


## Fractional differentiation composition operators from $S_p$ spaces to $H_q$ spaces

DEEPJYOTI BORGOHAIN  0000-0002-3565-8449

ABSTRACT. Let  $S_p$  be the space of functions analytic on the unit disk and whose derivatives belong to the Hardy space. In this article, we investigate the boundedness and compactness of the fractional differentiation composition operators from  $S_p$  spaces into Hardy spaces. Furthermore, we derive a sufficient condition for the boundedness of the fractional differentiation composition operators on  $S_p$  spaces. These results extends some well-known results in literature.

### 1. INTRODUCTION

Let  $\mathbb{D}$  be the open unit disk in the complex plane and  $H(\mathbb{D})$  be the set of all analytic functions on the unit disk. For  $1 \leq p \leq \infty$ , the Hardy space  $H_p$  is defined as follows:

$$H_p = \left\{ f \in H(\mathbb{D}) : \|f\|_{H_p} = \lim_{r \rightarrow 1} M_p(f, r) < \infty \right\},$$

where

$$M_p(f, r) = \begin{cases} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, & \text{if } p \in (0, \infty); \\ \sup_{\theta \in [0, 2\pi]} \{|f(re^{i\theta})|\}, & \text{if } p = \infty. \end{cases}$$

For  $1 \leq p \leq \infty$ , we denote by  $S_p$  the space of all analytic functions  $f$  on the unit disc  $\mathbb{D}$  whose derivative  $f'$  lies in  $H_p$ , endowed with the norm

$$(1) \quad \|f\|_{S_p} = |f(0)| + \|f'\|_{H_p}.$$

It is clear that  $S_p$  is a Banach space with respect to this norm. See [2, 7] for more information on this space.

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2020 *Mathematics Subject Classification.* 33C05, 30E20, 30H99, 47A30, 47B91, 47E05.

*Key words and phrases.* Gaussian hypergeometric function, boundedness, Hardy spaces.

*Full paper.* Received 6 Nov 2024, accepted 8 Apr 2024, available online 23 Apr 2025.

The classical/Gaussian hyper-geometric series is defined by the power series expansion

$$(2) \quad F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad |z| < 1.$$

Here  $a, b, c$  are complex numbers such that  $c \neq -m$ ,  $m = 0, 1, 2, \dots$ , and  $(a)_n$  is Pochhammer's symbol/shifted factorial defined by Appel's symbol

$$(a)_n = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \in \mathbb{N},$$

and  $(a)_0=1$  for  $a \neq 0$ . Obviously  $F(a, b; c; z) \in H(\mathbb{D})$ . Many properties of the hypergeometric series are found in standard textbooks such as [1, 16].

For any two analytic functions  $f$  and  $g$  represented by their power series expansions,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \in |z| < R,$$

the Hadamard product (or convolution) of  $f$  and  $g$  denoted by  $f * g$  and is defined by

$$(3) \quad (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

In particular for  $f, g \in H(\mathbb{D})$ , we have

$$(4) \quad (f * g)(\rho z) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{it}) g(z e^{-it}) dt, \quad 0 < \rho < 1.$$

If  $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$  and  $\beta > 0$ , then the fractional derivative  $f^{[\beta]}$  (see [9]) of order  $\beta$  is defined by

$$(5) \quad f^{[\beta]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\beta)}{\Gamma(n+1)} a_n z^n.$$

In terms of convolution, we also have

$$(6) \quad f^{[\beta]}(z) = \Gamma(1+\beta) (f(z) * F(1, 1+\beta; 1; z)).$$

For  $\beta = 0$ , we define

$$f^{[0]}(z) = f(z)$$

and for  $n \in \mathbb{N}$  we have

$$(7) \quad f^{[n]}(z) = \frac{d^n}{dz^n} (z^n f(z)).$$

The fractional differentiation composition operator, denoted by  $D_{\varphi, u}^{\beta}$  is defined as follows ([5, 6])

$$(8) \quad D_{\varphi, u}^{\beta} f(z) = u(z) f^{[\beta]}(\varphi(z)), \quad f \in H(\mathbb{D}),$$

where  $u(z) \in H(\mathbb{D})$  and  $\varphi(z)$  is nonconstant analytic self-map of  $\mathbb{D}$ .

This operator can be viewed as a generalization of a multiplication operator and a weighted composition operator. For  $\beta = 0$ ,  $D_{\varphi,u}^\beta$  equals the weighted composition operator defined by  $(uC_\varphi)(f)(z) = u(z)f(\varphi(z))$ ,  $z \in \mathbb{D}$ , which reduces to the composition operator  $C_\varphi$  for  $u(z) \equiv 1$ . During the last century, composition operators were studied between different spaces of analytic functions. If  $\beta = 1$ , we get the operator  $D_{\varphi,u}^1 = M_u C_\varphi + M_{u\varphi} C_\varphi D$ , which for  $u(z) \equiv 1$  gives  $D_{\varphi,1}^1 = C_\varphi + \varphi C_\varphi D$ , which for  $\varphi(z) = z$  gives  $D_{z,u}^1 = M_u + zM_u D$  and  $u(z) = \varphi'(z)$  gives  $D_{\varphi,\varphi'}^1 = M_{\varphi'} C_\varphi + \varphi D C_\varphi$ . For particular choices of  $\beta$ ,  $u$  and  $\varphi$ , we obtain many operators, which are product, addition and composition of multiplication and differentiation operators. Weighted composition operators find its usefulness in many different ways. For example, they are isometries of many Banach spaces of analytic functions. Novinger and Oberin proved that the isometries in  $S_p$  for  $1 \leq p < \infty, p \neq 2$  are given by

$$Tf(z) = \lambda_1 \left[ f(0) + \int_0^z W_{\varphi,\lambda_2(\varphi')^{1/p}}(f')(\xi) d\xi \right]$$

for  $f \in S_p$  where  $|\lambda_1| = |\lambda_2| = 1$  and  $\varphi$  is a self-analytic map of  $\mathbb{D}$ .

In 1978, the  $S_p$  spaces were introduced by Roan in [11], where he studied the boundedness and compactness of the composition operators  $C_\varphi$ , ( $1 \leq p < \infty$ ). Contreras and Hernández-Díaz characterized the boundedness, compactness of  $W_{\varphi,\psi}$  from  $S_p$  into  $S_q$ ,  $1 \leq p, q < \infty$  in terms of weighted composition operators on Hardy spaces (See [7]). For some recent results about different operators on  $S_p$  spaces one can see [3, 7, 10, 12]. Recently, Xie et al. [17] introduced a similar space  $B_p$  which consists of all  $f \in H(\mathbb{D})$  such that  $f'$  belongs to the Bergman space. In that paper the authors investigated the boundedness and compactness of the weighted composition operators in  $B_p$  spaces.

The operator  $D_{\varphi,u}^\beta$  was introduced by Naik and Borgohain in [5], where they studied the boundedness and compactness of this operator from mixed-norm spaces to weighted-type spaces. Recently, the author of this paper studied the boundedness of the operator  $D_{\varphi,u}^n$  from  $S_p$  spaces to weighted-type spaces.

In this paper we characterize the boundedness and compactness of the operator  $D_{\varphi,u}^\beta$  from  $S_p$  to  $H_q$  spaces. Throughout this paper  $C$  is any constant which may vary for different lines.

## 2. PRELIMINARY RESULTS

We collect some basic lemmas which are useful in the proof of the main results.

**Lemma 1** ([13], Proposition 1.4.10). *For  $\gamma > 1$  one has*

$$\int_0^{2\pi} \frac{d\theta}{|1 - z|^\gamma} \leq C \frac{1}{(1 - |z|)^{\gamma-1}}.$$

**Lemma 2** ([8], Proposition 3.1). *If  $f \in H_p$  ( $0 < p < \infty$ ), then*

$$|f(z)| \leq 2^{1/p} \frac{\|f\|_p}{(1 - r)^{1/p}}, \quad r = |z|.$$

Lemma 1 is true for  $f \in H_p$ . Here we will give a similar result which involves fractional derivative  $f^{[\beta]}$  of  $f \in S_p$ .

**Proposition 1.** *Suppose  $f \in S_p$  for  $0 < p \leq \infty$ .*

(a) *For  $1 \leq p \leq \infty$*

$$|f^{[0]}(z)| \leq C \|f\|_{S_p},$$

*and for  $\beta \geq 1$*

$$|f^{[\beta]}(z)| \leq \frac{2^{1/p} \Gamma(1 + \beta)}{1/p + \beta - 1} \frac{\|f\|_{S_p}}{(1 - |z|)^{1/p + \beta - 1}}.$$

(b) *For  $0 < p < 1$  and  $0 \leq \beta < \infty$*

$$|f^{[\beta]}(z)| \leq \frac{2^{1/p} \Gamma(1 + \beta)}{1/p - 1} \frac{\|f\|_{S_p}}{(1 - |z|)^{1/p + \beta - 1}}.$$

*Proof.* For all  $p \in (0, \infty]$ ,  $f \in S_p$  implies  $f' \in H_p$ . Hence, Lemma 1 gives

$$(9) \quad |f'(z)| \leq 2^{1/p} \frac{\|f\|_{S_p}}{(1 - |z|)^{1/p}}$$

Now, for any curve  $\alpha(t) = x(t) + iy(t)$ ,  $0 \leq t \leq 1$  from 0 to  $z = x + iy$ , we have

$$f(z) = \int_0^z f'(w) dw = \int_0^1 f'(\alpha(t)) \alpha'(t) dt.$$

It is clear that

$$(10) \quad |f(z)| = \left| z \int_0^1 f'(tz) dt \right| \leq |z| \int_0^1 |f'(tz)| dt.$$

(a) The proof for the case  $\beta = 0$  can be found in the proof of Theorem 2.1 of [7]. For  $\beta \geq 1$  the definition of fractional derivative gives

$$\begin{aligned} |f^{[\beta]}(z)| &= |\Gamma(1 + \beta)(f(z) * F(1, 1 + \beta; 1; z))| \\ &= \frac{\Gamma(1 + \beta)}{(2\pi)} \int_0^{2\pi} |f(ze^{-it})| \frac{1}{|1 - \rho e^{it}|^{1+\beta}} dt. \end{aligned}$$

Lemma 1 and inequality (9) give

$$\begin{aligned}
|f^{[\beta]}(z)| &= \frac{\Gamma(1+\beta)}{(2\pi)} \int_0^{2\pi} |f(ze^{-it})| \frac{1}{|1-\rho e^{it}|^{1+\beta}} dt \\
&\leq \frac{\Gamma(1+\beta)}{(2\pi)} \int_0^{2\pi} \left[ \int_0^1 |ze^{-it}| 2^{1/p} \frac{\|f\|_{S_p}}{(1-\xi|z|)^{1/p}} d\xi \right] \frac{1}{|1-\rho e^{it}|^{1+\beta}} dt \\
&\leq \frac{\Gamma(1+\beta)}{(2\pi)} \int_0^1 |z| 2^{1/p} \frac{\|f\|_{S_p}}{(1-\xi|z|)^{1/p}} \frac{1}{(1-\rho)^\beta} d\xi.
\end{aligned}$$

Taking  $\rho = \frac{1+\xi|z|}{2}$ , we have

$$\begin{aligned}
|f^{[\beta]}(z)| &\leq 2^{1/p+\beta} \|f\|_{S_p} \frac{\Gamma(1+\beta)}{(2\pi)} \int_0^1 |z| \frac{1}{(1-\xi|z|)^{1/p+\beta}} d\xi \\
&\leq \frac{2^{1/p+\beta} \Gamma(1+\beta)}{1/p+\beta-1} \frac{\|f\|_{S_p}}{(1-\xi|z|)^{1/p+\beta-1}} \\
&\leq \frac{2^{1/p+\beta} \Gamma(1+\beta)}{1/p+\beta-1} \frac{\|f\|_{S_p}}{(1-|z|)^{1/p+\beta-1}}.
\end{aligned}$$

(b) Equation (10) gives

$$|f(z)| \leq \int_0^1 \frac{2^{1/p} \|f\|_{S_p}}{(1-t|z|)^{1/p}} dt \leq \frac{2^{1/p}}{1/p-1} \frac{\|f\|_{S_p}}{(1-|z|)^{1/p-1}}.$$

Using the definition of fractional derivative, we have for  $0 \leq \beta < \infty$

$$\begin{aligned}
|f^{[\beta]}(z)| &\leq \int_0^{2\pi} |f(ze^{-it})| |F(1, 1+\beta; 1; \rho e^{it})| dt \\
&\leq \int_0^{2\pi} \frac{2^{1/p}}{1/p-1} \frac{\|f\|_{S_p}}{(1-|ze^{-it}|)^{1/p-1}} \frac{1}{|1-\rho e^{it}|^{\beta+1}} dt \\
&\leq \frac{2^{1/p}}{1/p-1} \int_0^{2\pi} \frac{\|f\|_{S_p}}{(1-|ze^{-it}|)^{1/p-1}} \frac{1}{|1-\rho e^{it}|^{\beta+1}} dt.
\end{aligned}$$

Taking  $\rho = \frac{1+|z|}{2}$  gives

$$\begin{aligned}
|f^{[\beta]}(z)| &\leq \frac{2^{1/p}}{1/p-1} \frac{\|f\|_{S_p}}{(1-|z|)^{1/p-1}} \frac{1}{|1-\rho|^\beta} \\
&\leq \frac{2^{1/p}}{1/p-1} \frac{\|f\|_{S_p}}{(1-|z|)^{1/p+\beta-1}}. \quad \square
\end{aligned}$$

**Theorem 1.** *The set of polynomials is dense in  $S_p$  for  $0 < p < \infty$ .*

*Proof.* Suppose  $f \in S_p$ . Then  $f' \in H_p$ . Since polynomials are dense in  $H_p$ , therefore there exists a sequence of polynomials  $\{p_n(z)\}_{n=1}^\infty$  in  $H_p$  such that  $\lim_{n \rightarrow \infty} \|p_n - f'\|_{H_p} = 0$ . Define

$$P_n(z) = f(0) + \int_0^z p_n(w) dw.$$

Then  $P_n(z)$  is a polynomial and

$$\begin{aligned} \lim_{n \rightarrow \infty} \|P_n - f\|_{S_p} &= \lim_{n \rightarrow \infty} [|P_n(0) - f(0)| + \|P'_n - f'\|_{H_p}] \\ &= \lim_{n \rightarrow \infty} \|p_n - f'\|_{H_p} = 0. \end{aligned} \quad \square$$

### 3. BOUNDEDNESS OF THE OPERATOR $D_{\varphi,u}^\beta : S_p \rightarrow H_q$

In this section we characterize the boundedness of the operator  $D_{\varphi,u}^\beta$  from  $S_p$  spaces to  $H_q$  spaces.

**Theorem 2.** *Let  $0 < q < \infty$  and  $f \in S_p$  for  $0 < p \leq \infty$ ,  $u(z) \in H(\mathbb{D})$  and  $\varphi(z)$  is an analytic self-map of  $\mathbb{D}$ . Then for  $1 \leq p \leq \infty$  and  $\beta \geq 1$  or  $0 < p < 1$  and  $\beta \geq 0$  the operator  $D_{\varphi,u}^\beta : S_p \rightarrow H_q$  is bounded if and only if*

$$(11) \quad \sup_{0 < r < 1} \left[ \frac{1}{2\pi} \int_0^{2\pi} \frac{|u(z)|^q}{(1 - |\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta \right]^{1/q} < \infty.$$

*Proof.* Suppose (11) holds. Then

$$\begin{aligned} \|D_{\varphi,u}^\beta f\|_{H_q} &= \sup_{0 < r < 1} \left[ \frac{1}{2\pi} \int_0^{2\pi} |D_{\varphi,u}^\beta f(z)|^q d\theta \right]^{1/q} \\ &= \sup_{0 < r < 1} \left[ \frac{1}{2\pi} \int_0^{2\pi} |u(z) f^{[\beta]}(\varphi(z))|^q d\theta \right]^{1/q}. \end{aligned}$$

Proposition 1 gives

$$\|D_{\varphi,u}^\beta f\|_{H_q} \leq C \|f\|_{S_p} \sup_{0 < r < 1} \left[ \frac{1}{2\pi} \int_0^{2\pi} \frac{|u(z)|^q}{(1 - |\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta \right]^{1/q} < C \|f\|_{S_p}.$$

Hence,  $D_{\varphi,u}^\beta : S_p \rightarrow H_q$  is bounded. Conversely, suppose  $D_{\varphi,u}^\beta : S_p \rightarrow H_q$  is bounded and for any  $w \in \mathbb{D}$  define the test function

$$f_w(z) = (1 - |w|^2) F \left( \frac{1}{p} + \beta, 1; 1 + \beta; \bar{w}z \right).$$

Then

$$\begin{aligned} f'_w(z) &= (1 - |w|^2) \frac{\frac{1}{p} + \beta}{1 + \beta} F \left( \frac{1}{p} + \beta + 1, 2; 2 + \beta; \bar{w}z \right) \\ &= (1 - |w|^2) \frac{\frac{1}{p} + \beta}{1 + \beta} \frac{1}{(1 - \bar{w}z)^{1/p+1}} F \left( 1 - \frac{1}{p}, \beta; 2 + \beta; \bar{w}z \right). \end{aligned}$$

Since  $2 + \beta - \left(1 - \frac{1}{p}\right) - \beta = 1 + \frac{1}{p} > 0$ , therefore  $F \left(1 - \frac{1}{p}, \beta; 2 + \beta; \bar{w}z\right)$  is bounded in  $\mathbb{D}$ . Hence, there is a constant  $C$  such that

$$|f'_w(z)| \leq C \frac{(1 - |w|^2)}{(1 - \bar{w}z)^{1/p+1}}.$$

Therefore,

$$\begin{aligned} M_p^p(f'_w, r) &= \frac{1}{2\pi} \int_0^{2\pi} |f'_w(re^\theta)| d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} C \frac{(1-|w|^2)^p}{|1-\bar{w}re^{i\theta}|^{1+p}} d\theta. \end{aligned}$$

Lemma 1 gives

$$M_p^p(f'_w, r) \leq C \frac{(1-|w|^2)^p}{(1-|w|r)^p} < \infty.$$

Hence  $f'_w \in H_p$  and therefore  $f_w \in S_p$ . The fractional derivative of  $f_w$  is

$$\begin{aligned} f_w^{[\beta]}(z) &= \Gamma(1+\beta) (f_w * F(1, 1+\beta; 1; z)) \\ &= \Gamma(1+\beta) \left( (1-|w|^2) F\left(\frac{1}{p} + \beta, 1; 1+\beta; \bar{w}z\right) * F(1, 1+\beta; 1; z) \right) \\ &= \Gamma(1+\beta)(1-|w|^2) \left( \sum_{n=0}^{\infty} \frac{(1/p + \beta)_n (1)_n}{(1)_n (1+\beta)_n} \bar{w}^n z^n * \sum_{n=0}^{\infty} \frac{(1+\beta)_n (1)_n}{(1)_n (1)_n} z^n \right) \\ &= \Gamma(1+\beta)(1-|w|^2) \left( \sum_{n=0}^{\infty} \frac{(1/p + \beta)_n}{(1)_n} \bar{w}^n z^n \right) \\ &= \Gamma(1+\beta) \frac{(1-|w|^2)}{(1-\bar{w}z)^{1/p+\beta}}. \end{aligned}$$

This gives

$$\begin{aligned} M_q^q(D_{\varphi, u}^\beta f_w, r) &= \frac{1}{2\pi} \int_0^{2\pi} |D_{\varphi, u}^\beta f_w(z)|^q d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} |u(z) f_w^{[\beta]}(\varphi(z))|^q d\theta \\ &= \Gamma(1+\beta) \frac{1}{2\pi} \int_0^{2\pi} \frac{|u(z)|^q (1-|w|^2)}{|1-\bar{w}\varphi(z)|^{q/p+q\beta}} d\theta. \end{aligned}$$

Taking  $w = \varphi(z)$  gives

$$M_q^q(D_{\varphi, u}^\beta f_w, r) = \Gamma(1+\beta) \frac{1}{2\pi} \int_0^{2\pi} \frac{|u(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta.$$

Since  $D_{\varphi, u}^\beta : S_p \rightarrow H_q$  is bounded, we have

$$\sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{|u(z)|^q}{(1-|\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta \right)^{1/q} < C \|f\|_{S_p} < \infty. \quad \square$$

In the next result, we give a sufficient condition for the operator  $D_{\varphi, u}^\beta : S_p \rightarrow S_q$  to be bounded.

**Theorem 3.** Let  $0 < p, q < \infty$ ,  $u(z) \in H(\mathbb{D})$  and  $\varphi$  is an analytic self-map of  $\mathbb{D}$ . In addition suppose  $f(z)$  satisfies the following properties:

$$(12) \quad \sup_{0 < r < 1} \left[ \frac{1}{2\pi} \int_0^{2\pi} \frac{|u'(z)|^q}{(1 - |\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta \right]^{1/q} < \infty$$

and

$$(13) \quad \sup_{0 < r < 1} \left[ \frac{1}{2\pi} \int_0^{2\pi} \frac{|u(z)\varphi'(z)|^q}{(1 - |\varphi(z)|^2)^{q/p+q\beta}} d\theta \right]^{1/q} < \infty.$$

Then the operator  $D_{\varphi,u}^\beta : S_p \rightarrow S_q$  is bounded.

*Proof.* Suppose (12) and (13) hold. Then Proposition 1 gives

$$\begin{aligned} M_q((D_{\varphi,u}^\beta f)', r) &= \left( \int_0^{2\pi} |(D_{\varphi,u}^\beta f)'(z)|^q d\theta \right)^{1/q} \\ &\leq \left( \int_0^{2\pi} |u'(z)f^{[\beta]}(z)|^q d\theta \right)^{1/q} + \left( \int_0^{2\pi} |u(z)\varphi'(z)f^{[\beta]+1}(z)|^q d\theta \right)^{1/q} \\ &\leq C \left( \int_0^{2\pi} \frac{|u'(z)|^q \|f\|_{S_p}^q}{(1 - |\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta \right)^{1/q} \\ &\quad + \left( \int_0^{2\pi} \frac{|u(z)\varphi'(z)|^q \|f\|_{S_p}^q}{(1 - |\varphi(z)|^2)^{q/p+q\beta}} d\theta \right)^{1/q} \\ &\leq C \left[ \left( \int_0^{2\pi} \frac{|u'(z)|^q}{(1 - |\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta \right)^{1/q} \right. \\ &\quad \left. + \left( \int_0^{2\pi} \frac{|u(z)\varphi'(z)|^q}{(1 - |\varphi(z)|^2)^{q/p+q\beta}} d\theta \right)^{1/q} \right] \|f\|_{S_p}. \end{aligned}$$

Hence,

$$\begin{aligned} \|(D_{\varphi,u}^\beta f)'\|_{H_q} &\leq C \left[ \sup_{0 < r < 1} \left( \int_0^{2\pi} \frac{|u'(z)|^q}{(1 - |\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta \right)^{1/q} \right. \\ &\quad \left. + \sup_{0 < r < 1} \left( \int_0^{2\pi} \frac{|u(z)\varphi'(z)|^q}{(1 - |\varphi(z)|^2)^{q/p+q\beta}} d\theta \right)^{1/q} \right] \|f\|_{S_p} \\ &\leq C \|f\|_{S_p} \end{aligned}$$

Hence,  $D_{\varphi,u}^\beta : S_p \rightarrow S_q$  is bounded.  $\square$

#### 4. COMPACTNESS OF THE OPERATOR $D_{\varphi,u}^\beta$

In this section we find conditions for the compactness of the operator  $D_{\varphi,u}^\beta$  from  $S_p$  spaces to  $H_q$  spaces. Now we state the following result whose proof can be obtained by adapting the proof of Lemma 9 of [5].

**Lemma 3.** Suppose  $\beta \geq 0$ ,  $0 < p < \infty$ ,  $u(z) \in H(\mathbb{D})$  and  $\varphi$  is an analytic self-map of  $\mathbb{D}$ . Then the operator  $D_{\varphi,u}^{\beta} : S_p \rightarrow H_q$  is compact if and only if  $D_{\varphi,u}^{\beta} : S_p \rightarrow H_q$  is bounded and for any bounded sequence  $\{f_k\}_{k \in \mathbb{N}}$  in  $S_p$  which converges to zero uniformly on compact subsets of  $\mathbb{D}$ , we have

$$\|D_{\varphi,u}^{\beta} f_k\|_{H_q} \rightarrow 0,$$

as  $k \rightarrow \infty$ .

**Theorem 4.** Suppose  $\beta \geq 0$ ,  $0 < p, q \leq \infty$ ,  $u(z) \in H(\mathbb{D})$  and  $\varphi(z)$  is an analytic self-map of  $\mathbb{D}$ . Then the operator  $D_{\varphi,u}^{\beta} : S_p \rightarrow H_q$  is compact if and only if  $D_{\varphi,u}^{\beta} : S_p \rightarrow H_q$  is bounded and

$$(14) \quad \lim_{|\varphi(z)| \rightarrow 1} \int_0^{2\pi} \frac{|u(z)|^q}{(1 - |\varphi(z)|^2)^{q/p + q(\beta-1)}} d\theta = 0.$$

*Proof.* Suppose,  $D_{\varphi,u}^{\beta} : S_p \rightarrow H_q$  is bounded and (14) holds. This implies that for every  $\varepsilon > 0$  there exists  $\rho \in (0, 1)$  such that when  $\rho < |\varphi(z)| < 1$ ,

$$\int_0^{2\pi} \frac{|u(z)|^q}{(1 - |\varphi(z)|^2)^{\frac{q}{p} + q(\beta-1)}} d\theta < \varepsilon.$$

Suppose  $g(z) = \frac{1}{\Gamma(1+\beta)}$ . Obviously,  $g(z) \in S_p$ . This implies that  $u(z) \in H_q$ .

Assume that  $\{h_k\}_{k \in \mathbb{N}}$  is bounded sequence in  $S_p$  and converges to zero uniformly on compact subsets of  $\mathbb{D}$  as  $k \rightarrow \infty$ . For  $\rho < |\varphi(z)| < 1$ ,

$$(15) \quad \int_0^{2\pi} |u(z) h_k^{[\beta]}(\varphi(z))|^q d\theta \leq C \|h_k\|_{S_p} \int_0^{2\pi} \frac{|u(z)|^q}{(1 - |\varphi(z)|^2)^{\frac{1}{p} + \beta - 1}} d\theta < LC\varepsilon.$$

For  $|\varphi(z)| \leq \rho$ , we have

$$h_k^{[\beta]}(\varphi(z)) = \frac{\Gamma(1+\beta)}{2\pi i} \int_{|w|=\xi} h_k(w) F\left(1, 1+\beta; 1; \frac{\varphi(z)}{w}\right) \cdot \frac{1}{w} dw, \quad |\varphi(z)| < |w|.$$

Hence,  $w = \xi e^{i\psi}$  gives us

$$\begin{aligned} h_k^{[\beta]}(\varphi(z)) &= \frac{\Gamma(1+\beta)}{2\pi} \int_0^{2\pi} h_k(w) F\left(1, 1+\beta; 1; \frac{\varphi(z)}{w}\right) d\psi \\ &= \frac{\Gamma(1+\beta)}{2\pi} \int_0^{2\pi} h_k(w) \frac{1}{\left(1 - \frac{\varphi(z)}{w}\right)^{1+\beta}} d\psi. \end{aligned}$$

A simple calculation gives us

$$\begin{aligned} |h_k^{[\beta]}(\varphi(z))| &\leq \frac{\Gamma(1+\beta)}{2\pi} \int_0^{2\pi} |h_k(w)| \frac{1}{\left|1 - \frac{\varphi(z)}{w}\right|^{1+\beta}} d\psi \\ &\leq \frac{\Gamma(1+\beta)}{2\pi(\xi-r)^{1+\beta}} \int_0^{2\pi} |h_k(w)| d\psi, \end{aligned}$$

where  $|\varphi(z)| = r$ . Therefore,

$$|h_k^{[\beta]}(\varphi(z))| \leq C \int_0^{2\pi} |h_k(w)| d\psi$$

in compact subsets of  $\mathbb{D}$ . Therefore, we have

$$\left[ \int_0^{2\pi} |u(z)|^q |h_k^{[\beta]}(\varphi(z))|^q d\theta \right]^{1/q} \leq \left[ \int_0^{2\pi} \left| u(z) \int_0^{2\pi} h_k(w) d\psi \right|^q d\theta \right]^{1/q}$$

Minkowski's inequality gives

$$\begin{aligned} \left[ \int_0^{2\pi} |u(z)|^q |h_k^{[\beta]}(\varphi(z))|^q d\theta \right]^{1/q} &\leq \int_0^{2\pi} \left[ \int_0^{2\pi} |u(z) h_k(w)|^q d\theta \right]^{1/q} d\psi \\ &\leq \|u\|_{H_q} \int_0^{2\pi} \left[ \int_0^{2\pi} |h_k(w)|^q d\theta \right]^{1/q} d\psi \rightarrow 0 \end{aligned}$$

on compact subsets of  $\mathbb{D}$ . It follows that  $\|D_{\varphi,u}^\beta h_k\|_{H_q} \rightarrow 0$  as  $k \rightarrow \infty$ .

Therefore, the operator  $D_{\varphi,u}^\beta : S_p \rightarrow H_q$  is compact.

Conversely suppose,  $D_{\varphi,u}^\beta : S_p \rightarrow H_q$  is compact. Then it is bounded. Suppose (14) is not true. Then there is a sequence  $\{z_k\}_{k \in \mathbb{N}}$  such that  $\varphi(z_k) \rightarrow 1$  as  $z_k \rightarrow \infty$  and  $\delta > 0$  such that

$$\int_0^{2\pi} \frac{|u(z)|^q}{(1 - |\varphi(z)|^2)^{q/p+q(\beta-1)}} d\theta \geq \delta, \quad k \in \mathbb{N}.$$

Let  $f_w$  be the test function defined in converse of Theorem 2 and let  $g_k(z) = f_{\varphi(z_k)}$ ,  $k \in \mathbb{N}$ .

Clearly

$$|g_k(z)| = |f_{\varphi(z_k)}| \leq C \frac{(1 - |\varphi(z_k)|^2)}{(1 - \overline{\varphi(z_k)}z)^{\frac{1}{p} + \beta}}.$$

As  $k \rightarrow \infty$ , we have  $\varphi(z_k) \rightarrow 1$ . Hence,  $|g_k|$ , that is  $g_k \rightarrow 0$  as  $k \rightarrow \infty$  uniformly on compact subsets of  $\mathbb{D}$ . Therefore,

$$\lim_{k \rightarrow \infty} \|D_{\varphi,u}^\beta g_k\|_{H_q} = 0.$$

But, from our assumption we have

$$\begin{aligned} \|D_{\varphi,u}^\beta g_k\|_{H_q} &= \sup_{0 < r < 1} \left[ \int_0^{2\pi} |u(z) g_k^{[\beta]}(\varphi(z))|^q d\theta \right]^{1/q} \\ &= \sup_{0 < r < 1} \left[ \int_0^{2\pi} \frac{|u(z)(1 - |\varphi(z_k)|^2)|^q}{(1 - \overline{\varphi(z_k)}\varphi(z_k))^{q/p+q\beta}} d\theta \right]^{1/q} \\ &\geq \sup_{0 < r < 1} \left[ \int_0^{2\pi} \frac{|u(z)|^q}{(1 - |\varphi(z_k)|^2)^{q/p+q(\beta-1)}} d\theta \right]^{1/q} \geq C\delta > 0, \end{aligned}$$

when  $k \rightarrow \infty$ , which is a contradiction.

Hence, (14) must be true. □

## 5. CONCLUSION

Fractional derivatives play a significant role in various fields of engineering and allied sciences. On the other hand, the study of operators involving fractional derivatives leads the use of fractional derivatives in a new direction, which involves generalizing well-known results in analytic function spaces. In this paper, we studied the boundedness of the operator  $D_{\varphi,u}^{\beta} : S_p \rightarrow H_q$ . A sufficient condition for the boundedness of the operator  $D_{\varphi,u}^{\beta} : S_p \rightarrow S_q$  is given. The necessary condition is open for evaluation.

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**DEEPJYOTI BORGOHAIN**

B.P. CHALIHA COLLEGE

NAGARBERA

ASSAM

INDIA

*E-mail address:* [deep1a2b@gmail.com](mailto:deep1a2b@gmail.com)