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Existence and uniqueness of solutions to a coupled system of implicit fractional differential equations

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Abstract. Using Perov's fixed point theorem, the authors establish the existence and uniqueness of solutions to the coupled system of implicit fractional differential equations

$$\begin{cases} {}^cD^\alpha x(t) = f_1(t, x(t), y(t), {}^cD^\alpha x(t)), & t \in J, \\ {}^cD^\beta y(t) = f_2(t, x(t), y(t), {}^cD^\beta y(t)), & t \in J, \\ x(0) = L_1[x], \quad x'(0) = L_2[x], \\ y(0) = L_3[y], \quad y'(0) = L_4[y], \end{cases}$$

where $\alpha, \beta \in [1, 2)$, $J = [0, 1]$, ${}^cD^\alpha$ and ${}^cD^\beta$ are Caputo fractional derivatives, $f_i : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions for $i = 1, 2$, and the functionals L_j, $j = 1, 2, 3, 4$, are Stieltjes integrals. A second existence result is obtained by using a vector version of a fixed point theorem for a sum of two operators due to Krasnosel'skii. There is also a study of the structure of the set of solutions to the problem. Examples illustrate the results.

1. Introduction

The motivation for this research derives from two main sources. The primary one is a very nice paper by Bolojan and Precup [6], who studied the first order nonlocal implicit system

$$\begin{cases} x'(t) = g_1(t, x(t), y(t)) + h_1(t, x'(t), y'(t)), & t \in [0, 1], \\ y'(t) = g_2(t, x(t), y(t)) + h_2(t, x'(t), y'(t)), & t \in [0, 1], \\ x(0) = \alpha[x], \\ y(0) = \beta[y], \end{cases}$$

where g_i, $h_i : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions for $i = 1, 2$, and α, $\beta : C[0, 1] \to \mathbb{R}$ are continuous linear functionals with $\alpha[1] \neq 1$ and $\beta[1] \neq 1$.

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Additional motivation for the present paper comes from a forthcoming paper by Oumansour, Kadari, Graef, and Ouahab, who considered the problem

$$\begin{cases} {}^cD^\alpha x(t) = g_1(t, x(t), y(t)) + h_1(t, {}^cD^\alpha x(t)), & t \in J, \\ {}^cD^\beta y(t) = g_2(t, x(t), y(t)) + h_2(t, {}^cD^\beta y(t)), & t \in J, \\ x(0) = L_1[x], \\ y(0) = L_2[y], \end{cases}$$

where α, $\beta \in [0,1)$, $J = [0,1]$, ${}^cD^\alpha$ and ${}^cD^\beta$ are Caputo fractional derivatives of order α and β, respectively, $g_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $h_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions for $i = 1, 2$, and the functionals L_1 and L_2 are given by Stieltjes integrals with $L_1[1] \neq 1$ and $L_2[1] \neq 1$.

The study of fractional differential equations and their applications has been a hot area of research in the last twenty years due in part to their applicability to problems in the physical and social sciences as can be found, for example, in [13, 14, 17]. Implicit systems have also been a popular area of research as can be attested to by the monograph [1] and the references contained therein.

Existence of solutions to fractional order differential equations and inclusions is an important area of study that has been considered by a number of authors using a variety of initial or boundary conditions; see, for example, [3–5, 12, 16, 20] and the references contained therein. Coupled systems of fractional differential equations with local and nonlocal boundary conditions have been considered in the papers [2, 4, 20, 22, 23, 25, 26]. In [11], the authors considered systems of fractional differential equations with Caputo fractional derivatives of orders in $(0, 1]$. They used Perov's fixed point theorem in vector Banach spaces to obtain existence of solutions and the compactness of the solutions sets.

Here, we consider the coupled implicit fractional differential system with nonlocal boundary conditions

$$\text{(P)} \qquad \begin{cases} {}^cD^\alpha x(t) = f_1(t, x(t), y(t), {}^cD^\alpha x(t)), & t \in J, \\ {}^cD^\beta y(t) = f_2(t, x(t), y(t), {}^cD^\beta y(t)), & t \in J, \\ x(0) = L_1[x], \quad x'(0) = L_2[x], \\ y(0) = L_3[y], \quad y'(0) = L_4[y], \end{cases}$$

where α, $\beta \in [1,2)$, $J = [0,1]$, ${}^cD^\alpha$ and ${}^cD^\beta$, are Caputo fractional derivatives, and $f_i : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous functions for $i = 1, 2$. The linear functionals L_j, $j = 1, 2, 3, 4$, are given by Stieltjes integrals,

$$L_j[v(t)] = \int_0^1 v(s)\, dA_j(s), \quad j = 1, \ldots, 4,$$

where $A_j : [0,1] \to \mathbb{R}$ are nondecreasing functions that are right continuous on $[0, 1)$, left-continuous at $t = 1$, and satisfy $A_j(0) = 0$; this makes the dA_j

positive Stieltjes measures. In order to insure that certain denominators do not vanish, we ask that

$$(1) \qquad (1 - L_{2i}[1])(1 - L_{2i-1}[t]) - L_{2i}[t]L_{2i-1}[1] \neq 0, \quad \text{for } i = 1, 2,$$

where $L_j[t] = \int_0^1 s\, dA_j(s)$. Notice that this is an improvement over the condition $L_j[1] \neq 1$ as is required, for example, in [6].

The Problem (P) is called *coupled* because the unknown functions $x(t)$ and $y(t)$ appear in both equations in the system, and it is *implicit* because the unknown function $x(t)$ appears on both sides of the first equation in the system. Having an increased order of the Caputo derivatives and the additional initial conditions makes the calculations considerably more complicated.

Section 2 below contains some general results and preliminary concepts. In Section 3, we formulate problem (P) as a fixed point problem. Section 4 contains our sufficient conditions for the existence and uniqueness of solutions to (P) by applying Perov's fixed point theorem. We end the paper with an example of our main results.

2. Preliminaries

In this section, we give the necessary concepts and notation needed for understanding what follows. We take the space $C(J, \mathbb{R}) \times C(J, \mathbb{R})$ and use the norm

$$\|(x, y)\|_{C\times C} := (\|x\|_C, \|y\|_C)^T ,$$

where

$$\|v\|_C := \sup_{[0,1]} |v(t)|.$$

The norms of the functionals L_j are given by

$$\|L_j\| = \sup_{\|v\|=1} \left| \int_0^1 v(s) dA_j(s) \right| .$$

Basic properties of Laplace transforms can be found in a variety of places, so we will use them freely here. The function

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

is the Laplace transform of the function $f(t)$. We can recover $f(t)$ from the Laplace transform $F(s)$ by using the inverse Laplace transform. As is well known, the Laplace transform of the convolution

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

of the functions f and g that are zero for $t < 0$ is the product of the Laplace transforms of the two functions, that is,

$$\mathcal{L}(f(t) * g(t))(s) = F(s)G(s),$$

assuming that both $F(s)$ and $G(s)$ exist. This property can be used to evaluate the Laplace transform of the Riemann-Liouville fractional integral. Also well known is that the Laplace transform of the n-th derivative of the function $f(t)$ is given by

$$\mathcal{L}(f^n(t))(s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^k(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0). \tag{2}$$

We will need the following definitions of the Riemann-Liouville fractional integral and the Caputo fractional derivative of a function.

Definition 1 ([15])**.** The Riemann-Liouville fractional integral $I^\epsilon h$ of order ϵ ($\epsilon > 0$) of the function h is defined by

$$I^\epsilon h(t) = \frac{1}{\Gamma(\epsilon)} \int_0^t (t-s)^{\epsilon-1} h(s) ds, \quad t > 0,$$

provided the right-hand side exists, where Γ is the Euler Gamma function given by $\Gamma(\epsilon) = \int_0^t t^{\epsilon-1} e^{-t} dt$, $\epsilon > 0$.

Definition 2 ([1, Definition 1.6], [15, Definition 2.1])**.** For a function $h \in AC^n(J, \mathbb{R})$, the Caputo fractional-order derivative of order ϵ of h is defined by

$${}^cD_0^\epsilon h(t) = \frac{1}{\Gamma(n-\epsilon)} \int_0^t (t-s)^{n-\epsilon-1} h^{(n)}(s) ds,$$

where $n-1 < \epsilon < n$ and $n = [\epsilon] + 1$.

We will also need some notions from vector metric spaces.

Definition 3 ([9, Definition 7.1], [21, Definition 2.1])**.** Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u, v) \geq 0$ for all u, $v \in X$, and if $d(u, v) = 0$, then u=v;
(ii) $d(u, v) = d(v, u)$ for all u, $v \in X$;
(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all u, v, $w \in X$.

A set X with a vector-valued metric d is called a *generalized metric space*. In this space, the notions of Cauchy sequence, convergence, completeness, and open and closed sets are similar to those in usual metric spaces. Here, if x, $y \in \mathbb{R}^n$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, \ldots, n$. The pair (X, d) is a generalized metric space with

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \vdots \\ d_n(x, y) \end{pmatrix}.$$

Notice that d is a generalized metric on X if and only if $d_i,\ i = 1, 2, \ldots, n$, are metrics on X.

Similarly, a *vector valued norm* on a linear space X is a mapping $\|\cdot\| : X \to \mathbb{R}^n_+$ with $\|x\| = 0$ only for $x = 0$, $\|\lambda x\| = |\lambda|\|x\|$ for $x \in X$ and $\lambda \in \mathbb{R}$, and $\|x + y\| \leq \|x\| + \|y\|$ for every x, $y \in X$. Associated to a vector valued norm $\|\cdot\|$ is a *vector valued metric* $d(x, y) := \|x - y\|$, and we say that $(X, \|\cdot\|)$ is a *generalized Banach space* if X is complete with respect to d.

Next, we define what is meant by a matrix that is convergent to zero. Here, $\mathcal{M}_{n\times n}(\mathbb{R}_+)$ denotes the set of all $n \times n$ matrices over $\mathbb{R}_+$.

Definition 4 ([21, Definition 2.4], [24, Definition 1.9])**.** A square matrix with real entries is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, all the eigenvalues of M are in the open unit disc $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denotes the identity matrix in $\mathcal{M}_{n\times n}(\mathbb{R}_+)$.

The following result gives some characterizations of a matrix that converges to zero.

Theorem 1 ([24])**.** *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$; the following assertions are equivalent:*

(a) *M is convergent to zero;*
(b) *$M^k \to 0$ as $k \to \infty$;*
(c) *The matrix $(I - M)$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots ;$$

(d) *The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Remark 1. Some simple examples of 2×2 matrices that converge to zero are:

(i) $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

(ii) $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

(iii) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $|a - b| < 1$, $a > 1$, $b > 0$.

Next, we define what we mean by a contractive operator.

Definition 5 ([9, Section 2.3])**.** Let (X, d) be a generalized metric space. An operator $N : X \to X$ is contractive associated with a generalized metric d on X, if there exists a convergent to zero matrix M such that

$$d(T(x), T(y)) \leq Md(x, y), \quad \text{for all } x, y \in X.$$

The following theorem is known as Perov's fixed point theorem.

Theorem 2 ([18], [19]). *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a contractive operator with matrix M. Then T has a unique fixed point u, and for each $u_0 \in X$,*

$$d(T^k(u_0), u) \le M^k(I - M)^{-1} d(u_0, T(u_0)), \quad \text{where } k \in \mathbb{N}.$$

3. Formulating the Fixed Point Problem

To apply Perov's theorem, we must first transform (P) into a fixed point problem. For $t \in J$, it is easy to see that

$${}^cD^\epsilon h(t) = I^{n-\epsilon} g(t) = \frac{1}{\Gamma(n-\epsilon)} \int_0^t (t-s)^{n-\epsilon-1} g(s) ds = (\psi_\epsilon * g)(t),$$

where $g(t) = h^{(n)}(t)$ and

$$\psi_\epsilon(t) = \begin{cases} \frac{1}{\Gamma(n-\epsilon)} t^{n-\epsilon-1}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

The Laplace transform of the function $t^{n-\epsilon-1}$ is

$$N(s) = \mathcal{L}\{t^{n-\epsilon-1}\} = \Gamma(n-\epsilon) s^{\epsilon - n}, \quad \text{for } n - \epsilon - 1 > -1.$$

If $n = 2$, as it would be for α, $\beta \in [1, 2)$, then using the Laplace transform of the convolution of $t^{n-\epsilon-1}$ and $g(t)$, we obtain the transform of the Caputo fractional derivative as

$$\mathcal{L}\{{}^cD^\epsilon x(t)\} = s^{-(1-\epsilon)} G_2(s), \tag{3}$$

where

$$G_2(s) = s^2 X(s) - s x'(0) - x(0) \tag{4}$$

by (2).

To obtain an equivalent fixed point form for (P), we set

$$u(t) = {}^cD^\alpha x(t) \quad \text{and} \quad v(t) = {}^cD^\beta y(t). \tag{5}$$

Applying the Laplace transform gives

$$\begin{aligned} \mathcal{L}\{u(t)\} &= \mathcal{L}\{{}^cD^\alpha x(t)\} \\ &= s^{-(1-\alpha)} G_2(s) \\ &= s^{1+\alpha} X(s) - s^\alpha x'(0) - s^{\alpha-1} x(0) = U(s). \end{aligned} \tag{6}$$

Similarly,

$$\mathcal{L}\{v(t)\} = s^{1+\beta} Y(s) - s^\beta y'(0) - s^{\beta-1} y(0) = V(s),$$

where $X(s)$, $U(s)$, $Y(s)$, and $V(s)$ are the Laplace transforms of $x(t)$, $u(t)$, $y(t)$, and $v(t)$, respectively.

It follows that

$$X(s) = s^{-1-\alpha} U(s) + s^{-1} x'(0) + s^{-2} x(0),$$

and taking the inverse transform gives

$$x(t)=\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^\alpha u(s)ds + x'(0)+tx(0). \tag{7}$$

From the condition $x(0)=L_1[x]$ in (P), we see that

$$\begin{aligned} x(0) &= L_1\left[\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^\alpha u(s)ds + x'(0)+tx(0)\right] \\ &= \frac{1}{\Gamma(\alpha+1)}L_1\left[\int_0^t (t-s)^\alpha u(s)ds\right] + L_1\left[x'(0)\right] + L_1\left[tx(0)\right] \\ &= \frac{1}{\Gamma(\alpha+1)}L_1\left[\int_0^t (t-s)^\alpha u(s)ds\right] + L_1\left[1\right]x'(0) + L_1\left[t\right]x(0), \end{aligned} \tag{8}$$

so

$$(1-L_1[t])\,x(0)=\frac{1}{\Gamma(\alpha+1)}L_1\left[\int_0^t (t-s)^\alpha u(s)ds\right] + L_1\left[1\right]x'(0).$$

Hence,

$$x(0)=\frac{1}{(1-L_1[t])}\left(\frac{1}{\Gamma(\alpha+1)}L_1\left[\int_0^t (t-s)^\alpha u(s)ds\right] + L_1\left[1\right]x'(0)\right). \tag{9}$$

A similar calculation gives

$$x'(0)=\frac{1}{(1-L_2[1])}\left(\frac{1}{\Gamma(\alpha+1)}L_2\left[\int_0^t (t-s)^\alpha u(s)ds\right] + L_2\left[t\right]x(0)\right). \tag{10}$$

Remark 2. If in fact, $L_1[t]=1$, then (8) could be solved for $x'(0)$ and the result would be the same as (10) with $L_1[t]=1$. That is, the solutions process would still work, but the resulting expression for $x(t)$ in (13) below would be somewhat simplified.

Solving the system consisting of (9) and (10), we obtain

$$\begin{aligned} x(0) &= \frac{(1-L_2[1])\frac{1}{\Gamma(\alpha+1)}L_1\left[\int_0^t (t-s)^\alpha u(s)ds\right]}{(1-L_1[t])(1-L_2[1])-L_1[1]L_2[t]} \\ &\quad + \frac{L_1[1]\frac{1}{\Gamma(\alpha+1)}L_2\left[\int_0^t (t-s)^\alpha u(s)ds\right]}{(1-L_1[t])(1-L_2[1])-L_1[1]L_2[t]} \end{aligned} \tag{11}$$

and

$$\begin{aligned} x'(0) &= \frac{(1-L_1[t])\frac{1}{\Gamma(\alpha+1)}L_2\left[\int_0^t (t-s)^\alpha u(s)ds\right]}{(1-L_2[1])(1-L_1[t])-L_2[t]L_1[1]} \\ &\quad + \frac{L_2[t]\frac{1}{\Gamma(\alpha+1)}L_1\left[\int_0^t (t-s)^\alpha u(s)ds\right]}{(1-L_2[1])(1-L_1[t])-L_2[t]L_1[1]}. \end{aligned} \tag{12}$$

Using (11) and (12) in (7) gives

$$
\begin{aligned}
x(t) = & \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^\alpha u(s)ds \\
& + \frac{(1-L_1[t])\frac{1}{\Gamma(\alpha+1)} L_2\left[\int_0^t (t-s)^\alpha u(s)ds\right]}{(1-L_2[1])(1-L_1[t]) - L_2[t]L_1[1]} \\
& + \frac{L_2[t]\frac{1}{\Gamma(\alpha+1)} L_1\left[\int_0^t (t-s)^\alpha u(s)ds\right]}{(1-L_2[1])(1-L_1[t]) - L_2[t]L_1[1]} \\
& + t\frac{(1-L_2[1])\frac{1}{\Gamma(\alpha+1)} L_1\left[\int_0^t (t-s)^\alpha u(s)ds\right]}{(1-L_1[t])(1-L_2[1]) - L_1[1]L_2[t]} \\
& + t\frac{L_1[1]\frac{1}{\Gamma(\alpha+1)} L_2\left[\int_0^t (t-s)^\alpha u(s)ds\right]}{(1-L_1[t])(1-L_2[1]) - L_1[1]L_2[t]}.
\end{aligned} \tag{13}
$$

In a completely analogous way we have:

$$
y(t) = \frac{1}{\Gamma(\beta+1)} \int_0^t (t-s)^\beta v(s)ds + y'(0) + ty(0),
$$

$$
y(0) = \frac{1}{(1-L_3[t])}\left(\frac{1}{\Gamma(\beta+1)} L_3\left[\int_0^t (t-s)^\beta v(s)ds\right] + L_3[1]y'(0)\right),
$$

$$
y'(0) = \frac{1}{(1-L_4[1])}\left(\frac{1}{\Gamma(\beta+1)} L_4\left[\int_0^t (t-s)^\beta v(s)ds\right] + L_4[t]y(0)\right),
$$

$$
\begin{aligned}
y(0) = & \frac{(1-L_4[1])\frac{1}{\Gamma(\beta+1)} L_3\left[\int_0^t (t-s)^\beta v(s)ds\right]}{(1-L_3[t])(1-L_4[1]) - L_3[1]L_4[t]} \\
& + \frac{L_3[1]\frac{1}{\Gamma(\beta+1)} L_4\left[\int_0^t (t-s)^\beta v(s)ds\right]}{(1-L_3[t])(1-L_4[1]) - L_3[1]L_4[t]},
\end{aligned}
$$

and

$$
\begin{aligned}
y'(0) = & \frac{(1-L_3[t])\frac{1}{\Gamma(\beta+1)} L_4\left[\int_0^t (t-s)^\beta v(s)ds\right]}{(1-L_4[1])(1-L_3[t]) - L_4[t]L_3[1]} \\
& + \frac{L_4[t]\frac{1}{\Gamma(\beta+1)} L_3\left[\int_0^t (t-s)^\beta v(s)ds\right]}{(1-L_4[1])(1-L_3[t]) - L_4[t]L_3[1]} \\
& + \frac{L_4[t]\frac{1}{\Gamma(\beta+1)} L_3\left[\int_0^t (t-s)^\beta v(s)ds\right]}{(1-L_4[1])(1-L_3[t]) - L_4[t]L_3[1]}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
y(t) = {} & \frac{1}{\Gamma(\beta+1)} \int_0^t (t-s)^\beta v(s)ds \\
& + \frac{(1-L_2[t])\frac{1}{\Gamma(\beta+1)} L_4\left[\int_0^t (t-s)^\beta v(s)ds\right]}{(1-L_4[1])(1-L_3[t]) - L_4[t]L_3[1]} \\
& + \frac{L_4[t]\frac{1}{\Gamma(\beta+1)} L_3\left[\int_0^t (t-s)^\beta v(s)ds\right]}{(1-L_4[1])(1-L_3[t]) - L_4[t]L_3[1]} \\
& + t\frac{(1-L_4[1])\frac{1}{\Gamma(\beta+1)} L_3\left[\int_0^t (t-s)^\beta v(s)ds\right]}{(1-L_3[t])(1-L_4[1]) - L_3[1]L_4[t]} \\
& + t\frac{L_3[1]\frac{1}{\Gamma(\beta+1)} L_4\left[\int_0^t (t-s)^\beta v(s)ds\right]}{(1-L_3[t])(1-L_4[1]) - L_3[1]L_4[t]}. \qquad (14)
\end{aligned}
$$

Let

$$
\begin{aligned}
& F_1(u,v)(t) \\
& = f_1\Bigg(t, I^{\alpha+1}u(t) + \frac{(1-L_1[t])\frac{1}{\Gamma(\alpha+1)}L_2[k_1(t)] + L_2[t]\frac{1}{\Gamma(\alpha+1)}L_1[k_1(t)]}{(1-L_2[1])(1-L_1[t]) - L_2[t]L_1[1]} \\
& \quad + t\frac{(1-L_2[1])\frac{1}{\Gamma(\alpha+1)}L_1[k_1(t)] + L_1[1]\frac{1}{\Gamma(\alpha+1)}L_2[k_1(t)]}{(1-L_1[t])(1-L_2[1]) - L_1[1]L_2[t]}, \\
& \quad I^{\beta+1}v(t) + \frac{(1-L_2[t])\frac{1}{\Gamma(\beta+1)}L_4[k_2(t)] + L_4[t]\frac{1}{\Gamma(\beta+1)}L_3[k_2(t)]}{(1-L_4[1])(1-L_3[t]) - L_4[t]L_3[1]} \\
& \quad + t\frac{(1-L_4[1])\frac{1}{\Gamma(\beta+1)}L_3[k_2(t)] + L_3[1]\frac{1}{\Gamma(\beta+1)}L_4[k_2(t)]}{(1-L_3[t])(1-L_4[1]) - L_3[1]L_4[t]}, u\Bigg) \qquad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
& F_2(u,v)(t) \\
& = f_2\Bigg(t, I^{\alpha+1}u(t) + \frac{(1-L_1[t])\frac{1}{\Gamma(\alpha+1)}L_2[k_1(t)] + L_2[t]\frac{1}{\Gamma(\alpha+1)}L_1[k_1(t)]}{(1-L_2[1])(1-L_1[t]) - L_2[t]L_1[1]} \\
& \quad + t\frac{(1-L_2[1])\frac{1}{\Gamma(\alpha+1)}L_1[k_1(t)] + L_1[1]\frac{1}{\Gamma(\alpha+1)}L_2[k_1(t)]}{(1-L_1[t])(1-L_2[1]) - L_1[1]L_2[t]}, \\
& \quad I^{\beta+1}v(t) + \frac{(1-L_2[t])\frac{1}{\Gamma(\beta+1)}L_4[k_2(t)] + L_4[t]\frac{1}{\Gamma(\beta+1)}L_3[k_2(t)]}{(1-L_4[1])(1-L_3[t]) - L_4[t]L_3[1]} \\
& \quad + t\frac{(1-L_4[1])\frac{1}{\Gamma(\beta+1)}L_3[k_2(t)] + L_3[1]\frac{1}{\Gamma(\beta+1)}L_4[k_2(t)]}{(1-L_3[t])(1-L_4[1]) - L_3[1]L_4[t]}, v\Bigg), \qquad (16)
\end{aligned}
$$

where

$$k_1(t) = \int_0^t (t-s)^{\alpha} u(s)ds \quad \text{and} \quad k_2(t) = \int_0^t (t-s)^{\beta} v(s)ds.$$

Problem (P) is then equivalent to the system

$$\begin{cases} u = F_1(u,v), \\ v = F_2(u,v). \end{cases} \tag{17}$$

We define the operator

$$T : C(J,\mathbb{R}) \times C(J,\mathbb{R}) \to C(J,\mathbb{R}) \times C(J,\mathbb{R})$$

by

$$T(u,v) = (T_1(u,v), T_2(u,v)), \quad (u,v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R}), \tag{18}$$

where

$$\begin{cases} T_1(u,v) = F_1(u,v), \\ T_2(u,v) = F_2(u,v). \end{cases} \tag{19}$$

System (17) can then be regarded as a fixed point problem for T.

4. Existence and uniqueness of solutions

We are now able to present our main existence and uniqueness result. In so doing, we need to make the following assumptions about our problem.

(C) $f_i : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for $i \in \{1,2\}$ are jointly continuous functions and there exist nonnegative numbers a_i, b_i, and c_i, for $i \in \{1,2\}$ such that for all u, v, w, $\bar{u}$, $\bar{v}$, $\bar{w} \in \mathbb{R}$ and $t \in [0,1]$,

$$|f_1(t,u,v,w) - f_1(t,\bar{u},\bar{v},\bar{w}))| \leq a_1|u-\bar{u}| + b_1|v-\bar{v}| + c_1|w-\bar{w}|,$$
$$|f_2(t,u,v,w) - f_2(t,\bar{u},\bar{v},\bar{w}))| \leq a_2|u-\bar{u}| + b_2|v-\bar{v}| + c_2|w-\bar{w}|.$$

Next, we define the constants

$$\bar{L}_i = L_i[t] = \int_0^1 s dA_i(s), \quad i = 1,\ldots,4.$$

Notice that with this notation, (1) becomes

$$(1 - L_{2i}[1])(1-\bar{L}_{2i-1}) - \bar{L}_{2i}L_{2i-1}[1] \neq 0, \quad \text{for } i = 1,2. \tag{20}$$

We also define the quantities

$$M_1 = \left(\frac{a_1}{\Gamma(\alpha+2)} + c_1\right) + \frac{\frac{a_1(1-\bar{L}_1)}{\Gamma(\alpha+1)}\|L_2\|\frac{1}{\alpha}}{(1-L_2[1])(1-\bar{L}_1) - \bar{L}_2 L_1[1]}$$
$$+ \frac{\frac{a_1\bar{L}_2}{\Gamma(\alpha+1)}\|L_1\|\frac{1}{\alpha}}{(1-L_2[1])(1-\bar{L}_1) - \bar{L}_2 L_1[1]} + \frac{\frac{a_1(1-L_2[1])}{\Gamma(\alpha+1)}\|L_1\|\frac{1}{\alpha}}{(1-\bar{L}_1)(1-L_2[1]) - L_1[1]\bar{L}_2}$$

$$+\frac{\frac{a_1 L_1[1]}{\Gamma(\alpha+1)}\|L_2\|_{\frac{1}{\alpha}}}{(1-\bar{L}_1)(1-L_2[1])-L_1[1]\bar{L}_2},$$

$$M_2 = +\frac{b_1}{\Gamma(\beta+2)} + \frac{\frac{b_1(1-\bar{L}_2)}{\Gamma(\beta+1)}\|L_4\|_{\frac{1}{\beta}}}{(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4 L_3[1]}$$
$$+\frac{\frac{b_1\bar{L}_4}{\Gamma(\beta+1)}\|L_3\|_{\frac{1}{\beta}}}{(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4 L_3[1]} + \frac{\frac{b_1(1-L_4[1])}{\Gamma(\beta+1)}\|L_3\|_{\frac{1}{\beta}}}{(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4}$$
$$+\frac{\frac{b_1 L_3[1]}{\Gamma(\beta+1)}\|L_4\|_{\frac{1}{\beta}}}{(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4},$$

$$M_3 = \left(\frac{a_2}{\Gamma(\alpha+2)} + c_2\right) + \frac{\frac{a_2(1-\bar{L}_1)}{\Gamma(\alpha+1)}\|L_2\|_{\frac{1}{\alpha}}}{(1-L_2[1])(1-\bar{L}_1)-\bar{L}_2 L_1[1]}$$
$$+\frac{\frac{a_2\bar{L}_2}{\Gamma(\alpha+1)}\|L_1\|_{\frac{1}{\alpha}}}{(1-L_2[1])(1-\bar{L}_1)-\bar{L}_2 L_1[1]} + \frac{\frac{a_2(1-L_2[1])}{\Gamma(\alpha+1)}\|L_1\|_{\frac{1}{\alpha}}}{(1-\bar{L}_1)(1-L_2[1])-L_1[1]\bar{L}_2}$$
$$+\frac{\frac{a_2 L_1[1]}{\Gamma(\alpha+1)}\|L_2\|_{\frac{1}{\alpha}}}{(1-\bar{L}_1)(1-L_2[1])-L_1[1]\bar{L}_2},$$

and

$$M_4 = \frac{b_2}{\Gamma(\beta+2)} + \frac{\frac{b_2(1-\bar{L}_2)}{\Gamma(\beta+1)}\|L_4\|_{\frac{1}{\beta}}}{(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4 L_3[1]}$$
$$+\frac{\frac{b_2\bar{L}_4}{\Gamma(\beta+1)}\|L_3\|_{\frac{1}{\beta}}}{(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4 L_3[1]} + \frac{\frac{b_2(1-L_4[1])}{\Gamma(\beta+1)}\|L_3\|_{\frac{1}{\beta}}}{(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4}$$
$$+\frac{\frac{b_2 L_3[1]}{\Gamma(\beta+1)}\|L_4\|_{\frac{1}{\beta}}}{(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4}.$$

Theorem 3. *Assume that condition* (C) *holds. If the matrix*

$$\mathfrak{M} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \tag{21}$$

converges to 0*, then the problem* (P) *has a unique solution.*

Proof. Let (u,v), $(\bar{u},\bar{v}) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$. Then, we have

$$|T_1(u,v)(t) - T_1(\bar{u},\bar{v})(t)| \leq |F_1(u,v)(t) - F_1(\bar{u},\bar{v})(t)|$$
$$\leq \frac{a_1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha} |u(s) - \bar{u}(s)| ds$$
$$+ a_1 \frac{(1-\bar{L}_1)\frac{1}{\Gamma(\alpha+1)}(L_2[k_1(t)] - L_2[\bar{k}_1(t)]) + \bar{L}_2 \frac{1}{\Gamma(\alpha+1)}(L_1[k_1(t)] - L_1[\bar{k}_1(t)])}{(1-L_2[1])(1-\bar{L}_1) - \bar{L}_2 L_1[1]}$$

$$
\begin{aligned}
&+ a_1 \frac{(1-L_2[1])\frac{1}{\Gamma(\alpha+1)}(L_1[k_1(t)] - L_1[\bar{k}_1(t)]) + L_1[1]\frac{1}{\Gamma(\alpha+1)}(L_2[k_1(t)] - L_2[\bar{k}_1(t)])}{(1-\bar{L}_1)(1-L_2[1]) - L_1[1]\bar{L}_2} \\
&+ \frac{b_1}{\Gamma(\beta+1)}\int_0^t (t-s)^{\beta}|v(s)-\bar{v}(s)|ds \\
&+ b_1 \frac{(1-\bar{L}_2)\frac{1}{\Gamma(\beta+1)}(L_4[k_2(t)] - L_4[\bar{k}_2(t)]) + \bar{L}_4\frac{1}{\Gamma(\beta+1)}(L_3[k_2(t)] - L_3[\bar{k}_2(t)])}{(1-L_4[1])(1-\bar{L}_3) - \bar{L}_4 L_3[1]} \\
&+ b_1 \frac{(1-L_4[1])\frac{1}{\Gamma(\beta+1)}(L_3[k_2(t)] - L_3[\bar{k}_2(t)]) + L_3[1]\frac{1}{\Gamma(\beta+1)}(L_4[k_2(t)] - L_4[\bar{k}_2(t)])}{(1-\bar{L}_3)(1-L_4[1]) - L_3[1]\bar{L}_4} \\
&+ c_1|u-\bar{u}| \\
\le{}& \frac{a_1 t^{\alpha+1}}{\Gamma(\alpha+2)}\|u-\bar{u}\| + \frac{\frac{a_1(1-\bar{L}_1)}{\Gamma(\alpha+1)} L_2[k_1(t)-\bar{k}_1(t)]}{(1-L_2[1])(1-\bar{L}_1) - \bar{L}_2 L_1[1]} \\
&+ \frac{\frac{a_1\bar{L}_2}{\Gamma(\alpha+1)} L_1[k_1(t)-\bar{k}_1(t)]}{(1-L_2[1])(1-\bar{L}_1) - \bar{L}_2 L_1[1]} + \frac{a_1\frac{(1-L_2[1])}{\Gamma(\alpha+1)} L_1[k_1(t)-\bar{k}_1(t)]}{(1-\bar{L}_1)(1-L_2[1]) - L_1[1]\bar{L}_2} \\
&+ \frac{\frac{a_1 L_1[1]}{\Gamma(\alpha+1)} L_2[k_1(t)-\bar{k}_1(t)])}{(1-\bar{L}_1)(1-L_2[1]) - L_1[1]\bar{L}_2} \\
&+ \frac{b_1 t^{\beta+1}}{\Gamma(\beta+2)}\|v-\bar{v}\| + \frac{\frac{b_1(1-\bar{L}_2)}{\Gamma(\beta+1)} L_4[k_2(t)-\bar{k}_2(t)]}{(1-L_4[1])(1-\bar{L}_3) - \bar{L}_4 L_3[1]} \\
&+ \frac{\frac{b_1\bar{L}_4}{\Gamma(\beta+1)} L_3[k_2(t)-\bar{k}_2(t)]}{(1-L_4[1])(1-\bar{L}_3) - \bar{L}_4 L_3[1]} + \frac{b_1\frac{(1-L_4[1])}{\Gamma(\beta+1)} L_3[k_2(t)-\bar{k}_2(t)]}{(1-\bar{L}_3)(1-L_4[1]) - L_3[1]\bar{L}_4} \\
&+ \frac{\frac{b_1 L_3[1]}{\Gamma(\beta+1)} (L_4[k_2(t)-\bar{k}_2(t)]}{(1-\bar{L}_3)(1-L_4[1]) - L_3[1]\bar{L}_4} \\
&+ c_1\|u-\bar{u}\|.
\end{aligned}
$$

Now in view of the facts that

$$
\|L_i[k_i-\bar{k}_i]\| \le \|L_i\|\|k_i-\bar{k}_i\|, \quad i=1,2,3,4, \tag{22}
$$

and

$$
\|k_1-\bar{k_1}\| \le \frac{1}{\alpha}\|u-\bar{u}\| \quad \text{and} \quad \|k_2-\bar{k_2}|| \le \frac{1}{\beta}\|v-\bar{v}\|,
$$

we see that

$$
\begin{aligned}
&\|T_1(u,v)(t) - T_1(\bar{u},\bar{v})(t)\| \\
&\le \left(\frac{a_1}{\Gamma(\alpha+2)} + c_1\right)\|u-\bar{u}\| + \frac{\frac{a_1(1-\bar{L}_1)}{\Gamma(\alpha+1)}\|L_2\|\frac{1}{\alpha}\|u-\bar{u}\|}{(1-L_2[1])(1-\bar{L}_1) - \bar{L}_2 L_1[1]} \\
&\quad + \frac{\frac{a_1\bar{L}_2}{\Gamma(\alpha+1)}\|L_1\|\frac{1}{\alpha}\|u-\bar{u}\|}{(1-L_2[1])(1-\bar{L}_1) - \bar{L}_2 L_1[1]} + \frac{\frac{a_1(1-L_2[1])}{\Gamma(\alpha+1)}\|L_1\|\frac{1}{\alpha}\|u-\bar{u}\|}{(1-\bar{L}_1)(1-L_2[1]) - L_1[1]\bar{L}_2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\frac{a_1L_1[1]}{\Gamma(\alpha+1)}\|L_2\|_{\frac{1}{\alpha}}\|u-\bar{u}\|}{(1-\bar{L}_1)(1-L_2[1])-L_1[1]\bar{L}_2}+\frac{b_1}{\Gamma(\beta+2)}\|v-\bar{v}\|\\
&+\frac{\frac{b_1(1-\bar{L}_2)}{\Gamma(\beta+1)}\|L_4\|_{\frac{1}{\beta}}\|v-\bar{v}\|}{(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4L_3[1]}+\frac{\frac{b_1\bar{L}_4}{\Gamma(\beta+1)}\|L_3\|_{\frac{1}{\beta}}\|v-\bar{v}\|}{(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4L_3[1]}\\
&+\frac{\frac{b_1(1-L_4[1])}{\Gamma(\beta+1)}\|L_3\|_{\frac{1}{\beta}}\|v-\bar{v}\|}{(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4}+\frac{\frac{b_1L_3[1]}{\Gamma(\beta+1)}\|L_4\|_{\frac{1}{\beta}}\|v-\bar{v}\|}{(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4}\\
\leq&\left\{\left(\frac{a_1}{\Gamma(\alpha+2)}+c_1\right)+\frac{\frac{a_1(1-\bar{L}_1)}{\Gamma(\alpha+1)}\|L_2\|_{\frac{1}{\alpha}}}{(1-L_2[1])(1-\bar{L}_1)-\bar{L}_2L_1[1]}\right.\\
&+\frac{\frac{a_1\bar{L}_2}{\Gamma(\alpha+1)}\|L_1\|_{\frac{1}{\alpha}}}{(1-L_2[1])(1-\bar{L}_1)-\bar{L}_2L_1[1]}+\frac{\frac{a_1(1-L_2[1])}{\Gamma(\alpha+1)}\|L_1\|_{\frac{1}{\alpha}}\|u-\bar{u}\|}{(1-\bar{L}_1)(1-L_2[1])-L_1[1]\bar{L}_2}\\
&\left.+\frac{\frac{a_1L_1[1]}{\Gamma(\alpha+1)}\|L_2\|_{\frac{1}{\alpha}}}{(1-\bar{L}_1)(1-L_2[1])-L_1[1]\bar{L}_2}\right\}\|u-\bar{u}\|\\
&+\left\{\frac{b_1}{\Gamma(\beta+2)}+\frac{\frac{b_1(1-\bar{L}_2)}{\Gamma(\beta+1)}\|L_4\|_{\frac{1}{\beta}}}{(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4L_3[1]}\right.\\
&+\frac{\frac{b_1\bar{L}_4}{\Gamma(\beta+1)}\|L_3\|_{\frac{1}{\beta}}}{(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4L_3[1]}+\frac{\frac{b_1(1-L_4[1])}{\Gamma(\beta+1)}\|L_3\|_{\frac{1}{\beta}}}{(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4}\\
&\left.+\frac{\frac{b_1L_3[1]}{\Gamma(\beta+1)}\|L_4\|_{\frac{1}{\beta}}}{(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4}\right\}\|v-\bar{v}\|
\end{aligned}
$$

We now see that we can write this as

$$\|T_1(u,v)-T_1(\bar{u},\bar{v})\|_C\leq M_1\|u-\bar{u}\|_C+M_2\|v-\bar{v}\|_C.$$

Similar to what we did above, we can obtain

$$
\begin{aligned}
&\|T_2(u,v)-T_2(\bar{u},\bar{v})\|_C\\
\leq&\left\{\left(\frac{a_2}{\Gamma(\alpha+2)}+c_2\right)+\frac{\frac{a_2(1-\bar{L}_1)}{\Gamma(\alpha+1)}\|L_2\|_{\frac{1}{\alpha}}}{(1-L_2[1])(1-\bar{L}_1)-\bar{L}_2L_1[1]}\right.\\
&+\frac{\frac{a_2\bar{L}_2}{\Gamma(\alpha+1)}\|L_1\|_{\frac{1}{\alpha}}}{(1-L_2[1])(1-\bar{L}_1)-\bar{L}_2L_1[1]}+\frac{\frac{a_2(1-L_2[1])}{\Gamma(\alpha+1)}\|L_1\|_{\frac{1}{\alpha}}}{(1-\bar{L}_1)(1-L_2[1])-L_1[1]\bar{L}_2}\\
&\left.+\frac{\frac{a_2L_1[1]}{\Gamma(\alpha+1)}\|L_2\|_{\frac{1}{\alpha}}}{(1-\bar{L}_1)(1-L_2[1])-L_1[1]\bar{L}_2}\right\}\|u-\bar{u}\|_C
\end{aligned}
$$

$$+\left\{\frac{b_2}{\Gamma(\beta+2)}+\frac{\frac{b_2(1-\bar{L}_2)}{\Gamma(\beta+1)}\|L_4\|_{\frac{1}{\beta}}}{(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4L_3[1]}\right.$$

$$+\frac{\frac{b_2\bar{L}_4}{\Gamma(\beta+1)}\|L_3\|_{\frac{1}{\beta}}}{(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4L_3[1]} \quad +\frac{\frac{b_2(1-L_4[1])}{\Gamma(\beta+1)}\|L_3\|_{\frac{1}{\beta}}}{(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4}$$

$$\left.+\frac{\frac{b_2L_3[1]}{\Gamma(\beta+1)}\|L_4\|_{\frac{1}{\beta}}}{(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4}\right\}\|v-\bar{v}\|_C.$$

This can be written as

$$\|T_2(u,v)-T_2(\bar{u},\bar{v})\|_C \le M_3\|u-\bar{u}\|+M_4\|v-\bar{v}\|.$$

We then have

$$\begin{bmatrix}\|T_1(u,v)-T_1(\bar{u},\bar{v})\|_C\\ \|T_2(u,v)-T_2(\bar{u},\bar{v})\|_C\end{bmatrix} \le \mathfrak{M}\begin{bmatrix}\|u-\bar{u}\|_C\\ \|v-\bar{v}\|_C\end{bmatrix},$$

where

$$\mathfrak{M}=\begin{pmatrix}M_1 & M_2\\ M_3 & M_4\end{pmatrix}.$$

This can be written in the equivalent form as

$$\|T(U)-T(\bar{U})\|_{C\times C} \le \mathfrak{M}\|U-\bar{U}\|_{C\times C}$$

for $U=(u,v)$ and $\bar{U}=(\bar{u},\bar{v})$. Since the matrix $\mathfrak{M}$ converges to 0, by Perov's theorem, system (P) has a unique solution $(u,v)\in C(J,\mathbb{R})\times C(J,\mathbb{R})$. □

To illustrate how this theorem can be applied, we consider the simple example of a coupled implicit fractional problem

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t)=\frac{90\sqrt{\pi}}{640}x(t)+\frac{45\sqrt{\pi}}{640}y(t)+\frac{1}{10}\left({}^cD^{\frac{3}{2}}x(t)\right), \quad t\in[0,1],\\ {}^cD^{\frac{3}{2}}y(t)=\frac{45\sqrt{\pi}}{640}x(t)+\frac{45\sqrt{\pi}}{640}y(t)+\frac{1}{5}({}^cD^{\frac{3}{2}}y(t)),\\ x(0)=\frac{1}{2} \quad x'(0)=0\\ y(0)=0, \quad y'(0)=\frac{1}{2}\end{cases}$$

Here we have $\alpha=\frac{3}{2}=\beta$, $L_1[x]=L_4[y]=\frac{1}{2}$, $L_2[x]=L_3[y]=0$. This problem can be regarded as being in the form of (P) with

$$f_1(t,u,v,w)=\frac{90\sqrt{\pi}}{640}u+\frac{45\sqrt{\pi}}{640}v+\frac{1}{10}w,$$
$$f_2(t,u,v,w)=\frac{45\sqrt{\pi}}{640}u+\frac{45\sqrt{\pi}}{640}v+\frac{1}{5}w.$$

Hence, condition (C) is satisfied with

$$a_1=\frac{90\sqrt{\pi}}{640},\ b_1=\frac{45\sqrt{\pi}}{640},\ c_1=\frac{1}{10},\ a_2=\frac{45\sqrt{\pi}}{640},\ b_2=\frac{45\sqrt{\pi}}{640},\ c_2=\frac{1}{5}.$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we see that $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, $\Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}$, $\Gamma(\frac{7}{2}) = \frac{15\sqrt{\pi}}{8}$. Consequently, we see that $\mathfrak{M}$ becomes

$$\mathfrak{M} = \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix}.$$

Since $\frac{3}{10} + \frac{1}{10} = .4 < 1$, in view of Remark 1(i), we see that the matrix $\mathfrak{M}$ converges to 0. By Theorem 3, this problem has a unique solution.

5. Another existence result

In this section we obtain another existence result by applying a fixed point theorem for a sum of two operators due to Krasnosel'skii, which we next state.

Theorem 4 ([6], Krasnosel'skii's fixed point theorem)**.** *Let $(X, \|\cdot\|)$ be a generalized Banach space, D be a nonempty, closed, bounded, and convex subset of X, and $T : D \to X$ satisfy*

(i) *$T = G + H$ with $G : D \to X$ completely continuous and $H : D \to X$ a generalized contraction, i.e., there exists a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}^+)$ with $\rho(M) < 1$, such that*

$$\|H(x) - H(y)\| \le M\|x - y\|, \quad \textit{for all } x, y \in D;$$

(ii) *$G(x) + H(y) \in D$ for all x, $y \in D$.*

Then T has at least one fixed point in D.

To obtain our result, we ask that the following condition holds.

(C_1) $f_i : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for $i \in \{1, 2\}$ are jointly continuous functions and there exist nonnegative numbers d_i, e_i, m_i, and ℓ_i for $i \in \{1, 2\}$ such that for all u, v, w, $\bar{u}$, $\bar{v}$, $\bar{w} \in \mathbb{R}$ and $t \in [0, 1]$,

$$|f_1(t, u, v, w)| \le d_1|u| + e_1|v| + m_1|w| + \ell_1$$

and

$$|f_2(t, u, v, w)| \le d_2|u| + e_2|v| + m_2|w| + \ell_2.$$

For convenience in what follows, we define the quantities

$$N_1 = d_1 \left(\frac{1}{\Gamma(\alpha+2)} + \frac{|1 - \bar{L}_1| \frac{1}{\alpha\Gamma(\alpha+1)} \|L_2\| + \|L_2\| \frac{1}{\alpha\Gamma(\alpha+1)} \|L_1\|}{|(1 - L_2[1])(1 - \bar{L}_1) - \bar{L}_2 L_1[1]|} \right.$$
$$\left. + \frac{|1 - L_2[1]| \frac{1}{\alpha\Gamma(\alpha+1)} \|L_1\| + \|L_1\| \frac{1}{\alpha\Gamma(\alpha+1)} \|L_2\|}{|(1 - \bar{L}_1)(1 - L_2[1]) - L_1[1]\bar{L}_2|} \right),$$

$$N_2 = e_1 \left(\frac{1}{\Gamma(\beta+2)} + \frac{|1 - \bar{L}_2| \frac{1}{\beta\Gamma(\beta+1)} \|L_4\| + \|L_4\| \frac{1}{\beta\Gamma(\beta+1)} \|L_3\||}{|(1 - L_4[1])(1 - \bar{L}_3) - \bar{L}_4 L_3[1]|} \right.$$

$$+\frac{|1-L_4[1]|\frac{1}{\beta\Gamma(\beta+1)}\|L_3\|+\|L_3\|\frac{1}{\beta\Gamma(\beta+1)}\|L_4\|}{|(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4|}\Bigg),$$

$$N_3=d_2\Bigg(\frac{1}{\Gamma(\alpha+2)}+\frac{|1-\bar{L}_1|\frac{1}{\alpha\Gamma(\alpha+1)}\|L_2\|+\|L_2\|\frac{1}{\alpha\Gamma(\alpha+1)}\|L_1\|}{|(1-L_2[1])(1-\bar{L}_1)-\bar{L}_2L_1[1]|}$$
$$+\frac{|1-L_2[1]|\frac{1}{\alpha\Gamma(\alpha+1)}\|L_1\|+\|L_1\|\frac{1}{\alpha\Gamma(\alpha+1)}\|L_2\|}{|(1-\bar{L}_1)(1-L_2[1])-L_1[1]\bar{L}_2|}\Bigg),$$

$$N_4=e_2\Bigg(\frac{1}{\Gamma(\beta+2)}+\frac{|1-\bar{L}_2|\frac{1}{\beta\Gamma(\beta+1)}\|L_4\|+\|L_4\|\frac{1}{\beta\Gamma(\beta+1)}\|L_3\||}{|(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4L_3[1]|}$$
$$+\frac{|1-L_4[1]|\frac{1}{\beta\Gamma(\beta+1)}\|L_3\|+\|L_3\|\frac{1}{\beta\Gamma(\beta+1)}\|L_4\|}{|(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4|}\Bigg).$$

We also define the matrices

$$\mathfrak{N}=\begin{pmatrix}N_1 & N_2\\ N_3 & N_4\end{pmatrix},\quad \widetilde{\mathfrak{N}}=\begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix},\quad \widehat{\mathfrak{N}}=\begin{pmatrix}\ell_1 & 0\\ 0 & \ell_2\end{pmatrix}.$$

Here is our main result in this section.

Theorem 5. *Assume that condition* (C_1) *holds. If the spectral radius of the matrix* $\mathfrak{N}+\widetilde{\mathfrak{N}}+\widehat{\mathfrak{N}}$ *is less than one, then the problem* (P) *has at least one solution.*

Proof. In order to apply Krasnosel'skii's fixed point theorem, Theorem 4 above, we need to define the operator $T=G+H:D\to X$ as follows:

$$G(t,u,v)=\begin{bmatrix}G_1(u,v)\\ G_2(u,v)\end{bmatrix}\quad\text{and}\quad H(u,v)=\begin{bmatrix}H_1(u)\\ H_2(v)\end{bmatrix},$$

where

$$\begin{aligned}G_1(u,v)&=f_1(t,u,v,\cdot), & G_2(u,v)&=f_2(t,u,v,\cdot),\\ H_1(u)&=f_1(\cdot,\cdot,\cdot,u), & H_2(v)&=f_2(\cdot,\cdot,\cdot,v).\end{aligned}\tag{23}$$

We will first show that solutions exist, which we will do in a series of steps.

Step 1: $H(u,v)=\begin{bmatrix}H_1(u)\\ H_2(v)\end{bmatrix}$ *is a generalized contraction mapping.* Let (u,v), $(\bar{u},\bar{v})\in C(J,\mathbb{R})\times C(J,\mathbb{R})$; by (C_1), we have

$$|H_1(u)(t)-H_1(\bar{u})(t)|=|f_1(\cdot,\cdot,\cdot,u(t))-f_1(\cdot,\cdot,\cdot,\bar{u}(t))|\le m_1|u(t)-\bar{u}(t)|,$$
$$|H_2(v)-H_2(\bar{v})|=|f_2(\cdot,\cdot,\cdot,v(t))-f_2(\cdot,\cdot,\cdot,\bar{v}(t))|\le m_2|v(t)-\bar{v}(t)|.$$

Taking the supremum, this implies

$$\|H_1(u,v) - H_1(\bar{u},\bar{v})\|_C \le m_1\|u-\bar{u}\|_C \tag{24}$$

and

$$\|H_2(u,v) - H_2(\bar{v},\bar{v})\|_C \le m_2\|v-\bar{v}\|_C. \tag{25}$$

If we view this as a vector norm, we see that

$$\|H(U) - H(\bar{U})\|_{C\times C} \le \widetilde{\mathfrak{N}}\|U-\bar{U}\|_{C\times C} \tag{26}$$

for $U = (u,v)$ and $\bar{U} = \bar{U} = (\bar{u},\bar{v})$. Since $\rho(\mathfrak{N}+\widetilde{\mathfrak{N}}+\widehat{\mathfrak{N}}) < 1$ and $\widetilde{\mathfrak{N}} < \mathfrak{N}+\widetilde{\mathfrak{N}}+\widehat{\mathfrak{N}}$, we have $\rho(\widetilde{\mathfrak{N}}) < 1$. Hence, H is a generalized contraction as is needed in Theorem 4(i).

Step 2: $G = G(u,v) = \begin{bmatrix} G_1(u,v) \\ G_2(u,v) \end{bmatrix}$ *is continuous.* Let (u_n,v_n) be a sequence such that $(u_n,v_n) \to (u,v)$ in $C(J,\mathbb{R})\times C(J,\mathbb{R})$; then for each $t\in[0,1]$,

$$\begin{aligned}
&|G_1(u_n,v_n)(t) - G_1(u,v)(t)|\\
&\le |f_1(t,u_n,v_n,\cdot) - f_1(t,u,v,\cdot)|\\
&\le I^{\alpha+1}u_n(t) + \frac{(1-\bar{L}_1)\frac{1}{\Gamma(\alpha+1)}L_2[k_{1,n}(t)] + \bar{L}_2\frac{1}{\Gamma(\alpha+1)}L_1[k_{1,n}(t)]}{(1-L_2[1])(1-\bar{L}_1) - \bar{L}_2L_1[1]}\\
&\quad + t\frac{(1-L_2[1])\frac{1}{\Gamma(\alpha+1)}L_1[k_{1,n}(t)] + L_1[1]\frac{1}{\Gamma(\alpha+1)}L_2[k_{1,n}(t)]}{(1-\bar{L}_1)(1-L_2[1]) - L_1[1]\bar{L}_2},\\
&\quad I^{\beta+1}v_n(t) + \frac{(1-\bar{L}_2)\frac{1}{\Gamma(\beta+1)}L_4[k_{2,n}(t)] + \bar{L}_4\frac{1}{\Gamma(\beta+1)}L_3[k_{2,n}(t)]}{(1-L_4[1])(1-\bar{L}_3) - \bar{L}_4L_3[1]}\\
&\quad + t\frac{(1-L_4[1])\frac{1}{\Gamma(\beta+1)}L_3[k_{2,n}(t)] + L_3[1]\frac{1}{\Gamma(\beta+1)}L_4[k_{2,n}(t)]}{(1-\bar{L}_3)(1-L_4[1]) - L_3[1]\bar{L}_4}\\
&\quad - I^{\alpha+1}u(t) - \frac{(1-\bar{L}_1)\frac{1}{\Gamma(\alpha+1)}L_2[k_{1,n}1(t)] + \bar{L}_2\frac{1}{\Gamma(\alpha+1)}L_1[k_{1,n}(t)]}{(1-L_2[1])(1-\bar{L}_1) - \bar{L}_2L_1[1]}\\
&\quad - t\frac{(1-L_2[1])\frac{1}{\Gamma(\alpha+1)}L_1[k_{1,n}(t)] + L_1[1]\frac{1}{\Gamma(\alpha+1)}L_2[k_{1,n}(t)]}{(1-\bar{L}_1)(1-L_2[1]) - L_1[1]\bar{L}_2},\\
&\quad - I^{\beta+1}v(t) - \frac{(1-\bar{L}_2)\frac{1}{\Gamma(\beta+1)}L_4[k_{2,n}(t)] + \bar{L}_4\frac{1}{\Gamma(\beta+1)}L_3[k_{2,n}(t)]}{(1-L_4[1])(1-\bar{L}_3) - \bar{L}_4L_3[1]}\\
&\quad - t\frac{(1-L_4[1])\frac{1}{\Gamma(\beta+1)}L_3[k_{2,n}(t)] + L_3[1]\frac{1}{\Gamma(\beta+1)}L_4[k_{2,n}(t)]}{(1-\bar{L}_3)(1-L_4[1]) - L_3[1]\bar{L}_4},
\end{aligned}$$

where

$$k_{1,n}(t) = \int_0^t (t-s)^\alpha u(s)ds, \quad k_{2,n}(t) = \int_0^t (t-s)^\beta v(s)ds$$

and

$$|k_{1,n}(t)-k_1(t)| \leq \int_0^t (t-s)^\alpha |u_n(s)-u(s)|ds \leq \frac{t^\alpha}{\alpha}\|u_n-u\|.$$

Similarly, we have

$$|k_{2,n}(t)-k_2(t)| \leq \frac{\tilde{t}^\beta}{\beta}\|v_n-v\|.$$

Hence,

$$k_{i,n} \to k_i, \quad \text{as} n \to \infty,\ i=1,2,$$

and so

$$\|G_1(u_n,v_n)-G_1(u,v)\| \to 0, \quad \text{as} n \to \infty, \tag{27}$$

i.e., G_1 is continuous.

A similar argument shows that G_2 is continuous, and so G is continuous.

Step 3: *G maps bounded sets in D into bounded sets in D.* It suffices to show that for any $R=(R_1,R_2)\in\mathbb{R}_+^2$, there exists a constant $l>0$ such that, for each $(u,v)\in D$ with $\|(u,v)\|\leq R$, we have

$$\|G(u,v)\|_{C\times C} \leq l = (l_1,l_2)^T.$$

From condition (C_1), we have

$$\begin{aligned}|G_1(u,v)(t)| \leq{}& d_1\left|I^{\alpha+1}u(t)+\frac{(1-\bar{L}_1)\frac{1}{\Gamma(\alpha+1)}L_2[k_1(t)]+\bar{L}_2\frac{1}{\Gamma(\alpha+1)}L_1[k_1(t)]}{(1-L_2[1])(1-\bar{L}_1)-\bar{L}_2L_1[1]}\right|\\ &+d_1\left|\frac{(1-L_2[1])\frac{1}{\Gamma(\alpha+1)}L_1[k_1(t)]+L_1[1]\frac{1}{\Gamma(\alpha+1)}L_2[k_1(t)]}{(1-\bar{L}_1)(1-L_2[1])-L_1[1]\bar{L}_2}\right|\\ &+e_1\left|I^{\beta+1}v(t)+\frac{(1-\bar{L}_2)\frac{1}{\Gamma(\beta+1)}L_4[k_2(t)]+\bar{L}_4\frac{1}{\Gamma(\beta+1)}L_3[k_2(t)]}{(1-L_4[1])(1-\bar{L}_3)-\bar{L}_4L_3[1]}\right|\\ &+e_1\left|\frac{(1-L_4[1])\frac{1}{\Gamma(\beta+1)}L_3[k_2(t)]+L_3[1]\frac{1}{\Gamma(\beta+1)}L_4[k_2(t)]}{(1-\bar{L}_3)(1-L_4[1])-L_3[1]\bar{L}_4}\right|+\ell_1.\end{aligned}$$

Since

$$|k_1(t)| = \left|\int_0^t (t-s)^{\alpha-1}u(s)ds\right| \leq \frac{1}{\alpha}\|u\|,$$

we have

$$\|k_1\| \leq \frac{1}{\alpha}R_1 \ \text{ and similarly } \ \|k_2\| \leq \frac{1}{\beta}R_2. \tag{28}$$

Hence, in view of (28) and (22),

$$\|G_1(u,v)\|_C \leq d_1\left(\frac{1}{\Gamma(\alpha+2)}+\frac{|1-\bar{L}_1|\frac{1}{\alpha\Gamma(\alpha+1)}\|L_2\|+\|L_2\|\frac{1}{\alpha\Gamma(\alpha+1)}\|L_1\|}{|(1-L_2[1])(1-\bar{L}_1)-\bar{L}_2L_1[1]|}\right.$$

$$+ \frac{|1 - L_2[1]| \frac{1}{\alpha\Gamma(\alpha+1)} \|L_1\| + \|L_1\| \frac{1}{\alpha\Gamma(\alpha+1)} \|L_2\|}{|(1 - \bar{L}_1)(1 - L_2[1]) - L_1[1]\bar{L}_2|} \Bigg) R_1$$

$$+ e_1 \Bigg(\frac{1}{\Gamma(\beta+2)} + \frac{|1 - \bar{L}_2| \frac{1}{\beta\Gamma(\beta+1)} \|L_4\| + \|L_4\| \frac{1}{\beta\Gamma(\beta+1)} \|L_3\||}{|(1 - L_4[1])(1 - \bar{L}_3) - \bar{L}_4 L_3[1]|}$$

$$+ \frac{|1 - L_4[1]| \frac{1}{\beta\Gamma(\beta+1)} \|L_3\| + \|L_3\| \frac{1}{\beta\Gamma(\beta+1)} \|L_4\|}{|(1 - \bar{L}_3)(1 - L_4[1]) - L_3[1]\bar{L}_4|} \Bigg) R_2 + \ell_1 := l_1. \tag{29}$$

In a completely analogous, way we can obtain

$$\|G_2(u, v)\|_C \leq d_2 \Bigg(\frac{1}{\Gamma(\alpha+2)} + \frac{|1 - \bar{L}_1| \frac{1}{\alpha\Gamma(\alpha+1)} \|L_2\| + \|L_2\| \frac{1}{\alpha\Gamma(\alpha+1)} \|L_1\|}{|(1 - L_2[1])(1 - \bar{L}_1) - \bar{L}_2 L_1[1]|}$$

$$+ \frac{|1 - L_2[1]| \frac{1}{\alpha\Gamma(\alpha+1)} \|L_1\| + \|L_1\| \frac{1}{\alpha\Gamma(\alpha+1)} \|L_2\|}{|(1 - \bar{L}_1)(1 - L_2[1]) - L_1[1]\bar{L}_2|} \Bigg) R_1$$

$$+ e_2 \Bigg(\frac{1}{\Gamma(\beta+2)} + \frac{|1 - \bar{L}_2| \frac{1}{\beta\Gamma(\beta+1)} \|L_4\| + \|L_4\| \frac{1}{\beta\Gamma(\beta+1)} \|L_3\||}{|(1 - L_4[1])(1 - \bar{L}_3) - \bar{L}_4 L_3[1]|}$$

$$+ \frac{|1 - L_4[1]| \frac{1}{\beta\Gamma(\beta+1)} \|L_3\| + \|L_3\| \frac{1}{\beta\Gamma(\beta+1)} \|L_4\|}{|(1 - \bar{L}_3)(1 - L_4[1]) - L_3[1]\bar{L}_4|} \Bigg) R_2 + \ell_2 := l_2. \tag{30}$$

Thus,

$$\|G(u, v)\|_C \leq (l_1, l_2),$$

which is what we needed to show.

Step 4: *G maps bounded sets into equicontinuous sets in* $C(J, \mathbb{R}) \times C(J, \mathbb{R})$. Let $D \subset C(J, \mathbb{R})$ be bounded, r_1, $r_2 \in [0, 1]$ with $r_1 < r_2$, and $(u, v) \in D$. Then,

$$|G_1(u, v)(r_2) - G_1(u, v)(r_1)| = |f_1(r_2, u, v, \cdot) - f_1(r_1, u, v, \cdot)|$$

$$\leq I^{\alpha+1} u(r_2) + \frac{(1 - \bar{L}_1) \frac{1}{\Gamma(\alpha+1)} L_2[k_1(r_2)] + \bar{L}_2 \frac{1}{\Gamma(\alpha+1)} L_1[k_1(r_2)]}{(1 - L_2[1])(1 - \bar{L}_1) - \bar{L}_2 L_1[1]}$$

$$+ r_2 \frac{(1 - L_2[1]) \frac{1}{\Gamma(\alpha+1)} L_1[k_1(r_2)] + L_1[1] \frac{1}{\Gamma(\alpha+1)} L_2[k_1(r_2)]}{(1 - \bar{L}_1)(1 - L_2[1]) - L_1[1]\bar{L}_2},$$

$$I^{\beta+1} v(r_2) + \frac{(1 - \bar{L}_2) \frac{1}{\Gamma(\beta+1)} L_4[k_2(r_2)] + \bar{L}_4 \frac{1}{\Gamma(\beta+1)} L_3[k_2(r_2)]}{(1 - L_4[1])(1 - \bar{L}_3) - \bar{L}_4 L_3[1]}$$

$$+ r_2 \frac{(1 - L_4[1]) \frac{1}{\Gamma(\beta+1)} L_3[k_2(r_2)] + L_3[1] \frac{1}{\Gamma(\beta+1)} L_4[k_2(r_2)]}{(1 - \bar{L}_3)(1 - L_4[1]) - L_3[1]\bar{L}_4}$$

$$- I^{\alpha+1} u(r_1) - \frac{(1 - \bar{L}_1) \frac{1}{\Gamma(\alpha+1)} L_2[k_1(r_1)] + \bar{L}_2 \frac{1}{\Gamma(\alpha+1)} L_1[k_1(r_1)]}{(1 - L_2[1])(1 - \bar{L}_1) - \bar{L}_2 L_1[1]}$$

$$- r_1 \frac{(1 - L_2[1]) \frac{1}{\Gamma(\alpha+1)} L_1[k_1(r_1)] + L_1[1] \frac{1}{\Gamma(\alpha+1)} L_2[k_1(r_1)]}{(1 - \bar{L}_1)(1 - L_2[1]) - L_1[1]\bar{L}_2},$$

$$- I^{\beta+1}v(r_1) - \frac{(1 - \bar{L}_2) \frac{1}{\Gamma(\beta+1)} L_4[k_2(r_1)] + \bar{L}_4 \frac{1}{\Gamma(\beta+1)} L_3[k_2(r_1)]}{(1 - L_4[1])(1 - \bar{L}_3) - \bar{L}_4 L_3[1]}$$

$$- r_1 \frac{(1 - L_4[1]) \frac{1}{\Gamma(\beta+1)} L_3[k_2(r_1)] + L_3[1] \frac{1}{\Gamma(\beta+1)} L_4[k_2(r_1)]}{(1 - \bar{L}_3)(1 - L_4[1]) - L_3[1]\bar{L}_4}.$$

We observe that

$$\begin{aligned} |I^{\alpha+1}u(r_2) - I^{\alpha+1}u(r_1)| &\leq \frac{1}{\Gamma(\alpha+1)} \int_0^{r_1} |(r_2 - s)^\alpha - (r_1 - s)^\alpha||u(s)|ds \\ &\quad + \frac{1}{\Gamma(\alpha+1)} \int_{r_1}^{r_2} (r_2 - s)^\alpha |u(s)|ds \\ &\leq \frac{R_1}{\alpha\Gamma(\alpha+1)} \left[r_2^{\alpha+1} - r_1^{\alpha+1} + (r_2 - r_1)^{\alpha+1}\right]. \end{aligned}$$

Similarly,

$$|I^{\beta+1}v(r_2) - I^{\beta+1}v(r_1)| \leq \frac{R_2}{\beta\Gamma(\beta+1)} \left[r_2^{\beta+1} - r_1^{\beta+1} + (r_2 - r_1)^{\beta+1}\right].$$

Hence, we can easily conclude that

$$|G_1(u,v)(r_2) - G_1(u,v)(r_1)| \to 0 \text{ as } r_2 \to r_1,$$

and a similar statement is true for G_2.

Steps 2–4 above show that the operator G is completely continuous.

Step 5: *There is a closed, bounded, convex set $D \subset C(J,\mathbb{R}) \times C(J,\mathbb{R})$ such that $G(D) + H(D) \subseteq D$.*

The inequalities (29) and (30) imply that

$$\begin{bmatrix} \|G_1(u,v)\|_C \\ \|G_2(u,v)\|_C \end{bmatrix} \leq \mathfrak{N} \begin{bmatrix} \|u\|_C \\ \|v\|_C \end{bmatrix} + \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix},$$

where

$$\mathfrak{N} = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}.$$

Equivalently, we have

$$\|G(u,v)\|_{C\times C} \leq \mathfrak{N}\|(u,v)\|_{C\times C} + \widehat{\mathfrak{N}}. \tag{31}$$

It follows from (26) that

$$\|H(u,v)\|_{C\times C} \leq \widetilde{\mathfrak{N}}\|(u,v)\|_{C\times C} + P \quad \text{for} \quad (u,v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R}), \tag{32}$$

where

$$P = \|H(0,0)\|_{C\times C}.$$

We need to find $R = (R_1, R_2) \in \mathbb{R}^2_+$ such that

$$\|G(u,v) + H(u,v)\|_{C\times C} \le R$$

for $(u,v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ with $\|(u,v)\|_{C\times C} \le R$. To do this, in view of (31) and (32), it suffices to show that

$$(\mathfrak{N})\, R + P + \widetilde{\mathfrak{N}} + \widehat{\mathfrak{N}} \le R.$$

Or equivalently,

$$P + \widetilde{\mathfrak{N}} + \widehat{\mathfrak{N}} \le (I - \mathfrak{N})\, R. \tag{33}$$

Since $\rho(\mathfrak{N}) < 1$, we know that $I - \mathfrak{N}$ is invertible, and its inverse $(I - \mathfrak{N})^{-1}$ is a nonnegative matrix (see Theorem 1 above). Thus, (33) is equivalent to

$$(I - \mathfrak{N})^{-1}\,(P + \widetilde{\mathfrak{N}} + \widetilde{\mathfrak{N}}) \le R,$$

so

$$G(D) + H(D) \subseteq D.$$

Therefore, by Theorem 4, problem (P) has at least one solution. □

6. Structure of the solution set

In this section we wish to examine the structure of the set of solutions to problem (P), i.e., the set

$$S(x_0, y_0) = \big\{(u,v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R}) : (u,v) \text{ is a solution of (P)}\big\}.$$

In order to do this, we will need the following concepts and results.

Theorem 6 ([7]). *Let E be a normed space, X be a metric space, and let $f : X \to E$ be a continuous map. Then for every $\epsilon > 0$, there is a locally Lipschitz function $f_\epsilon : X \to E$ such that*

$$\|f(x) - f_\epsilon(x)\| < \epsilon \text{ for all } x \in X.$$

We denote by $\mathcal{P}_{cv,cl}(X)$ the set of all subsets of X that are convex and closed.

Definition 6 ([10, Section 9.1]). The set A is contractible if there exists a continuous homotopy $H : A \times [0,1] \to A$ and $x_0 \in A$ such that

(a) $H(x,0) = x$ for every $x \in A$,
(b) $H(x,1) = x_0$ for every $x \in A$.

This says that A is contractible provided the identity map is homotopic to a constant map (A is homotopically equivalent to a point).

Remark 3. If $A \in \mathcal{P}_{cv,cl}(X)$, then A is contractible, but the class of contractible sets is much larger than the class of closed convex sets.

The following concept of an R_δ-set will be used in Section 4, where we discuss the topological structure of the solution set to our problem.

Definition 7 ([10, defiition 9.5]). A compact nonempty set X is called an R_δ-set if there exists a decreasing sequence of compact nonempty contractible sets $\{X_n\}_{n=1}^{\infty}$ such that

$$X = \bigcap_{n=1}^{\infty} X_n.$$

The next result gives a useful criteria for identifying an R_δ-set.

Theorem 7 ([7]). *Let (X, d) be a metric space, $(E, \|\cdot\|)$ be a Banach space and $F : X \to E$ be a proper map, i.e., F is continuous and for every compact set $K \subset E$, the set $F^{-1}(K)$ is compact. Assume that for each $\epsilon > 0$, a proper map $F_\epsilon : X \to E$ is given, and the following two conditions are satisfied:*

(a) $\|F_\epsilon(x) - F(x)\| < \epsilon$ *for every* $x \in X$*;*

(b) *for every $\epsilon > 0$ and $u \in E$ in a neighborhood of the origin such that $\|u\| < \epsilon$, the equation $F_\epsilon(x) = u$ has exactly one solution x_ϵ.*

Then the set $S = F^{-1}(0)$ is an R_δ-set.

Our next lemma gives a criteria for determining if a function is a proper map.

Lemma 1 ([8]). *Let E be a Banach space, $C \subset E$ be a nonempty, closed, and bounded subset of E, and $F : C \to E$ be a completely continuous map. Then $G = Id - F$ is a proper map.*

Our first result in this direction shows the compactness of the solution set.

Theorem 8. *Under the hypotheses of Theorem 5, the set*

$$S(x_0, y_0) = \big\{(u, v) \in C(J, \mathbb{R}) \times C(J, \mathbb{R}) : (u, v) \textit{ is solution of } (\mathrm{P})\big\}$$

is compact.

Proof. Let $\{(u_n, v_n)\}_{n\geq 1}$ be a sequence in $S(x_0, y_0)$. Then for $n \in \mathbb{N}$,

$$(u_n, v_n) = G(u_n, v_n) + H(u_n, v_n),$$

or

$$(u_n, v_n) - H(u_n, v_n) = G(u_n, v_n).$$

From Steps 2–4 in the proof of Theorem 5, we see that G is a completely continuous operator, and since $(I-H)^{-1}$ is continuous, the operator $(I-H)^{-1}G$ is compact. Since $\{(u_n, v_n)\}_{n\geq 1} \subset (I - H)^{-1}G(D)$, there is a subsequence $\{(u_{n_k}, v_{n_k})\}$ of $\{(u_n, v_n)\}$ that converges to say (u, v). Hence,

$$(I - H)^{-1}G(u_{n_k}, v_{n_k}) \to (I - H)^{-1}G(u, v), \quad \text{as } k \to \infty$$

by the continuity of $(I - H)^{-1}G$. Therefore,

$$(I - H)^{-1}G(u, v) = (u, v),$$

or

$$(u,v)-H(u,v)=G(u,v).$$

That is, there exists a subsequence of $\{(u_n,v_n)\}_{n\geq 1}$ converging to $(u,v)\in S(x_0,y_0)$, and so $S(x_0,y_0)$ is a compact set. This proves the theorem. □

Next, we show that the set $S(x_0,y_0)$ is in fact an R_δ set.

Theorem 9. *Under the hypotheses of Theorem 5, the set $S(x_0,y_0)$ is an R_δ set.*

Proof. Let

$$T:C(J,\mathbb{R})\times C(J,\mathbb{R})\to C(J,\mathbb{R})\times C(J,\mathbb{R})$$

be defined as in (18) and (19) and using (15), (16), and (23).

Then $Fix\,T=S(x_0,y_0)$, and by Step 5 in the proof of Theorem 5, it is clear that there exists $R=(R_1,R_2)\in\mathbb{R}^*_+$ such that

$$\|(u,v)\|_{C\times C}\leq R,\ \text{for all } (u,v)\in S(x_0,y_0).$$

Therefore, there exists $M^*=(M_1^*,M_2^*)>0$ such that

$$\|(x,y)\|_C=(\|x\|_C,\|y\|_C)\leq M^*.$$

For $i=1,2$, we define

$$\tilde{f}_i(t,u(t),v(t),w(t))=\begin{cases} f_i(t,u(t),v(t),\cdot), & \text{if } \|(u,v)\|_{C\times C}\leq (M_1^*,M_2^*),\\ f_i\left(t,\frac{M_1^*u(t)}{|u(t)|},\ \frac{M_2^*v(t)}{|v(t)|},\cdot\right), & \text{if } \|(u,v)\|_{C\times C}\geq (M_1^*,M_2^*),\end{cases}$$

and

$$\tilde{f}_1(\cdot,\cdot,\cdot,w(t))=\begin{cases} f_1\left(\cdot,\cdot,\cdot,w(t)\right), & \text{if } \|(u,v)\|_{C\times C}\leq (R_1,R_2),\\ f_1\left(\cdot,\cdot,\cdot,\frac{R_1u(t)}{|u(t)|}\right), & \text{if } \|(u,v)\|_{C\times C}\geq (R_1,R_2).\end{cases}$$

$$\tilde{f}_2(\cdot,\cdot,\cdot,w(t))=\begin{cases} f_2(\cdot,\cdot,\cdot,w(t)), & \text{if } \|(u,v)\|_{C\times C}\leq (R_1,R_2),\\ f_2\left(\cdot,\cdot,\cdot,\frac{R_2v(t)}{|v(t)|}\right), & \text{if } \|(u,v)\|_{C\times C}\geq (R_1,R_2).\end{cases}$$

with $u(t)={}^cD^\alpha x(t)$ and $v(t)={}^cD^\beta y(t)$.

Since (f_1,f_2) are continuous functions, $(\tilde{f}_1,\tilde{f}_2)$ are also continuous.

Now consider the problem

$$\begin{cases} {}^cD^\alpha x(t)=\tilde{f}_1(t,x(t),y(t),{}^cD^\alpha x(t)), & t\in J,\\ {}^cD^\beta y(t)=\tilde{f}_2(t,x(t),y(t),{}^cD^\beta y(t)), & t\in J,\\ x(0)=L_1[x],\quad x'(0)=L_2[x], \\ y(0)=L_3[y],\quad y'(0)=L_4[y], \end{cases}\tag{34}$$

and set

$$\tilde{S}(x_0,y_0)=\{(u,v)\in C(J,\mathbb{R})\times C(J,\mathbb{R}):(u,v)\text{ is solution of (34)}\}.$$

Let

$$\tilde{G}_1(u,v)(t)=\tilde{f}_1(t,u,v,\cdot)$$

and

$$\tilde{G}_2(u,v)(t) = \tilde{f}_2(t,u,v,\cdot),$$

where

$$\tilde{k}_1(t) = \int_0^t (t-s)^{\alpha-1}u(s)ds \quad \text{and} \quad \tilde{k}_2(t) = \int_0^t (t-s)^{\beta-1}v(s)ds.$$

Also, we define

$$\tilde{H}_1(u,v)(t) = \tilde{f}_1(\cdot,\cdot,\cdot,u) \quad \text{and} \quad \tilde{H}_2(u,v)(t) = \tilde{f}_2(\cdot,\cdot,\cdot,v).$$

Now problem (34) is equivalent to

$$\begin{cases} u = \tilde{G}_1(u,v) + \tilde{H}_1(u,v), \\ v = \tilde{G}_2(u,v) + \tilde{H}_2(u,v). \end{cases} \tag{35}$$

It is easy to see that

$$S(x_0,y_0) = \tilde{S}(x_0,y_0) = Fix\,\tilde{T},$$

where $\tilde{T}(u,v) = (\tilde{T}_1(u,v), \tilde{T}_2(u,v))$ for $(x,y) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ and

$$\begin{cases} \tilde{T}_1(u,v) = \tilde{G}_1(u,v) + \tilde{H}_1(u,v), \\ \tilde{T}_2(u,v) = \tilde{G}_2(u,v) + \tilde{H}_2(u,v). \end{cases}$$

Clearly, $\tilde{T}$ is uniformly bounded, and as we did above, we can prove that $\tilde{T}(u,v) : C(J,\mathbb{R}) \times C(J,\mathbb{R}) \to C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is completely continuous.

Now consider a perturbation of the identity map, namely

$$\tilde{F}(u,v) = (u,v) - \tilde{T}(u,v).$$

We then have

$$(I-\tilde{H})^{-1}\tilde{F}(u,v) = (u,v) - (I-\tilde{H})^{-1}\tilde{G}(u,v).$$

As in Steps 2–4 in the proof of Theorem 5, we can prove that $\tilde{G}(u,v)$ is completely continuous, and since $I-\tilde{H}$ is continuous, $(I-\tilde{H})^{-1}\tilde{G}(u,v)$ is a completely continuous map and $(I-\tilde{H})^{-1}\tilde{F}(u,v)$ is a proper map. From the compactness of $(I-\tilde{H})^{-1}\tilde{G}$, it is easy to show that all conditions of Theorem 7 are satisfied, and so $((I-\tilde{H})^{-1}\tilde{F})^{-1}(0)$ is an R_δ set. Since $(I-\tilde{H})^{-1} \neq 0$, the solution set $Fix\,\tilde{T} = \tilde{S}(x_0,y_0) = \tilde{F}^{-1}(0)$ is an R_δ set. This proves the theorem. □

To illustrate our results in Sections 5 and 6, we again consider the problem

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = \frac{90\sqrt{\pi}}{640}x(t) + \frac{45\sqrt{\pi}}{640}y(t) + \frac{1}{10}\left({}^cD^{\frac{3}{2}}x(t)\right), \quad t \in [0,1], \\ {}^cD^{\frac{3}{2}}y(t) = \frac{\sqrt{45\pi}}{640}x(t) + \frac{45\sqrt{\pi}}{640}y(t) + \frac{1}{5}({}^cD^{\frac{4}{3}}y(t)), \\ x(0) = \frac{1}{2}, \quad x'(0) = 0, \\ y(0) = 0, \quad y'(0) = \frac{1}{2}. \end{cases} \tag{36}$$

As before, $\alpha = \frac{3}{2} = \beta$, $L_1[x] = L_4[y] = \frac{1}{2}$, $L_2[x] = L_3[y] = 0$. Also, (C_1) is satisfied with

$$d_1 = \frac{90\sqrt{\pi}}{640},\ e_1 = \frac{45\sqrt{\pi}}{640},\ m_1 = \frac{1}{10},\ d_2 = \frac{45\sqrt{\pi}}{640},\ e_2 = \frac{45\sqrt{\pi}}{640},\ m_2 = \frac{1}{5}.$$

The matrices become

$$\mathfrak{N} = \begin{pmatrix} \frac{88}{640} & \frac{44}{640} \\ \frac{44}{640} & \frac{44}{640} \end{pmatrix}, \quad \widetilde{\mathfrak{N}} = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}, \quad \widehat{\mathfrak{N}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of $\mathfrak{N} + \widetilde{\mathfrak{N}} + \widehat{\mathfrak{N}}$ are 0.324 and 0.183, so by Theorem 5 problem (36) has a least one solution. By Theorems 8 and 9, the set of solutions to (36) is a compact R_δ-set.

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