal components of the tangent vector field $\dot{\alpha} = E$ of $\alpha(t)$, respectively, then α is a geodesic if and only if along α the following equations hold:*

$$\begin{aligned} \cos^2 \Theta \mathcal{V}\nabla_{\dot{\alpha}} Y_1 &= \mathcal{T}_{Y_1} \omega \phi Y_1 + \mathcal{A}_{Y_2} \omega \phi Y_1 + \phi \mathcal{T}_{Y_1} \omega Y_1 + B \mathcal{H}\nabla_{Y_1} \omega Y_1 \\ &+ B \mathcal{T}_{Y_1} B Y_2 + \phi \mathcal{V}\nabla_{Y_2} \omega Y_1 + \phi \mathcal{T}_{Y_1} C Y_2 + \omega \mathcal{T}_{Y_1} C Y_2 \\ &+ B \mathcal{H}\nabla_{Y_1} C Y_2 + \phi \mathcal{A}_{Y_2} \omega Y_1 + B \mathcal{A}_{Y_2} B Y_2 + \phi \mathcal{V}\nabla_{Y_2} B Y_2 \\ &+ B \mathcal{H}\nabla_{Y_2} C Y_2 + \phi \mathcal{A}_{Y_2} C Y_2, \end{aligned}$$

$$\begin{aligned} \cos^2 \Theta (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2}) Y_1 &= \mathcal{H}\nabla_{Y_1} \omega \phi Y_1 + \mathcal{H}\nabla_{Y_2} \omega \phi Y_1 + \omega \mathcal{T}_{Y_1} \omega Y_1 + C \mathcal{H}\nabla_{Y_1} \omega Y_1 \\ &+ C \mathcal{T}_{Y_1} B Y_2 + \omega \mathcal{V}\nabla_{Y_2} \omega Y_1 + \omega \mathcal{T}_{Y_1} C Y_2 + C \mathcal{H}\nabla_{Y_1} C Y_2 \\ &+ C \mathcal{H}\nabla_{Y_2} \omega Y_1 + \omega \mathcal{A}_{Y_2} \omega Y_1 + C \mathcal{A}_{Y_2} B Y_2 + \omega \mathcal{V}\nabla_{Y_2} B Y_2 \\ &+ C \mathcal{H}\nabla_{Y_2} C Y_2 + \omega \mathcal{A}_{Y_2} C Y_2. \end{aligned}$$

Proof. Let $\alpha : I \rightarrow N_1$ be a regular curve on N_1 . Since $Y_1(t)$ and $Y_2(t)$ are the vertical and horizontal parts of the tangent vector field $\dot{\alpha}(t)$, i.e., $\dot{\alpha}(t) = Y_1(t) + Y_2(t)$. Using equations (4), (5), (13)-(16), (19), (20) and Lemma 1, we get

$$\begin{aligned}
 \nabla_{\dot{\alpha}} \dot{\alpha} &= -J(\nabla_{\dot{\alpha}} J \dot{\alpha}) \\
 &= -J(\nabla_{Y_1} \phi Y_1 + \nabla_{Y_1} \omega Y_1 + \nabla_{Y_2} \phi Y_1 + \nabla_{Y_2} \omega Y_1 \\
 &\quad + \nabla_{Y_1} B Y_2 + \nabla_{Y_1} C Y_2 + \nabla_{Y_2} B Y_2 + \nabla_{Y_2} C Y_2), \\
 &= -\nabla_{Y_1} \phi^2 Y_1 - \nabla_{Y_1} \omega \phi Y_1 - \nabla_{Y_2} \phi^2 Y_1 - \nabla_{Y_2} \omega \phi Y_1 \\
 &\quad - J(\mathcal{T}_{Y_1} \omega Y_1 + \mathcal{H} \nabla_{Y_1} \omega Y_1 + \mathcal{T}_{Y_1} B Y_2 + \mathcal{V} \nabla_{Y_2} \omega Y_1 + \mathcal{T}_{Y_1} C Y_2 \\
 &\quad + \mathcal{H} \nabla_{Y_1} C Y_2 + \mathcal{H} \nabla_{Y_2} \omega Y_1 + \mathcal{A}_{Y_2} \omega Y_1 + \mathcal{A}_{Y_2} B Y_2 + \mathcal{V} \nabla_{Y_2} B Y_2 \\
 &\quad + \mathcal{H} \nabla_{Y_2} C Y_2 + \mathcal{A}_{Y_2} C Y_2) \\
 &= \cos^2 \Theta \mathcal{V} \nabla_{\dot{\alpha}} Y_1 + \cos^2 \Theta (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2}) Y_1 - \mathcal{T}_{Y_1} \omega \phi Y_1 - \mathcal{H} \nabla_{Y_1} \omega \phi Y_1 \\
 &\quad - \mathcal{H} \nabla_{Y_2} \omega \phi Y_1 - \mathcal{A}_{Y_2} \omega \phi Y_1 - \phi \mathcal{T}_{Y_1} \omega Y_1 - \omega \mathcal{T}_{Y_1} \omega Y_1 - B \mathcal{H} \nabla_{Y_1} \omega Y_1 \\
 &\quad - C \mathcal{H} \nabla_{Y_1} \omega Y_1 - B \mathcal{T}_{Y_1} B Y_2 - C \mathcal{T}_{Y_1} B Y_2 - \phi \mathcal{V} \nabla_{Y_2} \omega Y_1 - \omega \mathcal{V} \nabla_{Y_2} \omega Y_1 \\
 &\quad - \phi \mathcal{T}_{Y_1} C Y_2 - \omega \mathcal{T}_{Y_1} C Y_2 - B \mathcal{H} \nabla_{Y_1} C Y_2 - C \mathcal{H} \nabla_{Y_1} C Y_2 - \mathcal{H} \nabla_{Y_2} \omega Y_1 \\
 &\quad - C \mathcal{H} \nabla_{Y_2} \omega Y_1 - \phi \mathcal{A}_{Y_2} \omega Y_1 - \omega \mathcal{A}_{Y_2} \omega Y_1 - B \mathcal{A}_{Y_2} B Y_2 - C \mathcal{A}_{Y_2} B Y_2 \\
 &\quad - \phi \mathcal{V} \nabla_{Y_2} B Y_2 - \omega \mathcal{V} \nabla_{Y_2} B Y_2 - B \mathcal{H} \nabla_{Y_2} C Y_2 - C \mathcal{H} \nabla_{Y_2} C Y_2 \\
 &\quad - \phi \mathcal{A}_{Y_2} C Y_2 - \omega \mathcal{A}_{Y_2} C Y_2.
 \end{aligned}$$

Taking the vertical and horizontal components in above equation, we get

$$\begin{aligned}
 \mathcal{V} \nabla_{\dot{\alpha}} \dot{\alpha} &= \cos^2 \Theta \mathcal{V} \nabla_{\dot{\alpha}} Y_1 - \mathcal{T}_{Y_1} \omega \phi Y_1 - \mathcal{A}_{Y_2} \omega \phi Y_1 - \phi \mathcal{T}_{Y_1} \omega Y_1 \\
 &\quad - B \mathcal{H} \nabla_{Y_1} \omega Y_1 - B \mathcal{T}_{Y_1} B Y_2 - \phi \mathcal{V} \nabla_{Y_2} \omega Y_1 - \phi \mathcal{T}_{Y_1} C Y_2 \\
 &\quad - \omega \mathcal{T}_{Y_1} C Y_2 - B \mathcal{H} \nabla_{Y_1} C Y_2 - \phi \mathcal{A}_{Y_2} \omega Y_1 - B \mathcal{A}_{Y_2} B Y_2 \\
 &\quad - \phi \mathcal{V} \nabla_{Y_2} B Y_2 - B \mathcal{H} \nabla_{Y_2} C Y_2 - \phi \mathcal{A}_{Y_2} C Y_2,
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{H} \nabla_{\dot{\alpha}} \dot{\alpha} &= \cos^2 \Theta (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2}) Y_1 - \mathcal{H} \nabla_{Y_1} \omega \phi Y_1 - \mathcal{H} \nabla_{Y_2} \omega \phi Y_1 - \omega \mathcal{T}_{Y_1} \omega Y_1 \\
 &\quad - C \mathcal{H} \nabla_{Y_1} \omega Y_1 - C \mathcal{T}_{Y_1} B Y_2 - \omega \mathcal{V} \nabla_{Y_2} \omega Y_1 - \omega \mathcal{T}_{Y_1} C Y_2 \\
 &\quad - C \mathcal{H} \nabla_{Y_1} C Y_2 - C \mathcal{H} \nabla_{Y_2} \omega Y_1 - \omega \mathcal{A}_{Y_2} \omega Y_1 - C \mathcal{A}_{Y_2} B Y_2 \\
 &\quad - \omega \mathcal{V} \nabla_{Y_2} B Y_2 - C \mathcal{H} \nabla_{Y_2} C Y_2 - \omega \mathcal{A}_{Y_2} C Y_2.
 \end{aligned}$$

Now, α is a geodesic on N_1 if and only if $\mathcal{V} \nabla_{\dot{\alpha}} \dot{\alpha} = 0$ and $\mathcal{H} \nabla_{\dot{\alpha}} \dot{\alpha} = 0$, which is completes proof. $\square$

Theorem 2. Let F be a slant submersion from a Kähler manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2) . Then F is a Clairaut slant submersion with $r = e^h$ if and only if

$$\begin{aligned}
 g_1(\mathcal{V} \nabla_{\dot{\alpha}} \phi Y_1 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2}) C Y_2 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2}) \omega Y_1, B Y_2) + g_1(\mathcal{H} \nabla_{\dot{\alpha}} \omega Y_1 \\
 + (\mathcal{T}_{Y_1} + \mathcal{A}_{Y_2}) B Y_2 + (\mathcal{A}_{Y_2} + \mathcal{T}_{Y_1}) \phi Y_1, C Y_2) + g_1(Y_1, Y_1) \frac{dh}{dt} = 0,
 \end{aligned}$$

where $\alpha : I \rightarrow N_1$ is a geodesic on N_1 and Y_1, Y_2 are vertical and horizontal components of $\dot{\alpha}(t)$.

Proof. Let $\alpha : I \rightarrow N_1$ be a geodesic on N_1 with $Y_1(t) = \mathcal{V}\dot{\alpha}(t)$ and $Y_2(t) = \mathcal{H}\dot{\alpha}(t)$ denote the angle in $[0, F]$ between $\dot{\alpha}(t)$ and $Y_2(t)$. Assuming $\nu = \|\dot{\alpha}(t)\|^2$ then we get

$$(33) \quad g_1(Y_1(t), Y_1(t)) = \nu \sin^2 \theta(t),$$

$$(34) \quad g_1(Y_2(t), Y_2(t)) = \nu \cos^2 \theta(t).$$

Now, differentiating (33), we get

$$(35) \quad \begin{aligned} \frac{d}{dt} g_1(Y_1(t), Y_1(t)) &= 2\nu \cos \theta(t) \sin \theta(t) \frac{d\theta}{dt}, \\ \nu \cos \theta(t) \sin \theta(t) \frac{d\theta}{dt} &= g_1(\mathcal{V}\nabla_{\dot{\alpha}} Y_1, Y_1), \\ \nu \cos^2 \Theta \cos \theta(t) \sin \theta(t) \frac{d\theta}{dt} &= g_1(\cos^2 \Theta \mathcal{V}\nabla_{\dot{\alpha}} Y_1, Y_1). \end{aligned}$$

On the other hand, using Lemma 4 and equation (35), we get

$$(36) \quad \begin{aligned} &\nu \cos^2 \Theta \cos \theta(t) \sin \theta(t) \frac{d\theta}{dt} \\ &= g_1(\mathcal{T}_{Y_1} \omega \phi Y_1 + \mathcal{A}_{Y_2} \omega \phi Y_1 + \phi \mathcal{T}_{Y_1} \omega Y_1 + B\mathcal{H}\nabla_{Y_1} \omega Y_1 \\ &\quad + B\mathcal{T}_{Y_1} B Y_2 + \phi \mathcal{V}\nabla_{Y_2} \omega Y_1 + \phi \mathcal{T}_{Y_1} C Y_2 + \omega \mathcal{T}_{Y_1} C Y_2 \\ &\quad + B\mathcal{H}\nabla_{Y_1} C Y_2 + \phi \mathcal{A}_{Y_2} \omega Y_1 + B\mathcal{A}_{Y_2} B Y_2 + \phi \mathcal{V}\nabla_{Y_2} B Y_2 \\ (37) \quad &+ B\mathcal{H}\nabla_{Y_2} C Y_2 + \phi \mathcal{A}_{Y_2} C Y_2, Y_1). \end{aligned}$$

Moreover, F is a Clairaut slant submersion with $r = e^h$ if and only if $\frac{d}{dt}(e^{h\circ\alpha} \sin \theta) = 0$, i.e., $e^{h\circ\alpha}(\cos \theta \frac{d\theta}{dt} + \sin \theta \frac{dh}{dt}) = 0$. By multiplying this with non-zero factor $\nu \cos^2 \Theta \sin \theta$, we have

$$(38) \quad \nu \cos^2 \Theta \cos \theta \sin \theta \frac{d\theta}{dt} = -\nu \cos^2 \Theta \sin^2 \theta \frac{dh}{dt}.$$

Thus, from equations (17), (36) and (38), we have

$$(39) \quad \begin{aligned} &-\cos^2 \Theta \|\dot{Y}_1\|^2 g_1(\nabla h, Y_2) \\ &= g_1(\mathcal{T}_{Y_1} \omega \phi Y_1 + \mathcal{A}_{Y_2} \omega \phi Y_1 + \phi \mathcal{T}_{Y_1} \omega Y_1 + B\mathcal{H}\nabla_{Y_1} \omega Y_1 \\ &\quad + B\mathcal{T}_{Y_1} B Y_2 + \phi \mathcal{V}\nabla_{Y_2} \omega Y_1 + \phi \mathcal{T}_{Y_1} C Y_2 + \omega \mathcal{T}_{Y_1} C Y_2 \\ &\quad + B\mathcal{H}\nabla_{Y_1} C Y_2 + \phi \mathcal{A}_{Y_2} \omega Y_1 + B\mathcal{A}_{Y_2} B Y_2 + \phi \mathcal{V}\nabla_{Y_2} B Y_2 \\ &\quad + B\mathcal{H}\nabla_{Y_2} C Y_2 + \phi \mathcal{A}_{Y_2} C Y_2, Y_1), \end{aligned}$$

which completes the proof. $\square$

4. CURVATURE RELATIONS

In this section, we are going to obtain curvature relations of Clairaut slant Riemannian submersions [7].

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with corresponding curvature relation R and R^* , respectively. Let $\pi : (N_1, g_1) \rightarrow (N_2, g_2)$ be a Riemannian submersion and $\hat{R}$ the curvature tensor of fibers of F . If X_1, X_2, X_3, X_4 are horizontal and Z_1, Z_2, Z_3, Z_4 vertical vectors, then

$$(40) \quad R(Z_1, Z_2, Z_3, Z_4) = \hat{R}(Z_1, Z_2, Z_3, Z_4) - g_1(\mathcal{T}_{Z_1}Z_4, \mathcal{T}_{Z_2}Z_3) \\ + g_1(\mathcal{T}_{Z_2}Z_4, \mathcal{T}_{Z_1}Z_3),$$

$$(41) \quad R(Z_1, Z_2, Z_3, X_1) = g_1((\nabla_{Z_1}\mathcal{T})(Z_2, Z_3), X_1) \\ - g_1((\nabla_{Z_2}\mathcal{T})(Z_1, Z_3), X_1),$$

$$(42) \quad R(X_1, X_2, X_3, Z_1) = -g_1((\nabla_{X_3}\mathcal{A})(X_1, X_2), Z_1) - g_1(\mathcal{A}_{X_1}X_2, \mathcal{T}_{Z_1}X_3) \\ + g_1(\mathcal{A}_{X_2}X_3, \mathcal{T}_{Z_1}X_1) + g_1(\mathcal{A}_{X_3}X_1, \mathcal{T}_{Z_1}X_2),$$

$$(43) \quad R(X_1, X_2, X_3, X_4) = R^*(X_1, X_2, X_3, X_4) + 2g_1(\mathcal{A}_{X_1}X_2, \mathcal{A}_{X_2}X_3) \\ - g_1(\mathcal{A}_{X_2}X_3, \mathcal{A}_{X_1}X_4) + g_1(\mathcal{A}_{X_1}X_3, \mathcal{A}_{X_2}X_4),$$

$$(44) \quad R(X_1, X_2, Z_1, Z_2) = -g_1((\nabla_{Z_1}\mathcal{A})(X_1, X_2), Z_2) \\ + g_1((\nabla_{Z_2}\mathcal{A})(X_1, X_2), Z_1) - g_1(\mathcal{A}_{X_1}Z_1, \mathcal{A}_{X_2}Z_2) \\ + g_1(\mathcal{A}_{X_1}Z_2, \mathcal{A}_{X_2}Z_1) + g_1(\mathcal{T}_{Z_1}X_1, \mathcal{T}_{Z_2}X_2) \\ - g_1(\mathcal{T}_{Z_2}X_1, \mathcal{T}_{Z_1}X_2),$$

$$(45) \quad R(X_1, Z_1, X_2, Z_2) = -g_1((\nabla_{X_1}\mathcal{T})(Z_1, Z_2), X_2) \\ - g_1((\nabla_{Z_1}\mathcal{T})(X_1, X_2), Z_2) + g_1(\mathcal{T}_{Z_1}X_1, \mathcal{T}_{Z_2}X_2) \\ - g_1(\mathcal{A}_{X_1}Z_1, \mathcal{A}_{X_2}Z_2).$$

Lemma 5. *Let F be a slant submersion from a Kähler manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2) . Then*

$$R(V_1, V_2, V_3, V_4) \\ = \cos^4 \Theta (\hat{R}(V_1, V_2, V_3, V_4) - g_1(\mathcal{T}_{V_1}V_4, \mathcal{T}_{V_2}V_3) + g_1(\mathcal{T}_{V_2}V_4, \mathcal{T}_{V_1}V_3)) \\ - g_1((\nabla_{\omega\phi V_2}\mathcal{T})(V_1, V_3), \omega\phi V_4) - g_1((\nabla_{V_1}\mathcal{A})(\omega\phi V_2, \omega\phi V_4), V_3) \\ + g_1(\mathcal{T}_{V_1}\omega\phi V_2, \mathcal{T}_{V_3}\omega\phi V_4) - g_1(\mathcal{A}_{\omega\phi V_2}V_1, \mathcal{A}_{\omega\phi V_4}V_3) \\ - \cos^2 \Theta g_1((\nabla_{V_1}\mathcal{T})(V_2, V_3), \omega\phi V_4) + \cos^2 \Theta g_1((\nabla_{V_2}\mathcal{T})(V_1, V_3), \omega\phi V_4) \\ + \cos^2 \Theta g_1((\nabla_{V_3}\mathcal{T})(V_4, V_1), \omega\phi V_2) - \cos^2 \Theta g_1((\nabla_{V_4}\mathcal{T})(V_3, V_1), \omega\phi V_2) \\ + \cos^2 \Theta g_1((\nabla_{V_2}\mathcal{T})(V_1, \phi V_3), \omega V_4) - \cos^2 \Theta g_1((\nabla_{V_1}\mathcal{T})(V_2, \phi V_3), \omega V_4) \\ + \cos^2 \Theta (-g_1((\nabla_{V_1}\mathcal{A})(\omega V_3, \omega V_4), V_2))$$

$$\begin{aligned}
& + g_1((\nabla_{V_2}\mathcal{A})(\omega V_3, \omega V_4), V_1) - g_1(\mathcal{A}_{\omega V_3}V_1, \mathcal{T}_{V_2}\omega V_4) \\
& + g_1(\mathcal{A}_{\omega V_3}V_2, \mathcal{T}_{\omega V_4}V_1) + g_1(\mathcal{T}_{V_1}\omega V_3, \mathcal{T}_{V_2}\omega V_4) \\
& - g_1(\mathcal{T}_{V_2}\omega V_3, \mathcal{T}_{V_1}\omega V_4)) + g_1((\nabla_{\omega V_4}\mathcal{T})(\phi V_3, V_1), \omega \phi V_2) \\
& + g_1((\nabla_{\phi V_3}\mathcal{A})(\omega V_4, \omega \phi V_2), V_1) - g_1(\mathcal{T}_{\phi V_3}\omega V_4, \mathcal{T}_{V_1}\omega \phi V_2) \\
& + g_1(\mathcal{A}_{\omega V_4}\phi V_3, \mathcal{A}_{\omega \phi V_2}V_1) - g((\nabla_{\omega \phi V_2}\mathcal{A})(\omega V_3, \omega V_4), V_1) \\
& - g_1(\mathcal{A}_{\omega V_3}\omega V_4, \mathcal{A}_{V_1}\omega \phi V_2) + g_1(\mathcal{A}_{\omega V_4}\omega \phi V_2, \mathcal{A}_{V_1}\omega V_3) \\
& + g_1(\mathcal{A}_{\omega \phi V_2}\omega V_3, \mathcal{T}_{V_1}\omega V_4) + \cos^2 \Theta(g_1((\nabla_{V_3}\mathcal{T})(V_4, \phi V_1), \omega V_2) \\
& - g_1((\nabla_{V_4}\mathcal{T})(V_3, \phi V_1), \omega V_2)) + \cos^2 \Theta(-g((\nabla_{V_3}\mathcal{A})(\omega V_1, \omega V_2), V_4) \\
& + g_1((\nabla_{V_4}\mathcal{A})(\omega V_1, \omega V_2), V_3) - g_1(\mathcal{A}_{\omega V_1}V_3, \mathcal{A}_{\omega V_2}V_4) \\
& + g_1(\mathcal{A}_{\omega V_1}V_4, \mathcal{A}_{\omega V_2}V_3) + g_1(\mathcal{T}_{V_3}\omega V_1, \mathcal{T}_{V_4}\omega V_2) \\
& - g_1(\mathcal{T}_{V_4}\omega V_1, \mathcal{T}_{V_3}\omega V_2)) + g_1((\nabla_{\omega V_2}\mathcal{T})(\phi V_1, V_3), \omega \phi V_4) \\
& + g_1((\nabla_{\phi V_1}\mathcal{A})(\omega V_2, \omega \phi V_2), V_3) - g_1(\mathcal{T}_{\phi V_1}\omega V_2, \mathcal{T}_{V_3}\omega \phi V_2) \\
& + g_1(\mathcal{A}_{\omega V_2}\phi V_1, \mathcal{A}_{\omega \phi V_4}Z) - g_1((\nabla_{\omega \phi V_4}\mathcal{A})(\omega V_1, \omega V_2), V_3) \\
& - g_1(\mathcal{A}_{\omega V_1}\omega V_2, \mathcal{T}_{V_3}\omega \phi V_4) + g_1(\mathcal{A}_{\omega V_2}\omega \phi V_4, \mathcal{T}_{V_3}\omega V_1) \\
& + g_1(\mathcal{A}_{\omega \phi V_4}\omega V_1, \mathcal{T}_{V_3}\omega V_2) - g_1((\nabla_{\omega V_2}\mathcal{T})(\phi V_1, \phi V_3), \omega V_4) \\
& - g_1((\nabla_{\phi V_1}\mathcal{A})(\omega V_2, \omega V_4), \phi V_3) + g_1(\mathcal{T}_{\phi V_1}\omega V_2, \mathcal{T}_{\phi V_3}\omega V_4) \\
& - g_1(\mathcal{A}_{\omega V_2}\phi V_1, \mathcal{A}_{\omega V_4}\phi V_3) + g((\nabla_{\omega V_2}\mathcal{A})(\omega V_3, \omega V_4), \phi V_1) \\
& + g_1(\mathcal{A}_{\omega V_3}\omega V_4, \mathcal{T}_{\phi V_3}\omega V_2) - g_1(\mathcal{A}_{\omega V_4}\omega V_2, \mathcal{T}_{\phi V_1}\omega V_3) \\
& - g_1(\mathcal{A}_{\omega V_2}\omega V_3, \mathcal{T}_{\phi V_1}\omega V_4) + g((\nabla_{\omega V_4}\mathcal{A})(\omega V_1, \omega V_2), \phi V_3) \\
& + g_1(\mathcal{A}_{\omega V_1}\omega V_2, \mathcal{T}_{\phi V_3}\omega V_4) - g_1(\mathcal{A}_{\omega V_2}\omega V_4, \mathcal{T}_{\phi V_3}\omega V_1) \\
& - g_1(\mathcal{A}_{\omega V_2}\omega V_2, \mathcal{T}_{\phi V_3}\omega V_2) + R^*(\omega V_1, \omega V_2, \omega V_3, \omega V_4) \\
& + 2g_1(\mathcal{A}_{\omega V_1}\omega V_2, \mathcal{A}_{\omega V_3}\omega V_4) - g(\mathcal{A}_{\omega V_2}\omega V_3, \mathcal{A}_{\omega V_1}\omega V_4) \\
& + g(\mathcal{A}_{\omega V_1}\omega V_3, \mathcal{A}_{\omega V_2}\omega V_4),
\end{aligned}$$

$$\begin{aligned}
& R(Z_1, Z_2, Z_3, Z_4) \\
& = \widehat{R}(BZ_1, BZ_2, BZ_3, BZ_4) - g_1(\mathcal{T}_{BZ_1}BZ_4, \mathcal{T}_{BZ_2}BZ_3) \\
& + g_1(\mathcal{T}_{BZ_2}BZ_4, \mathcal{T}_{BZ_1}BZ_3) + g_1((\nabla_{BZ_1}\mathcal{T})(BZ_2, BZ_3), CZ_4) \\
& - g_1((\nabla_{BZ_2}\mathcal{T})(BZ_1, BZ_3), CZ_4) - g_1((\nabla_{BZ_1}\mathcal{T})(BZ_2, BZ_4), CZ_3) \\
& + g_1((\nabla_{BZ_2}\mathcal{T})(BZ_1, BZ_4), CZ_3) - g_1((\nabla_{BZ_1}\mathcal{A})(CZ_3, CZ_4), BZ_2) \\
& + g_1((\nabla_{BZ_2}\mathcal{A})(CZ_3, CZ_4), BZ_1) + g_1(\mathcal{A}_{CZ_3}BZ_1, \mathcal{A}_{CZ_4}BZ_2) \\
& - g_1(\mathcal{A}_{CZ_3}BZ_2, \mathcal{A}_{CZ_4}BZ_1) + g_1(\mathcal{T}_{BZ_1}CZ_3, \mathcal{T}_{BZ_2}CZ_4) \\
& - g_1(\mathcal{T}_{BZ_2}CZ_3, \mathcal{T}_{BZ_1}CZ_4) + g_1((\nabla_{BZ_3}\mathcal{T})(BZ_4, BZ_1), CZ_2) \\
& - g_1((\nabla_{BZ_4}\mathcal{T})(BZ_3, BZ_1), CZ_2) - g_1((\nabla_{CZ_2}\mathcal{T})(BZ_1, BZ_3), CZ_4) \\
& - g_1((\nabla_{BZ_1}\mathcal{A})(CZ_2, CZ_4), BZ_3) + g_1(\mathcal{T}_{BZ_1}CZ_2, \mathcal{T}_{BZ_3}CZ_4)
\end{aligned}$$

$$\begin{aligned}
& -g_1(\mathcal{A}_{CZ_2}BZ_1, \mathcal{A}_{CZ_4}BZ_3) + g_1((\nabla_{CZ_2}\mathcal{T})(BZ_1, BZ_4), CZ_3) \\
& + g_1((\nabla_{BZ_1}\mathcal{A})(CZ_2, CZ_3), BZ_4) - g_1(\mathcal{T}_{BZ_1}CZ_2, \mathcal{T}_{BZ_4}CZ_3) \\
& - g_1(\mathcal{A}_{CZ_2}BZ_1, \mathcal{A}_{CZ_3}BZ_4) + g_1((\nabla_{CZ_2}\mathcal{A})(CZ_3, CZ_4), BZ_1) \\
& + g_1(\mathcal{A}_{CZ_1}CZ_4, \mathcal{T}_{BZ_1}CZ_2) - g_1(\mathcal{A}_{CZ_4}CZ_2, \mathcal{T}_{BZ_1}CZ_3) \\
& - g_1(\mathcal{A}_{CZ_2}CZ_3, \mathcal{T}_{BZ_1}CZ_4) - g_1((\nabla_{BZ_3}\mathcal{T})(BZ_4, BZ_2), CZ_1) \\
& + g_1((\nabla_{BZ_4}\mathcal{T})(BZ_3, BZ_2), CZ_1) + g_1((\nabla_{CZ_1}\mathcal{T})(BZ_2, BZ_3), CZ_4) \\
& + g_1((\nabla_{BZ_2}\mathcal{A})(CZ_1, CZ_4), BZ_3) - g_1(\mathcal{T}_{BZ_2}CZ_1, \mathcal{T}_{BZ_3}CZ_4) \\
& + g_1(\mathcal{A}_{CZ_1}BZ_2, \mathcal{A}_{CZ_4}BZ_3) + g_1((\nabla_{CZ_1}\mathcal{T})(BZ_2, BZ_3), CZ_4) \\
& + g_1((\nabla_{BZ_2}\mathcal{A})(CZ_1, CZ_4), BZ_3) - g_1(\mathcal{T}_{BZ_2}CZ_1, \mathcal{T}_{BZ_3}CZ_4) \\
& + g_1(\mathcal{A}_{CZ_1}BZ_2, \mathcal{A}_{CZ_4}BZ_3) - g_1((\nabla_{CZ_1}\mathcal{A})(CZ_3, CZ_4), BZ_2) \\
& - g_1(\mathcal{A}_{CZ_3}CZ_4, \mathcal{T}_{BZ_2}CZ_1) + g_1(\mathcal{A}_{CZ_4}CZ_1, \mathcal{T}_{BZ_2}CZ_3) \\
& + g_1(\mathcal{A}_{CZ_1}CZ_3, \mathcal{T}_{BZ_2}CZ_4) - g_1((\nabla_{BZ_3}\mathcal{A})(CZ_1, CZ_2), BZ_4) \\
& + g_1((\nabla_{BZ_4}\mathcal{A})(CZ_1, CZ_2), BZ_3) - g_1(\mathcal{A}_{CZ_1}BZ_3, \mathcal{A}_{CZ_2}BZ_4) \\
& + g_1(\mathcal{A}_{CZ_1}BZ_4, \mathcal{A}_{CZ_2}BZ_3) + g_1(\mathcal{T}_{BZ_3}CZ_1, \mathcal{T}_{BZ_4}CZ_2) \\
& - g_1(\mathcal{T}_{BZ_4}CZ_1, \mathcal{T}_{BZ_3}CZ_2) + g_1((\nabla_{CZ_4}\mathcal{A})(CZ_1, CZ_2), BZ_3) \\
& + g_1(\mathcal{A}_{CZ_1}CZ_2, \mathcal{T}_{BZ_3}CZ_4) - g_1(\mathcal{A}_{CZ_2}CZ_4, \mathcal{T}_{BZ_3}CZ_1) \\
& + g_1(\mathcal{A}_{CZ_4}CZ_1, \mathcal{T}_{BZ_3}CZ_2) - g_1((\nabla_{CZ_3}\mathcal{A})(CZ_1, CZ_2), BZ_4) \\
& - g_1(\mathcal{A}_{CZ_1}CZ_2, \mathcal{T}_{BZ_4}CZ_3) + g_1(\mathcal{A}_{CZ_2}CZ_3, \mathcal{T}_{BZ_4}CZ_1) \\
& + g_1(\mathcal{A}_{CZ_3}CZ_2, \mathcal{T}_{BZ_4}CZ_2) + R^*(CZ_1, CZ_2, CZ_3, CZ_4) \\
& + 2g_1(\mathcal{A}_{CZ_1}CZ_2, \mathcal{A}_{CZ_3}CZ_4) - g_1(\mathcal{A}_{CZ_2}CZ_3, \mathcal{A}_{CZ_1}CZ_4) \\
& + g_1(\mathcal{A}_{CZ_1}CZ_3, \mathcal{A}_{CZ_2}CZ_4),
\end{aligned}$$

$$\begin{aligned}
& R(Z_1, V_1, Z_2, V_2) \\
& = \cos^4 \Theta (-g_1((\nabla_{Z_1}\mathcal{T})(V_1, V_2), Z_2) - g_1((\nabla_{V_1}\mathcal{A})(Z_1, Z_2), V_2) \\
& \quad + g_1(\mathcal{T}_{V_1}Z_1, \mathcal{T}_{V_2}Z_2) - g_1(\mathcal{A}_{Z_1}V_1, \mathcal{A}_{Z_2}V_2)) \\
& \quad - \cos^2 \Theta (-g_1((\nabla_{Z_1}\mathcal{A})(Z_2, \omega\phi V_2), V_1) - g_1(\mathcal{A}_{Z_2}\omega\phi V_2, \mathcal{T}_{V_1}Z_1) \\
& \quad + g_1(\mathcal{A}_{\omega\phi V_2}Z_1, \mathcal{T}_{V_1}Z_2) + g_1(\mathcal{A}_{Z_1}Z_2, \mathcal{T}_{V_1}\omega\phi V_2)) \\
& \quad + R^*(Z_1, \omega\phi V_1, Z_2, \omega\phi V_2) + 2g_1(\mathcal{A}_{Z_1}\omega\phi V_1, \mathcal{A}_2\omega\phi V_2) \\
& \quad - g_1(\mathcal{A}_{\omega\phi V_1}Z_2, \mathcal{A}_{Z_1}\omega\phi V_2) + g(\mathcal{A}_{Z_1}Z_2, \mathcal{A}_{\omega\phi V_1}\omega\phi V_2) \\
& \quad - \cos^2 (-g_1(\nabla_{Z_2}\mathcal{A})(Z_1, \omega\phi V_1), V_2) - g_1(\mathcal{A}_{Z_1}\omega\phi V_1, \mathcal{T}_{V_2}Z_2) \\
& \quad + g_1(\mathcal{A}_{\omega\phi V_1}Z_2, \mathcal{T}_{V_2}Z_1) + g_1(\mathcal{A}_{Z_2}Z_1, \mathcal{T}_{V_2}\omega\phi V_1)) \\
& \quad - \cos^2 \Theta (-g_1((\nabla_{\omega V_2}\mathcal{T})(BZ_2, V_1), Z_1) - g_1((\nabla_{BZ_2}\mathcal{A})(\omega V_2, Z_1), V_1) \\
& \quad + g_1(\mathcal{T}_{BZ_2}\omega V_2, \mathcal{T}_{V_1}Z_1) - g_1(\mathcal{A}_{\omega V_2}BZ_2, \mathcal{A}_{Z_1}V_1)) \\
& \quad + \cos^2 \Theta (-g_1((\nabla_{Z_1}\mathcal{A})(CZ_2, \omega V_2), V_1) - g_1(\mathcal{T}_{CZ_2}\omega V_2, \mathcal{T}_{V_1}Z_1)
\end{aligned}$$

$$\begin{aligned}
& + g_1(\mathcal{A}_{\omega V_2} Z_1, \mathcal{T}_{V_1} C Z_2) + g_1(\mathcal{A}_{Z_1} C Z_2, \mathcal{T}_{V_1} \omega V_2) \\
& - g_1((\nabla_{\omega V_2} \mathcal{A})(Z_1, \omega \phi V_1), B Z_2) - g_1(\mathcal{A}_{Z_1} \omega \phi V_1, \mathcal{T}_{B Z_2} \omega V_2) \\
& + g_1(\mathcal{A}_{\omega \phi V_1} \omega V_2, \mathcal{T}_{B Z_2} Z_1) + g_1(\mathcal{A}_{\omega V_2} Z_1, \mathcal{T}_{B Z_2} \omega \phi V_1) \\
& - R^*(C Z_2, \omega V_2, Z_1, \omega \phi V_1) - 2g_1(\mathcal{A}_{C Z_2} \omega V_2, \mathcal{A}_{Z_1} \omega \phi V_1) \\
& - g_1(\mathcal{A}_{\omega V_2} Z_1, \mathcal{A}_{C Z_2} \omega \phi V_1) + g_1(\mathcal{A}_{C Z_2} Z_1, \mathcal{A}_{\omega V_2} \omega \phi V_1) \\
& - \cos^2 \Theta (-g_1((\nabla_{\omega V_1} \mathcal{T})(B Z_1, V_2), Z_2) - g_1((\nabla_{V_2} \mathcal{A})(\omega V_1, Z_2), B Z_1)) \\
& + g_1(\mathcal{T}_{V_2} Z_2, \mathcal{T}_{B Z_1} \omega V_1) - g_1(\mathcal{A}_{\omega V_1} Z_2, \mathcal{A}_{B Z_1} V_2) \\
& + \cos^2 \Theta (-g_1((\nabla_{Z_2} \mathcal{A})(C Z_1, \omega V_1), V_2) - g_1(\mathcal{A}_{C Z_1} \omega V_1, \mathcal{T}_{V_2} Z_2)) \\
& + g_1(\mathcal{A}_{\omega V_1} Z_2, \mathcal{T}_{V_2} C Z_1) + g_1(\mathcal{A}_{Z_2} C Z_1, \mathcal{T}_{V_2} \omega V_1) \\
& - g_1((\nabla_{\omega V_1} \mathcal{A})(Z_2, \omega \phi V_2), B Z_1) - g_1(\mathcal{A}_{Z_2} \omega \phi V_2, \mathcal{T}_{B Z_1} \omega V_1) \\
& + g_1(\mathcal{A}_{\omega \phi V_2} \omega V_1, \mathcal{T}_{B Z_1} Z_2) + g_1(\mathcal{A}_{\omega V_1} Z_2, \mathcal{T}_{B Z_1} \omega \phi V_2) \\
& - R^*(C Z_1, \omega V_1, Z_2, \omega \phi V_2) + 2g_1(\mathcal{A}_{C Z_1} \omega V_1, \mathcal{A}_{Z_2} \omega \phi V_2) \\
& - g_1(\mathcal{A}_{\omega V_1} Z_2, \mathcal{A}_{C Z_2} \omega \phi V_2) + g_1(\mathcal{A}_{C Z_1} Z_2, \mathcal{A}_{\omega V_1} \omega \phi V_2) \\
& - g_1((\nabla_{\omega V_1} \mathcal{T})(B Z_1, B Z_2), \omega V_1) - g_1((\nabla_{B Z_1} \mathcal{A})(\omega V_1, \omega V_2), B Z_2) \\
& + g_1(\mathcal{T}_{B Z_1} \omega V_1, \mathcal{T}_{B Z_2} \omega V_2) - g_1(\mathcal{A}_{\omega V_1} B Z_1, \mathcal{A}_{\omega V_2} B Z_2) \\
& + g_1((\nabla_{\omega V_1} \mathcal{T})(C Z_1, \omega V_2), B Z_1) + g_1(\mathcal{A}_{C Z_2} \omega V_2, \mathcal{T}_{B Z_1} \omega V_1) \\
& - g_1(\mathcal{A}_{\omega V_2} \omega V_1, \mathcal{T}_{B Z_1} C Z_2) - g_1(\mathcal{A}_{\omega V_1} C Z_2, \mathcal{T}_{B Z_1} \omega V_2) \\
& + g_1((\nabla_{\omega V_2} \mathcal{T})(C Z_2, \omega V_1), B Z_2) + g_1(\mathcal{A}_{C Z_1} \omega V_1, \mathcal{T}_{B Z_2} \omega V_2) \\
& - g_1(\mathcal{A}_{\omega V_1} \omega V_2, \mathcal{T}_{B Z_2} C Z_1) - g_1(\mathcal{A}_{\omega V_2} C Z_1, \mathcal{T}_{B Z_2} \omega V_1) \\
& + R^*(C Z_1, \omega V_1, C Z_2, \omega V_1) + 2g_1(\mathcal{A}_{C Z_1} \omega V_1, \mathcal{T}_{C Z_2} \omega V_2) \\
& - g_1(\mathcal{A}_{\omega V_1} C Z_2, \mathcal{A}_{C Z_1} \omega V_2) + g_1(\mathcal{A}_{C Z_1} C Z_2, \mathcal{A}_{\omega V_1} \omega V_2).
\end{aligned}$$

for $Z_1, Z_2 \in \Gamma(\ker F_*)^\perp$ and $V_1, V_2 \in \Gamma(\ker F_*)$.

Proof. Using equations (6)-(10), (40)-(45) and Lemma 1, we can easily get Lemma 5. $\square$

5. EXAMPLE

Example 1. Let N_1 be an Euclidean space given by

$$N_1 = \{(x_1, x_2, x_3, x_4) \in R^4 : (x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0)\}.$$

We define the Riemannian metric g_1 on N_1 given by

$$g_1 = e^{2x_4} dx_1^2 + e^{2x_4} dx_2^2 + e^{2x_4} dx_3^2 + dx_4^2$$

and the complex structure on J and N_1 defined as

$$J(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3).$$

Let $N_2 = \{(v_1, v_2, v_3) \in R^3\}$ be a Riemannian manifold with Riemannian metric g_2 on N_2 given by $g_2 = e^{2x_4} dv_1^2 + dv_2^2$.

Define a map $F : R^4 \rightarrow R^2$ by $F(x_1, x_2, x_3, x_4) = (\frac{x_1 - x_3}{\sqrt{2}}, x_4)$. Then we have

$$(\ker F_*) = \langle X_1 = (e_1 + e_3), X_2 = e_2 \rangle,$$

and

$$(\ker F_*)^\perp = \langle V_1 = (e_1 - e_3), V_2 = e_4 \rangle,$$

where

$$\left\{ e_1 = e^{-x_4} \frac{\partial}{\partial x_1}, \quad e_2 = e^{-x_4} \frac{\partial}{\partial x_2}, \quad e_3 = e^{-x_4} \frac{\partial}{\partial x_3}, \quad e_4 = \frac{\partial}{\partial x_4} \right\},$$

$$\left\{ e_1^* = \frac{\partial}{\partial v_1}, \quad e_2^* = \frac{\partial}{\partial v_2} \right\}$$

are bases on $T_q N_1$ and $T_{F(q)} N_2$ respectively, for all $p \in N_1$. By direct computations, we can see that

$$F_*(V_1) = \sqrt{2}e^{-x_4}e_1^*, F_*(V_2) = e_2^*$$

and

$$g_1(V_i, V_j) = g_2(F_*V_i, F_*V_j),$$

for all $V_i, V_j \in \Gamma(\ker F_*)^\perp$, $i, j = 1, 2$. Therefore F is a slant submersioon with slant angle $\Theta = \frac{\pi}{4}$.

Now, we will find smooth function h on N_1 satisfying $\mathcal{T}_X X = g_1(X, X)\nabla f$, for all $X \in \Gamma(\ker F_*)$. Since covariant derivative for vector fields $E = E_i \frac{\partial}{\partial x_i}$, $F = F_j \frac{\partial}{\partial x_j}$ on N_1 is defined as

$$(46) \quad \nabla_E F = E_i F_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + E_i \frac{\partial F_j}{\partial x_i} \frac{\partial}{\partial x_j},$$

where the covariant derivative of basis vector fields $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial x_i}$ is defined by

$$(47) \quad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \frac{\partial}{\partial x_k},$$

and Christoffel symbols are defined by

$$(48) \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{1jl}}{\partial x_i} + \frac{\partial g_{1il}}{\partial x_j} - \frac{\partial g_{1ij}}{\partial x_k} \right).$$

Now, we get

$$(49) \quad g_{1ij} = \begin{bmatrix} e^{2x_4} & 0 & 0 & 0 \\ 0 & e^{2x_4} & 0 & 0 \\ 0 & 0 & e^{2x_4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad g_1^{ij} = \begin{bmatrix} e^{-2x_4} & 0 & 0 & 0 \\ 0 & e^{-2x_4} & 0 & 0 \\ 0 & 0 & e^{-2x_4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By using (48) and (49), we get

$$(50) \quad \begin{aligned} \Gamma_{11}^1 &= 0, & \Gamma_{11}^2 &= 0, & \Gamma_{11}^3 &= 0, & \Gamma_{11}^4 &= -e^{-2x_4}, \\ \Gamma_{22}^1 &= 0, & \Gamma_{22}^2 &= 0, & \Gamma_{22}^3 &= 0, & \Gamma_{22}^4 &= -e^{-2x_4}, \end{aligned}$$

$$\begin{aligned}
\Gamma_{33}^1 &= 0, & \Gamma_{33}^2 &= 0, & \Gamma_{33}^3 &= 0, & \Gamma_{33}^4 &= -e^{-2x_4}, \\
\Gamma_{12}^1 &= \Gamma_{12}^2 = \Gamma_{12}^3 = \Gamma_{12}^4 = 0, \\
\Gamma_{21}^1 &= \Gamma_{21}^2 = \Gamma_{21}^3 = \Gamma_{21}^4 = 0, \\
\Gamma_{13}^1 &= \Gamma_{13}^2 = \Gamma_{13}^3 = \Gamma_{13}^4 = 0, \\
\Gamma_{31}^1 &= \Gamma_{31}^2 = \Gamma_{31}^3 = \Gamma_{31}^4 = 0, \\
\Gamma_{23}^1 &= \Gamma_{23}^2 = \Gamma_{23}^3 = \Gamma_{23}^4 = 0, \\
\Gamma_{32}^1 &= \Gamma_{32}^2 = \Gamma_{32}^3 = \Gamma_{32}^4 = 0.
\end{aligned}$$

Using equations (47) and (50), we obtain

$$\begin{aligned}
(51) \quad \nabla_{e_1} e_1 &= \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = -\frac{\partial}{\partial x_4}, \\
\nabla_{e_1} e_2 &= \nabla_{e_1} e_3 = \nabla_{e_2} e_1 = \nabla_{e_2} e_3 = 0, \\
\nabla_{e_3} e_1 &= \nabla_{e_3} e_2 = 0,
\end{aligned}$$

Therefore

$$(52) \quad \nabla_{X_1} X_1 = \nabla_{e_1+e_3} e_1 + e_3 = -2\frac{\partial}{\partial x_4}, \quad \nabla_{X_2} X_2 = \nabla_{e_2} e_2 = -\frac{\partial}{\partial x_4}.$$

Now, we have

$$\begin{aligned}
T_X X &= T_{\lambda_1 X_1 + \lambda_2 X_2} \lambda_1 X_1 + \lambda_2 X_2, \lambda_1, \lambda_2 \in R, \\
(53) \quad T_X X &= \lambda_1^2 T_{X_1} X_1 + \lambda_2^2 T_{X_2} X_2 + 2\lambda_1 \lambda_2 T_{X_1} X_2.
\end{aligned}$$

Using (52), we obtain

$$(54) \quad T_{X_1} X_1 = -2\frac{\partial}{\partial x_4}, \quad T_{X_2} X_2 = -\frac{\partial}{\partial x_4}, \quad T_{X_1} X_2 = 0.$$

Next, using (53) and (54), we get

$$(55) \quad \mathcal{T}_X X = -(2\lambda_1^2 + \lambda_2^2) \frac{\partial}{\partial x_4}.$$

Since $X = \lambda_1 X_1 + \lambda_2 X_2$, so

$$g_1(\lambda_1 X_1 + \lambda_2 X_2, \lambda_1 X_1 + \lambda_2 X_2) = 2\lambda_1^2 + \lambda_2^2.$$

For any smooth function h on R^4 , the gradient of h with respect to the metric g_1 is given by

$$\nabla h = \sum_{i,j=1}^4 g_1^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}.$$

Hence

$$\nabla h = e^{-2x_4} \frac{\partial h}{\partial x_1} \frac{\partial}{\partial x_1} + e^{-2x_4} \frac{\partial h}{\partial x_2} \frac{\partial}{\partial x_2} + e^{-2x_4} \frac{\partial h}{\partial x_3} \frac{\partial}{\partial x_3} + \frac{\partial h}{\partial x_4} \frac{\partial}{\partial x_4}.$$

Hence $\nabla h = \frac{\partial}{\partial x_4}$ for the function $h = x_4$. Then it is easy to see that $T_X X = -g_1(X, X)\nabla h$, thus by **Theorem 1**, is a Clairaut slant Riemannian submersion.

Example 2. Let N_1 be an Euclidean space given by

$$N_1 = \{(x_1, x_2, x_3, x_4) \in R^4 : (x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0)\}.$$

We define the Riemannian metric g_1 on N_1 given by

$$g_1 = e^{2x_4} dx_1^2 + e^{2x_4} dx_2^2 + e^{2x_4} dx_3^2 + dx_4^2$$

and the complex structure on J and N_1 defined as

$$J(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3).$$

Let $N_2 = \{(v_1, v_2, v_3) \in R^3\}$ be a Riemannian manifold with Riemannian metric g_2 on N_2 given by $g_2 = e^{2x_4} du_1^2 + du_2^2$.

Define a map $F : R^4 \rightarrow R^2$ by $F(x_1, x_2, x_3, x_4) = (\frac{x_2 - \sqrt{3}x_3}{2}, x_4)$. Then we have

$$(\ker F_*) = \langle V_1 = e_1, V_2 = \sqrt{3}e_2 + e_3 \rangle,$$

and

$$(\ker F_*)^\perp = \langle H_1 = e_2 - \sqrt{3}e_3, H_2 = e_4 \rangle,$$

where

$$\left\{ e_1 = e^{-x_4} \frac{\partial}{\partial x_1}, \quad e_2 = e^{-x_4} \frac{\partial}{\partial x_2}, \quad e_3 = e^{-x_4} \frac{\partial}{\partial x_3}, \quad e_4 = \frac{\partial}{\partial x_4} \right\},$$

$$\left\{ e_1^* = \frac{\partial}{\partial u_1}, \quad e_2^* = \frac{\partial}{\partial u_2} \right\}$$

are bases on $T_q N_1$ and $T_{F(q)} N_2$ respectively, for all $p \in N_1$. By direct computations, we can see that

$$F_*(H_1) = 2e^{-x_4} \frac{\partial}{\partial u_1}, F_*(H_2) = \frac{\partial}{\partial u_1}$$

and

$$g_1(H_i, H_j) = g_2(F_* H_i, F_* H_j),$$

for all $H_i, H_j \in \Gamma(\ker F_*)^\perp$, $i = 1, 2$. Therefore F is a slant submersion with slant angle $\Theta = \frac{\pi}{6}$.

Now, we will find smooth function R^4 on satisfying $\mathcal{T}_V V = g_1(V, V)\nabla h$ for all $V \in \Gamma(\ker F_*)$.

Using the given complex structure, we find

$$(56) \quad \begin{aligned} [e_1, e_1] &= [e_2, e_2] = [e_3, e_3] = [e_4, e_4] = 0, \\ [e_1, e_2] &= 0, [e_1, e_3] = 0, [e_1, e_4] = e_1, \\ [e_2, e_3] &= 0, [e_2, e_4] = e_2, [e_3, e_4] = e_3. \end{aligned}$$

The Levi-Civita connection ∇ of the metric g_1 is given by the Koszul's formula, which is

$$(57) \quad \begin{aligned} 2g_1(\nabla_X Z, V) &= Xg_1(Z, V) + Zg_1(V, X) - Vg_1(X, Z) \\ &\quad - g_1([X, Z], V) - g_1([Z, V], X) + g_1([V, X], Z). \end{aligned}$$

Using (56) and (57), we have

$$(58) \quad \begin{aligned} \nabla_{e_1} e_1 &= -e_4, \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_4, \\ \nabla_{e_2} e_3 &= 0, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -e_4. \end{aligned}$$

Therefore

$$(59) \quad \begin{aligned} \nabla_{V_1} V_1 &= \nabla_{e_1} e_1 = -e_4, \quad \nabla_{V_1} V_2 = \nabla_{e_1} \sqrt{3}e_2 + e_3 = 0 \\ \nabla_{V_2} V_1 &= \nabla_{\sqrt{3}e_2 + e_3} e_1 = 0, \quad \nabla_{\sqrt{3}e_2 + e_3} \sqrt{3}e_2 + e_3 = -4e_4. \end{aligned}$$

Now, we have

$$\mathcal{T}_V V = \nabla_{\lambda_1 V_1 + \lambda_2 V_2} \lambda_1 V_1 + \lambda_2 V_2, \lambda_1 \lambda_2 \in R.$$

$$(60) \quad \mathcal{T}_V V = \lambda_1^2 \mathcal{T}_{V_1} V_1 + \lambda_2^2 \mathcal{T}_{V_2} V_2 + \lambda_1 \lambda_2 \mathcal{T}_{V_1} V_2 + \lambda_1 \lambda_2 \mathcal{T}_{V_2} V_1.$$

Using (59), we obtain

$$(61) \quad \mathcal{T}_{V_1} V_1 = -e_4, \mathcal{T}_{V_1} V_2 = 0, \mathcal{T}_{V_2} V_1 = 0, \quad \mathcal{T}_{V_2} V_2 = -4e_4.$$

Using (60) and (61), we get

$$(62) \quad \mathcal{T}_V V = -(\lambda_1^2 + 4\lambda_2^2) \frac{\partial}{\partial x_4}.$$

Since $V = \lambda_1 V_1 + \lambda_2 V_2$, so

$$g_1(\lambda_1 V_1 + \lambda_2 V_2, \lambda_1 V_1 + \lambda_2 V_2) = \lambda_1^2 + 4\lambda_2^2.$$

For any smooth function h on R^4 the gradient of h with respect to the metric g_1 is given by

$$\nabla h = e^{-2x_4} \frac{\partial h}{\partial x_1} \frac{\partial}{\partial x_1} + e^{-2x_4} \frac{\partial h}{\partial x_2} \frac{\partial}{\partial x_2} + e^{-2x_4} \frac{\partial h}{\partial x_3} \frac{\partial}{\partial x_3} + \frac{\partial h}{\partial x_4} \frac{\partial}{\partial x_4}.$$

Hence $\nabla h = \frac{\partial}{\partial x_4}$ for the function $h = x_4$. Then it is easy to see that $\mathcal{T}_V V = g_1(V, V) \nabla h$, thus by **Theorem 1**, it is a Clairaut slant Riemannian submersion.

6. CONCLUSION

We introduce Clairaut slant submersions from Almost Hermitian manifolds onto Riemannian manifolds in the present paper. We discuss the geometrical properties of Clairaut slant submersions from Kähler manifolds onto Riemannian manifolds. With the help of **Theorem 1**, we prove F is a Clairaut slant Riemannian submersion in Euclidean space with almost complex structure. Further, we are studying about this discussed submersion with almost contact structure in Euclidean space.

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