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Functional impulsive fractional differential inclusions involving the Caputo-Hadamard derivative

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Abstract. This paper establishes sufficient conditions for the existence of solutions to fractional impulsive functional differential inclusions, utilizing fixed-point theorems for multivalued mappings.

1. Introduction

In this paper, we investigate the existence of solutions for a class of fractional impulsive differential inclusions:

$$ {}^{CH}D^r y(t) \in F(t, y_t), \quad t \in J = [a, T],\ t \neq t_k,\ a > 0,\ k = 1, \dots, m; \tag{1} $$

$$ \Delta y \,|_{t=t_k} = I_k(y(t_k^-)), \quad t = t_k,\ k = 1, \dots, m; \tag{2} $$

$$ y(t) = \phi(t), \quad t \in (a - r, a], \tag{3} $$

where ${}^{CH}D^r$ is the Caputo-Hadamard fractional derivative, $0 < r \leq 1$ and $F : J \times C([a-r, a], \mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\phi \in C([a-r, a], \mathbb{R})$ is given function, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m$, are continuous functions, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim\limits_{\varepsilon \to 0^+} y(t_k + \varepsilon)$ and $y(t_k^-) = \lim\limits_{\varepsilon \to 0^-} y(t_k + \varepsilon)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1, \dots, m$.

For any continuous function y defined on $[a-r, T]$ and any $t \in J$, we denote by y_t the element of $C_r := C([a-r, a], \mathbb{R})$, defined by

$$ y_t(\theta) = y(t + \theta), \quad \theta \in [a - r, a]. $$

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Hence, $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t. Also, C_τ is endowed with the norm

$$\|\phi\|_{C_\tau} := \sup\{|\phi(\theta)| : a-\tau \leq \theta \leq a\}.$$

Fractional calculus is one of the new fields of current investigation. In particular, fractional differential operators are used in a much better way than ordinary differential operators to gives some models of some physical phenomena. It has been noted that much of the work on this subject is focused on the fractional differential equations of Riemann-Liouville and Caputo.

The Hadamard fractional derivative, introduced in 1892 [23], is another type of fractional derivative that appears side by side with the Riemann-Liouville and Caputo derivatives in the literature, which varies from the previous ones in the type of derivative that contains the arbitrary logarithmic function. Further details can be found in [1–3,6,30,32,33]. Next, Jarad et al, proposed a Caputo-type modification of the Hadamard fractional derivative in [31].

In [19], the authors studied an initial value problem (IVP) fractional functional and neutral functional differential equations with Riemman-Liouville derivative and infnite delay. Recently, some researchers have concentrated on fractional impulsive differential equations with Hadamard and Caputo-Hadamard derivatives (see [13, 14, 16, 17, 25, 26] and references therein). Initial value problems for fractional impulsive functional and neutral functional differential equations and inclusions with Caputo-Hadamard derivative were investigated in [8, 9, 15, 25].

The rest of this paper is organized as follows. In Section 2 we present some basic definitions and preliminary results that will be used to prove our main results. In Section 3 we give two result for the problem (1)–(3). The first is based on the nonlinear alternative of Leray-Schauder when the right hand side is convex and the second result is based on a fixed point theorem due to Covitz and Nadler [21], when the right hand side is not convex. Finally, an example is given to illustrate our results.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts that will be used in the rest of this paper.

Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Let $L^1(J,\mathbb{R})$ be the Banach space of functions $y : J \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_a^T |y(t)|dt.$$

Further, let $AC([a,b],R)$ be the space of functions $y : J \to \mathbb{R}$, which are absolutely continuous, $(X, \|\cdot\|)$ is a Banach space and:

$$\begin{aligned} P_{cl}(X) &= \big\{Y \in \mathcal{P}(X) : Y \text{ is closed}\big\}, \\ P_b(X) &= \big\{Y \in \mathcal{P}(X) : Y \text{ is bounded}\big\}, \\ P_{cp}(X) &= \big\{Y \in \mathcal{P}(X) : Y \text{ is compact}\big\}, \\ P_{cp,c}(X) &= \big\{Y \in \mathcal{P}(X) : Y \text{is compact and convex}\big\}. \end{aligned}$$

A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. A multivalued map G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\big\{\sup\{|y| : y \in G(x)\}\big\}$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \to d(y, G(t)) = \inf\big\{|y - z| : z \in G(t)\big\}$$

is measurable.

Definition 1. A multivalued map $F : J\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if:

(1) $t \to F(t,u)$ is measurable for each $u \in \mathbb{R}$.
(2) $u \to F(t,u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in AC(J,\mathbb{R})$, define the set of selections of F by

$$S_{F,y} = \big\{v \in L^1([a,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ a.e } t \in [a,T]\big\}.$$

Let (X,d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\big\}.$$

Definition 2. A multivalued operator $N : X \to P_{cl}(X)$ is called

(1) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y), \quad \text{for each } x,y \in X,$$

(2) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Theorem 1 (Nonlinear alternative of Leray Schauder)**.** *Let X be a Banach space and C a nonempty closed convex subset of X. Let U a a nonempty convex subset of C with $0 \in U$ and $T : \bar{U} \to P_{cp,c}(X)$ is a upper semicontinuous compact map. Then either*

(i) *T has a fixed point in U, or*

(ii) *there exist $u \in \partial U$ and $\lambda \in [0,1]$ for which $u \in \lambda T(u)$.*

Lemma 1 ([21])**.** *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \varnothing$.*

For more details on multivalued maps see the following references: Aubin and Cellina [10], Aubin and Frankowska [11], Deimling [22] and Castaing and Valadier [20].

Definition 3 ([30])**.** The Hadamard fractional integral of order $r > 0$ for a function $\rho \in L^1(J,\mathbb{R})$ is defined by

$$I^r\rho(t) = \frac{1}{\Gamma(r)} \int_a^t \left(\log \frac{t}{s}\right)^{r-1} \frac{\rho(s)}{s} ds,$$

provided the integral exists.

Definition 4 ([30])**.** The Hadamard fractional derivative of order $r > 0$ applied to the function $\rho \in AC^n_\delta(J,\mathbb{R})$ is defined by

$$(D^r_a\rho)(t) = \delta^n(I^{n-r}_a\rho)(t),$$

where $n-1 < r < n,\ n = [r]+1$, and $[r]$ is the integer part of r.

Lemma 2 ([5])**.** *Let $y \in AC^n_\delta(J,\mathbb{R})$ or $C^n_\delta(J,\mathbb{R})$. Then*

$$I^r({}^{HC}D^r y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{t}{a}\right)^k.$$

Definition 5 ([30])**.** The Hadamard fractional derivative of order $r > 0$ applied to the function $\rho \in AC^n_\delta(J,\mathbb{R})$ is defined by

$$(D^r_a\rho)(t) = \delta^n(I^{n-r}_a\rho)(t),$$

where $n-1 < r < n$, $n = [r]+1$, and $[r]$ is the integer part of r.

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$$I^r({}^{HC}D^r y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{t}{a}\right)^k.$$

3. Main result

Consider the following space

$$\Omega=\left\{\begin{array}{l} y:[a-\tau,T]\to\mathbb{R},\ \ y\in AC_\delta((t_k,t_{k+1}],\mathbb{R}),\ k=1,\dots,m,\\ \text{and there exist } y(t_k^+) \text{ and } y(t_k^-),\ k=1,\dots,m,\\ \text{with } y(t_k^-)=y(t_k),\ y(t)=\phi(t),\ t\in(a-r,a] \end{array}\right\}.$$

This space is a Banach space with the norm

$$\|y\|_\Omega=\sup_{t\in J}|y(t)|.$$

Set $J'=J\setminus\{t_1,\dots,t_m\}$.

Definition 6. A function $y\in\Omega$ is said to be a solution of (1)–(3) if there exists a function $\nu\in L^1([a,T],\mathbb{R})$ with $\nu(t)\in F(t,y_t)$ for each $t\in J$, such that ${}^{CH}D^r y(t)=\nu(t)$ on J', and the conditions (2)–(3).

Lemma 4. *Let $0<r\le 1$ and let $\sigma:J\to\mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t)=\begin{cases} \phi(t), & \text{if } t\in[a-r,a]\\ \dfrac{1}{\Gamma(r)}\displaystyle\int_a^t\left(\log\frac{t}{s}\right)^{r-1}\sigma(t)\frac{ds}{s}, & \text{if } t\in[a,t_1],\\ \dfrac{1}{\Gamma(r)}\displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\left(\log\frac{t_i}{s}\right)^{r-1}f(t,y_t)\frac{ds}{s} & \\ \quad+\dfrac{1}{\Gamma(r)}\displaystyle\int_{t_i}^{t}\left(\log\frac{t}{s}\right)^{r-1}\sigma(t)\frac{ds}{s} & \\ \quad+\displaystyle\sum_{i=1}^{k}I_i(y(t_i^-)), & \text{if } t\in(t_k,t_{k+1}]. \end{cases}\tag{4}$$

where $k=1,\dots,m$, if y is a solution of the fractional IVP

$${}^{CH}D^r_{t_k}y(t)=\sigma(t),\quad \text{for each } t\in J_k,\tag{5}$$

$$\Delta y|_{t=t_k}=I_k(y(t_k^-)),\quad k=1,\dots,m,\tag{6}$$

$$y(t)=\phi(t),\quad t\in(a-\tau,a].\tag{7}$$

Proof. Let y be a solution of (5)–(7).

For $t\in[a,t_1]$, from Lemma 3, we have

$$y(t)=\phi(a)+\frac{1}{\Gamma(r)}\int_a^t\left(\log\frac{t}{s}\right)^{r-1}\sigma(t)\frac{ds}{s},$$

then

$$y(t_1^+) = \phi(a) + \frac{1}{\Gamma(r)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} + I_1(y(t_1^-)).$$

For $t \in (t_1, t_2]$, by applying Lemma 3, we have

$$\begin{aligned} y(t) &= y(t_1^+) + \frac{1}{\Gamma(r)} \int_{t_1}^{t} \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} \\ &= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(r)} \int_{t_1}^{t} \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} \\ &= \phi(a) + I_1(y(t_1^-)) + \frac{1}{\Gamma(r)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(r)} \int_{t_1}^{t} \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s}. \end{aligned}$$

If $t \in (t_2, t_3]$, we have

$$\begin{aligned} y(t) &= y(t_2^+) + \frac{1}{\Gamma(r)} \int_{t_2}^{t} \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} \\ &= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\beta)} \int_{t_2}^{t} \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} \\ &= I_2(y(t_2^-)) + I_1(y(t_1^-)) + \frac{1}{\Gamma(r)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(r)} \int_{t_2}^{t} \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s}. \end{aligned}$$

For $t \in (t_k, t_{k+1}]$, from Lemma 3, we can obtain

$$\begin{aligned} y(t) = \frac{1}{\Gamma(r)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} \\ + \frac{1}{\Gamma(r)} \int_{t_i}^{t} \left(\log \frac{t}{s}\right)^{r-1} \sigma(s) \frac{ds}{s} \\ + \sum_{i=1}^{k} I_i(y(t_i^-)). \end{aligned}$$ □

Further, we assume that F is a compact and convex valued multivalued map.

Theorem 2. *Assume the following hypotheses hold:*

(H1) *The function $F : J \times C_\tau \to P_{cp,c}(\mathbb{R})$ is a Carathéodory multivalued map.*

(H2) *There exist a function $p \in C(J, \mathbb{R}^+)$ and a continuous non-decreasing function $\psi : [0,\infty) \to (0,\infty)$, such that*

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|\nu|, \nu \in F(t,u)\} \le p(t)\psi(|u|),$$
$$\text{for each } (t,u) \in (J, C_\tau).$$

(H3) *There exists a continuous non-decreasing function $\psi^* : [0,\infty) \to (0,\infty)$ such that*

$$\|I_k(u)\| \le \psi^*(|u|), \quad \text{for each } u \in \mathbb{R}.$$

(H4) *There exists a a constant $M > 0$, where $p_f = \sup_{t\in J} |p(t)|$, such that*

$$\frac{M}{\psi(M)\dfrac{\left(\log \dfrac{T}{a}\right)^r (m+1)p_f}{\Gamma(r+1)} + m\psi^*(M)} > 1. \tag{8}$$

(H5) *There exists $l \in L^1(J, \mathbb{R}^+)$, such that*

$$H_d(F(t,u), F(t,\bar{u})) \le l(t)|u - \bar{u}|_{C_\tau}, \quad \text{for every } u, \bar{u} \in C_\tau.$$

Then the IVP (1)–(3) has at least one solution on $[a-\tau, T]$.

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the multivalued operator $N : \Omega \to \Omega$ is defined by

$$N(y) = \left\{ \rho \in \Omega : \rho(t) = \begin{cases} \phi(t), \qquad \text{if } t \in [a-\tau, T]), \\ \dfrac{1}{\Gamma(r)} \displaystyle\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\log \frac{t_i}{s}\right)^{r-1} \nu(s)\frac{ds}{s} \\ + \dfrac{1}{\Gamma(r)} \displaystyle\int_{t_i}^{t} \left(\log \frac{t}{s}\right)^{r-1} \nu(s)\frac{ds}{s} \\ + \displaystyle\sum_{i=1}^{k} I_i(y(t_i^-)), \ \text{ if } t \in [t_k, t_{k+1}], \ \text{for } \nu \in S_{F,y}. \end{cases} \right.$$

Clearly, from Lemma 4, the fixed points of the operator N are solutions of the problem (1)–(3).

We shall show that N satisfies the assumptions of the nonlinear alternatives of Leray-Shauder.

The proof will be given in several steps.

Step 1. *$N(y)$ is convex for each $y \in \Omega$.*
Indeed if $\rho_1, \rho_2 \in N(y)$, then there exists $\nu_1, \nu_2 \in S_{F,y}$ such that for each $t \in J$, we have

$$\rho_i(t) = \frac{1}{\Gamma(r)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-1} \nu_i(s) \frac{ds}{s} + \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{r-1} \nu_i(s) \frac{ds}{s} + \sum_{a<t_k<t} I_k\left(y\left(t_k^-\right)\right), \quad i = 1, 2.$$

Let $0 \le \lambda \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} &(\lambda\rho_1 + (1-\lambda)\rho_2)(t) \\ &= \frac{1}{\Gamma(r)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-1} \left[\lambda\nu_1 + (1-\lambda)\nu_2\right](s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{r-1} \left[\lambda\nu_1 + (1-\lambda)\nu_2\right](s) \frac{ds}{s} + \sum_{a<t_k<t} I_k\left(y\left(t_k^-\right)\right). \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$\lambda\rho_1 + (1-\lambda)\rho_2 \in N(y).$$

Step 2. *N maps bounded sets into bounded sets in Ω.*
Let $B_\eta = \{y \in \Omega : \|y\|_\infty \le \eta\}$ be a bounded set in Ω and $y \in B_\eta$. Then for each $\rho \in N(y)$, there exists $\nu \in S_{F,y}$ such that, for each $t \in J$, we have

$$\rho(t) = \frac{1}{\Gamma(r)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-1} \nu(s) \frac{ds}{s} + \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{r-1} \nu(s) \frac{ds}{s} + \sum_{a<t_k<t} I_k\left(y\left(t_k^-\right)\right).$$

By (H2) and (H3), we have

$$\begin{aligned} |\rho(t)| &\le \frac{1}{\Gamma(r)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{r-1} |\nu(s)| \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{r-1} |\nu(s)| \frac{ds}{s} + \sum_{a<t_k<t} |I_k(y(t_k^-))| \\ &\le \frac{1}{\Gamma(r+1)} \sum_{a<t_k<t} \left(\log \frac{t_k}{t_{k-1}}\right)^{r} p_f \psi\left(\|y\|_\infty\right) \\ &\quad + \frac{1}{\Gamma(r+1)} \left(\log \frac{t}{t_k}\right)^{r} p_f \psi(\|y\|_\infty) + m\psi^*(\|y\|_\infty) \\ &\le \frac{(m+1)p_f}{\Gamma(r+1)} \left(\log \frac{T}{a}\right)^{r} \psi(\|y\|_\infty) + m\psi^*(\|y\|_\infty). \end{aligned}$$

Thus

$$\|\rho\|_\infty \leq \frac{(m+1)p_f}{\Gamma(r+1)}\left(\log\frac{T}{a}\right)^r \psi(\eta)+m\psi^*(\eta):=\ell.$$

Step 3. *N maps bounded sets into equicontinuous sets of Ω.*
Let $\tau_1,\tau_2\in J,\tau_1<\tau_2$, and let $y\in B_\eta$ be a bounded set of Ω as in Step 2 and $\rho\in N(y)$. Then there exists $\nu\in B_\eta$ such that

$$\begin{aligned}
|\rho(\tau_2)-\rho(\tau_1)| \leq{} & \frac{1}{\Gamma(r)}\sum_{\tau_1<t_k<\tau_2}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}\nu(s)\frac{ds}{s}\\
&+\frac{1}{\Gamma(r)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{r-1}\nu(s)\frac{ds}{s}\\
&+\frac{1}{\Gamma(r)}\int_{t_k}^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{r-1}-\left(\log\frac{\tau_1}{s}\right)^{r-1}\right]\nu(s)\frac{ds}{s}\\
&+\sum_{\tau_1<t_k<\tau_2} I_k\left(y\left(t_k^-\right)\right).
\end{aligned}$$

As $\tau_1\to\tau_2$, the right hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we can conclude that N is completely continuous.

Step 4. N has a closed graph.
Let $\rho_n\to\rho_*$ and $y_n\to y_*$. We will prove that $\rho_*\in N(y_*)$. Now $\rho_n\in N(y_n)$ implies there exists $\nu_n\in S_{F,y_n}$, such that for each $t\in J$

$$\begin{aligned}
\rho_n(t)={} & \frac{1}{\Gamma(r)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}\nu_n(s)\frac{ds}{s}\\
&+\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}\nu_n(s)\frac{ds}{s}+\sum_{a<t_k<t} I_k\left(y\left(t_k^-\right)\right).
\end{aligned}$$

We must show that there exists $\nu_*\in S_{F,y_*}$, show that, for each $t\in J$,

$$\begin{aligned}
\rho_*(t)={} & \frac{1}{\Gamma(r)}\sum_{a<t_k<t}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-1}\nu_*(s)\frac{ds}{s}\\
&+\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}\nu_*(s)\frac{ds}{s}+\sum_{a<t_k<t} I_k\left(y\left(t_k^-\right)\right).
\end{aligned}$$

Since $F(t,\cdot)$ is upper semi-continuous, then for every $\epsilon>0$, there exists a natural number $n_0(\epsilon)$, for every $n\geq n_0$, we have

$$\nu_n(t)\in F\left(t,y_{n,t}\right)\subset F\left(t,y_{*,t}\right))+\epsilon B(0,1),\quad \text{a.e. } t\in J.$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $\nu_{n_m}(\cdot)$ such that

$$\nu_{n_m}(\cdot) \to \nu_*(\cdot), \quad \text{as } m \to \infty$$

and

$$\nu_*(t) \in F\left(t, y_{*,t}\right), \quad \text{a.e. } t \in J.$$

For every $w \in F\left(t, y_{*,t}\right)$, we have

$$|\nu_{n_m}(t) - \nu_*(t)\| \leq |\nu_{n_m}(t) - w| + |w - \nu_*(t)|.$$

Then

$$|\nu_{n_m}(t) - \nu_*(t)| \leq d(\nu_{n_m}(t), F(t, y_*(t)).$$

We obtain an analogous relation by interchanging the roles of ν_{n_m} and ν_*, and it follows that

$$|\nu_{n_m}(t) - \nu_*(t)| \leq H_d(F(t,(y_n)_t), F(t,(y_*)_t) \leq l(t)\|(y_n)_t - (y_*)_t\|_{C_\tau}.$$

Then

$$\begin{aligned}
|\rho_{n_m}(t) - \rho_*(t)| &\leq \frac{1}{\Gamma(r)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log\frac{t_k}{s}\right)^{r-1} |\nu_{n_m}(s) - \nu_*(s)| \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(r)} \int_{t_k}^{t} \left(\log\frac{t}{s}\right)^{r-1} |\nu_{n_m}(s) - \nu_*(s)| \frac{ds}{s} \\
&\quad + \sum_{a<t_k<t} |I_k(y_{n_m}(t_k^-)) - I_k(y_*(t_k^-))| \\
&\leq \frac{m}{\Gamma(r+1)} \left(\log\frac{T}{a}\right)^{r} \int_a^T l(s)ds\|y_{n_m} - y_*\|_\infty \\
&\quad + \frac{1}{\Gamma(r+1)} \left(\log\frac{T}{a}\right)^{r} \int_a^T l(s)ds\|y_{n_m} - y_*\|_\infty \\
&\quad + \sum_{a<t_k<t} |I_k(y_{n_m}(t_k^-)) - I_k(y_*(t_k^-))|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\rho_{n_m}(t) - \rho_*(t)\|_\infty &\leq \frac{m+1}{\Gamma(r+1)} \left(\log\frac{T}{a}\right)^{r} \int_a^T l(s)ds\|y_{n_m} - y_*\|_\infty \\
&\quad + \sum_{a<t_k<t} \left|I_k\left(y_{n_m}\left(t_k^-\right)\right) - I_k\left(y_*\left(t_k^-\right)\right)\right| \to 0, \quad \text{as } m \to \infty.
\end{aligned}$$

Step 5. *A priori bounds on solutions.*
Let $y \in \Omega$ be such that $y \in \lambda N(y)$ with $\lambda \in (0,1]$. Then there exists $\nu \in S_{F,y}$ for each $t \in J$, we have

$$|y(t)| \leq \frac{1}{\Gamma(r)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log\frac{t_k}{s}\right)^{r-1} p(s)\psi(|y_s|)\frac{ds}{s}$$

$$+\frac{1}{\Gamma(r)}\int_{t_k}^{t}\left(\log\frac{t}{s}\right)^{r-1}p(s)\psi(|y_s|)\frac{ds}{s}+\sum_{a<t_k<t}\psi^*(|y(t_k^-)|)$$
$$\leq\frac{(m+1)p_f}{\Gamma(r+1)}\left(\log\frac{T}{a}\right)^{r}\psi\left(\|y\|_\infty\right)+m\psi^*\left(\|y\|_\infty\right).$$

Thus

$$\frac{\|y\|_\infty}{\dfrac{(m+1)\left(\log\frac{T}{a}\right)^{r}p_f}{\Gamma(r+1)}\psi\left(\|y\|_\infty\right)+m\psi^*\left(\|y\|_\infty\right)}\leq 1.$$

Then, by condition (8), there exists $M>0$ such that $\|y\|_\infty\neq M$.

Let $U=\{y\in\Omega:\ \|y\|_\infty<M\}$. The operator $N:\bar{U}\mathcal{P}(\Omega)$ is upper semi-continuous and completely continuous. From the choice of U, there is no $y\in\partial U$ such that $y\in\lambda N(y)$ for some $\lambda\in(0,1]$. As a consequence of the nonlinear alternative of Leray-Schauder, we deduce that N has a fixed point $y\in\bar{U}$, which is a solution of the problem (1)–(3).

This completes the proof. □

3.1. **The nonconvex case.** We present now a result for the problem (1)–(2) with a nonconvex valued right hand side. Our consideration are based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler [21]. The proof will be given in two steps.

Theorem 3. *Assume (H5) and the following hypotheses hold:*

(H6) *$F:J\times C_\tau\to\mathcal{P}(\mathbb{R})$ has the property that $F(\cdot,u):J\to\mathcal{P}(\mathbb{R})$ is measurable, conx and integrably bounded for each $u\in\mathbb{R}$.*

(H7) *There exists a constant $l^*>0$, let $l=\sup_{t\in J}\{l(t)\}$ such that*

$$|I_k(u)-I_k(\bar{u})|\leq l^*|u-\bar{u}|,\quad \textit{for each } u,\bar{u}\in\mathbb{R},$$

if

$$\left[\frac{(m+1)l\left(\log\dfrac{T}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}+ml^*\right]<1, \tag{9}$$

then the VIP (1)–(3) has at least one solution on $[a-\tau,T]$.

Proof.

Step 1. *$N(y)\in\mathcal{P}(\Omega)$ for each $y\Omega$.*

Indeed, let $\{y_n\}_{n\geq 1}\in N(y)$ be such that $y_n\to\bar{y}$ in $C([a-\tau,T],\mathbb{R})$, and there exists $\nu_n\in S_{F,y}$ such that, for each $t\in J$,

$$y_n(t) = \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} \nu_n(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \nu_n(s) \frac{ds}{s} + \sum_{a<t_k<t} I_k(y(t_k^-)).$$

Using the fact that F has compact values and from (H5), we may pass to a subsequence if necessary to get that ν_n converges weakly to some $\nu \in L^1_w(J, \mathbb{R})$ (the space endowed with the weak topology). An application of Mazur's theorem implies that $\{(\nu_n)\}$ converges strongly to ν and hence $\nu S_{F,y}$. Then for each $t \in J$,

$$y_n(t) \to \bar{y}(t) = \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} \nu(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \nu(s) \frac{ds}{s} + \sum_{a<t_k<t} I_k(y(t_k^-)).$$

So, $\bar{y} \in N(y)$.

Step 2. *There exist $\gamma < 1$ such that*

$$H_d(N(y), N(\bar{y})) \leq \gamma \|y - \bar{y}\|_\infty$$

for each $y, \bar{y} \in \Omega$.
Then, there exists $\nu_1 \in F(t, y_t)$ for each such $t \in J$

$$\rho_1(t) = \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} \nu_1(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \nu_1(s) \frac{ds}{s} + \sum_{a<t_k<t} I_k(y(t_k^-)).$$

From (H5) it follows that

$$H_d(F(t, y_t), F(t, \bar{y}_t)) \leq l(t)|y_t - \bar{y}_t|.$$

Hence, there exists $w \in F(t, \bar{y}_t)$. such that

$$|\nu_1(t) - w| \leq l(t)|y_t - \bar{y}_t|, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \big\{w \in \mathbb{R} : |\nu_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|\big\}.$$

Since the multivalued operator $\nu(t) = U(t) \cap F(t, \bar{y}_t)$ is measurable, there exists a function $\nu_2(t)$ which is a measurable selection for ν. So, $\nu_2(t) \in F(t, \bar{y}_t)$, and for each $t \in J$

$$|\nu_1(t) - \nu_2(t)| \leq l(t)|y_t - \bar{y}_t|, \quad t \in J.$$

Let us define for each $t \in J$,

$$\rho_2(t) = \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} \nu_2(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \nu_2(s) \frac{ds}{s} + \sum_{a<t_k<t} I_k(\bar{y}(t_k^-)).$$

Then for each $t \in J$,

$$|\rho_1(t) - \rho_2(t)| \leq \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} |\nu_1(s) - \nu_2(s)| \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} |\nu_1(s) - \nu_2(s)| \frac{ds}{s} + \sum_{a<t_k<t} |I_k((y(t_k^-)) - I_k(\bar{y}(t_k^-))|.$$

Thus

$$\begin{aligned} |\rho_1(t) - \rho_2(t)| &\leq \frac{l}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \left(\log \frac{t_k}{s}\right)^{\alpha-1} |y(s) - \bar{y}(s)| \frac{ds}{s} \\ &\quad + \frac{l}{\Gamma(\alpha)} \int_{t_k}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} |y(s) - \bar{y}(s)| \frac{ds}{s} \\ &\quad + \sum_{a<t_k<t} l^* |y(t_k^-) - \bar{y}(t_k^-)| \\ &\leq \frac{ml \left(\log \frac{T}{a}\right)^{\alpha}}{\Gamma(\alpha+1)} \|y - \bar{y}\|_\infty + \frac{l \left(\log \frac{T}{a}\right)^{\alpha}}{\Gamma(\alpha+1)} \|y - \bar{y}\|_\infty + ml^* \|y - \bar{y}\|_\infty. \end{aligned}$$

Therefore,

$$\|\rho_1 - \rho_2\|_\infty \leq \left[\frac{(m+1) l \left(\log \frac{T}{a}\right)^{\alpha}}{\Gamma(\alpha+1)} + ml^* \right] \|y - \bar{y}\|_\infty.$$

For an analogous relation, obtained by interchanging the roles of y and $\bar{y}$, it follows that

$$H_d(N(y), N(\bar{y})) \leq \left[\frac{(m+1) l \left(\log \frac{T}{a}\right)^{\alpha}}{\Gamma(\alpha+1)} + ml^* \right] \|y - \bar{y}\|_\infty.$$

So, by (9), N is a contraction and thus, by Lemma 1, N has a fixed point y which is solution to (1)–(3).

The proof is complete. □

4. EXAMPLE

We still consider the following fractional differential inclusion:

$$(10) \qquad {}^{CH}\!D^r y(t) \in F(t, y_t), \quad \text{for a.e. } t \in J = [1, e],\ t \neq \frac{7}{4},$$

$$(11) \qquad \Delta y|_{t=\frac{7}{4}} = \frac{1}{8} y\left(\frac{7}{4}\right),$$

$$(12) \qquad y(t) = \phi(t), \quad t \in J = [1-\tau, e],$$

where ${}^{CH}\!D^r$ is the Caputo-Hadammard fractional derivative $F : [1, e] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map, $P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $\varphi \in C([1-\tau, T], \mathbb{R})$ with $\varphi(a) = 0$. Set

$$F(t, y) = \big\{ v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y),\ \ f_1, f_2 : [1-\tau, e] \times \mathbb{R} \longrightarrow \mathbb{R} \big\}.$$

We assume that for each $t \in J$, $f_1(t, .)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R}\colon\ f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in J$, $f_2(t, .)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R}\colon f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$).

Hence the condition (H2) holds with

$$F(t, y) \leq \max(|f_1(t, y), f_2(t, y)|) \leq p(t)\psi(|y|), \quad t \in J,\ y \in \mathbb{R}.$$

It is clear that F is compact and convex valued, and also upper semi-continuous. Hence, the condition (H2) holds if there exists $\psi^* : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$\|I_k(u)\| \leq \psi^*(|u|), \quad \text{for each } u \in \mathbb{R}.$$

Finally, we assume that condition (H2) there exists a number $M > 0$ such that

$$\frac{M}{|\phi(a)| + \psi(M) \dfrac{\left(\log \dfrac{T}{a}\right)^r (m+1) p_f}{\Gamma(r+1)} + m\psi^*(M)} > 1,$$

where $p_f = \sup_{t \in J} |p(t)|$.

Then by Theorem 2 are satisfied, the problem (10)–(12) has at least one solution on $[1-\tau, e]$.

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