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K-d-frames and their duals

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Abstract. In this paper, we define a linear bounded operator for double sequences and give a new generalization of frame called K-d-frame. We establish that K-d-frame is square summable in norm for finite dimensional separable Hilbert spaces and prove some results on properties of frame operators and K-d-duals.

1. Introduction

"A sequence $\{x_n\}_{n=1}^{\infty}$ is called a frame for $\mathcal{H}$, if there exist positive constants A and B such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathcal{H}.$$

A and B are called lower and upper frame bounds respectively."
Frames provide infinite representations of vectors after removing the uniqueness property from bases in a Hilbert spaces. Redundancy becomes the main property of frames which makes them more applicable than bases. The applications of frames in a various fields viz. signal and image processing [8], filter bank theory [12], harmonic analysis [10], wireless communications [11] make the study of frames more interesting. For more literature review one may refer to ([1, 3, 4, 6, 14]). Because of applicability of frames in different areas of study, researchers have introduced various concepts of frames like fusion frames [5], continuous fusion frames [7], generalized frames [17], K-frames [15] and d-frames [2] etc.

Throughout this paper, $\mathcal{H}$ denotes Hilbert/separable Hilbert space, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ a collection of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ (if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, then it is denoted by $\mathcal{B}(\mathcal{H})$). For $K \in \mathcal{B}(\mathcal{H}), R(K)$ is the range space of K. K^* is an adjoint of K and $K^\dagger$ is a pseudo- inverse of K.

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Considering the fact that every Bessel sequence in a Hilbert space need not necessarily be a frame, recently Biswas et al. [2] gave a new generalization of frame with the help of double sequences.

Infact, Biswas et al. [2] gave the following definition of d-frame.

Definition 1. [2] A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ is said to be a d-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \le \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \le B\|x\|^2, \quad \text{for all}\, x \in \mathcal{H}, \tag{1}$$

here, constants A and B are called lower and upper d-frame bounds respectively. If $A = B$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called tight d-frame. If $A = B = 1$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called Parseval d-frame.

If only the right-hand inequality holds in equation (1), then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called a double Bessel sequence or d-Bessel sequence for $\mathcal{H}$.

On the other hand, Gavruta [9] introduced the concept of K-frame to study atomic systems with respect to a bounded linear operator K in a Hilbert space. L. Gavruta [9] gave the following definition of K-frame.

Definition 2 ([9])**.** Let $\mathcal{H}$ be a separable Hilbert space and $K \in \mathcal{B}(\mathcal{H})$. A sequence $\{x_n\}_{n=1}^{\infty}$ is called K-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^*x\|^2 \le \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \le B\|x\|^2, \quad \text{for all } x \in \mathcal{H},$$

here, constants A and B are called lower and upper K-frame bounds respectively.

It is remarkable that K-frames are more general than ordinary frames (see [13, 16]).

Motivated by this fact, we extend and generalize d-frames with the help of linear bounded operator K and introduce K-d-frames. Further, we extend the results available in the literature for K-d-frames.

2. K-d-Frame

Definition 3. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a double sequence in separable Hilbert space $\mathcal{H}$ and $K \in \mathcal{B}(\mathcal{H})$. Then, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called a K-d-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|K^*x\|^2 \le \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \le B\|x\|^2, \quad \text{for all}\, x \in \mathcal{H}, \tag{2}$$

here, constants A and B are called lower and upper K-d-frame bounds respectively.

(i) If $A\|K^*x\|^2 = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called a tight K-d-frame.

(ii) If $A = 1$, the above equality becomes $\|K^*x\|^2 = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called Parseval K-d-frame.

Remark 1. If only the right hand inequality holds in equation (2), then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called a K-d-Bessel sequence for $\mathcal{H}$.

Remark 2. For $K = I$, K-d-frames are d-frames.

Remark 3. Every K-frame is a K-d-frame.

Theorem 1. *Every d-frame is a K-d-frame. But converse need not to be true.*

Proof. By definition of d-frame,

$$A\|x\|^2 \leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \leq B\|x\|^2, \quad \text{for all}\, x \in \mathcal{H}. \tag{3}$$

Let $K \in \mathcal{B}(\mathcal{H})$ such that

$$\|K^*x\| \leq c\|x\|, \text{ implies } A\|K^*x\|^2 \leq Ac\|x\|^2. \tag{4}$$

On multiplying equation (3) by c,

$$Ac\|x\|^2 \leq c \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \leq Bc\|x\|^2 \tag{5}$$

on combining equations (4) and (5),

$$A\|K^*x\|^2 \leq c \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \leq Bc\|x\|^2$$

$$\frac{A}{c}\|K^*x\|^2 \leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \leq B\|x\|^2.$$

Hence, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a K-d-frame for $\mathcal{H}$. □

Now, we give the following two examples for the converse of the theorem.

Example 1. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$ and let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a double sequence such that,

$$x_{ij} = \begin{cases} e_{i+1} + e_i, & i = j; \\ 0, & \text{otherwise}; \end{cases}$$

and $Ke_n = e_{n+1} + e_n$, for all $n \in \mathbb{N}$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a K-d-frame for $\mathcal{H}$ with K-d-frame bounds $A = 1, B = 4$ but not a d-frame due to non-existence of its lower bound.

Example 2. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$. Consider $\{x_{ij}\}_{i,j\in\mathbb{N}}$ such that,

$$x_{ij} = \begin{cases} \frac{e_i}{i}, & i = j; \\ 0, & \text{otherwise.} \end{cases}$$

$Ke_n = \frac{e_n}{n}$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a K-d-frame for $\mathcal{H}$ with K-d-frame bounds $A = 1, B = 1$, but not a d-frame due to its lower bound which does not exist.

Here, we remark that one can construct K-d-frames from the given d-frames/frames by taking a suitable linear bounded operator K in a Hilbert space. We illustrate this fact by following examples.

Recall that an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ is a Parseval frame for a separable Hilbert space $\mathcal{H}$.

Example 3. Construct a double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ such that,

$$x_{ij} = \begin{cases} e_i, & i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Since, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a Parseval d-frame, so for $K \in \mathcal{B}(\mathcal{H})$, taking $Ke_1 = e_1$, $Ke_2 = e_1$, $Ke_3 = e_2$, $\ldots$, $Ke_n = e_{n-1}$, $\ldots$, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ becomes a K-d-frame for $\mathcal{H}$ with K-d-frame bounds $A = 1/2, B = 1$.

Example 4. Construct a double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ such that,

$$x_{ij} = \begin{cases} e_i, & i = j \text{ and } i = j + 1; \\ e_j, & j = i + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Since, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a d-frame with bounds 1 and 3 (lower and upper bounds respectively), considering $Ke_1 = e_1$, $Ke_2 = e_1$, $Ke_3 = e_2$, $\ldots$, $Ke_n = e_{n-1}$, $\ldots$, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ becomes a K-d-frame for $\mathcal{H}$ with K-d-frame bounds $A = 1/2, B = 3$.

We give the following result to show that K-d-frame is square summable in norm for a finite dimensional separable Hilbert space $\mathcal{H}$.

Theorem 2. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be an K-d-frame for $\mathcal{H}$ and $K \in \mathcal{B}(\mathcal{H})$. If dimension of $\mathcal{H}$ is finite, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is square summable in norm.*

Proof. Let the dimension of $\mathcal{H}$ is k (say) finite and $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-frame such that

$$A\|K^*x\|^2 \le \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \le B\|x\|^2, \quad \text{for all } x \in \mathcal{H}.$$

Let $\{e_r\}_{r=1}^{k}$ be an orthonormal basis for $\mathcal{H}$. We have

$$\|x_{ij}\|^2 = \sum_{r=1}^{k} |\langle e_r, x_{ij}\rangle|^2, \quad \text{for all}\, i,j \in \mathbb{N} (\text{by Parseval's identity}).$$

Hence,

$$\begin{aligned} \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \|x_{ij}\|^2 &= \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \sum_{r=1}^{k} |\langle e_r, x_{ij}\rangle|^2 \\ &= \sum_{r=1}^{k} \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle e_r, x_{ij}\rangle|^2 \\ &\leq \sum_{r=1}^{k} B\|e_r\|^2 \\ &= Bk. \end{aligned}$$

So, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is square summable. □

For the separable Hilbert space having infinite dimension, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ need not be square summable in norm. We can see it in the following example.

Example 5. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$. Consider $\{x_{ij}\}_{i,j\in\mathbb{N}}$ by

$$x_{ij} = \begin{cases} e_i, & i = j = 1; \\ e_{i-1} + e_i, & i = j > 1; \\ 0, & i \neq j. \end{cases}$$

$K : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator such that $Ke_n = e_{n+1} + e_n$, for all $n \in \mathbb{N}$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a K-d-frame for $\mathcal{H}$.

But,

$$\lim_{n\to\infty} \sum_{i=2}^{n} \|e_{i-1} + e_i\|^2 = \lim_{n\to\infty} \sum_{i=2}^{n} |\langle e_{i-1} + e_i, e_{i-1} + e_i\rangle|^2 = \lim_{n\to\infty} \sum_{i=2}^{n} 4 = \infty.$$

Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in separable Hilbert space $\mathcal{H}$ is a K-d-frame, so it is a d-Bessel sequence.

So, we define the operators $T : \ell^2(\mathbb{N} \times \mathbb{N}) \to \mathcal{H}$ by

$$T(\{a_{ij}\}_{i,j\in\mathbb{N}}) = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} a_{ij} x_{ij}, \quad \text{for all } \{a_{ij}\}_{i,j\in\mathbb{N}} \in \ell^2(\mathbb{N} \times \mathbb{N})$$

and $T^* : \mathcal{H} \to \ell^2(\mathbb{N} \times \mathbb{N})$ by

$$T^* x = \{\langle x, x_{ij}\rangle\}_{i,j\in\mathbb{N}}, \quad \text{for all } x \in \mathcal{H}.$$

Then, $\mathcal{S} = TT^*$ be a frame operator from $\mathcal{H} \to \mathcal{H}$ such that

$$\mathcal{S}x = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, x_{ij}\rangle x_{ij}, \quad \text{for all } x \in \mathcal{H}.$$

Theorem 3. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a double Bessel sequence in $\mathcal{H}$ and $K \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent:*

- (i) *$\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a K-d-frame for $\mathcal{H}$ with lower and upper bounds A and B respectively,*
- (ii) *there exists A such that $A\|K^*x\|^2 \leq \|T^*x\|^2$,*
- (iii) *there exists $A > 0$ such that $\mathcal{S} = TT^* \geq AKK^*$.*

Proof. (i) $\implies$ (ii)

$$A\|K^*x\|^2 \leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \leq B\|x\|^2$$

$$\begin{aligned} \Big\langle \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, x_{ij}\rangle x_{ij}, x\Big\rangle &= \langle TT^*x, x\rangle \\ &= \langle T^*x, T^*x\rangle \\ &= \|T^*x\|^2 \\ &\geq A\|K^*x\|^2. \end{aligned}$$

(ii) $\implies$ (iii)

$$A\|K^*x\|^2 \leq \|T^*x\|^2$$

$$\begin{aligned} \langle AKK^*x, x\rangle = A\langle K^*x, K^*x\rangle &= A\|K^*x\|^2 \\ &= \|T^*x\|^2 \\ &= \langle TT^*x, x\rangle, \end{aligned}$$

this implies

$$AKK^* \leq TT^*.$$

(iii) $\implies$ (i) Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a double Bessel sequence and $\mathcal{S} \geq AKK^*$.

$$\begin{aligned} \langle AKK^*x, x\rangle = A\|K^*x\|^2 \leq \langle \mathcal{S}x, x\rangle &= \langle TT^*x, x\rangle \\ &= \langle \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, x_{ij}\rangle x_{ij}, x\rangle \\ &= \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \\ &\leq B\|x\|^2, \text{ for all}\, x \in \mathcal{H}. \end{aligned}$$

Hence, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a K-d-frame for $\mathcal{H}$. ☐

Corollary 1. *Let* $\{x_{ij}\}_{i,j\in\mathbb{N}}$ *be a tight* K*-d-frame for* $\mathcal{H}$ *with bound* A*, then*

1. $\mathcal{S} = AKK^*$.
2. $\|T\| = \sqrt{A}\|K\|$.

Proof.

1. This is obvious from Theorem 3.
2. $A\|K^*x\|^2 = \|T^*x\|^2$

$$\begin{aligned}\|T\| = \|T^*\| = \sup_{\|x\|=1, x\in\mathcal{H}} \|T^*x\| &= \sup_{\|x\|=1, x\in\mathcal{H}} \sqrt{A}\|K^*\| \\ &= \sqrt{A}\|K^*\| \\ &= \sqrt{A}\|K\|.\end{aligned}$$

□

In general frame operator of a K-d-frame is not invertible on $\mathcal{H}$, but with the help of the following definition, we can show that it is invertible on a closed subspace $R(K) \subset \mathcal{H}$.

Definition 4 ([3])**.** Let $\mathcal{H}$ be a Hilbert space, and suppose that $K \in \mathcal{B}(\mathcal{H})$ has a closed range. Then, there exists a pseudo-inverse $K^\dagger \in \mathcal{B}(\mathcal{H})$ such that

$$N(K^\dagger) = R(K)^\perp, \quad R(K)^\dagger = N(K^\perp), \quad KK^\dagger = I,$$

and it is uniquely determined for all $x \in R(K)$. In fact, if K is invertible, then $K^{-1} = K^\dagger$.

Theorem 4. *The frame operator* $\mathcal{S}$ *of* K*-d-frame is invertible if range space of* K*, i.e.,* $R(K)$ *is closed subspace of* $\mathcal{H}$*.*

Proof. Since $R(K)$ is closed subspace of $\mathcal{H}$, so by Definition 4 there exists a pseudo-inverse $K^\dagger$ of K such that

$$KK^\dagger = I,$$

implies

$$(\, K^\dagger)^* K^* = I^*.$$

Hence,

$$\begin{aligned}\|x\| = \|(\, K^\dagger)^* K^* x\| &\leq \|K^\dagger\|\,\|K^*x\| \\ \|K^\dagger\|^{-1}\,\|x\| &\leq \|K^*x\| \leq \|K\|\,\|x\| \qquad (6) \\ \|K^*x\|^2 &\geq \|K^\dagger\|^{-2}\,\|x\|^2.\end{aligned}$$

Using the definition of K-d-frame

$$A\|K^\dagger\|^{-2}\,\|x\| \leq \|\mathcal{S}x\| = \Big\| \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, x_{ij}\rangle x_{ij} \Big\| \leq B\|x\|, \quad \text{for all } x \in R(K),$$

thus $\mathcal{S} : R(K) \to \mathcal{S}(R(K))$ is a homeomorphism.

And we get

$$B^{-1}\|x\| \leq \|\mathcal{S}^{-1}x\| \leq A^{-1}\|K^\dagger\|^2\,\|x\|, \quad \text{for all } x \in \mathcal{S}(R(K)).$$

□

Theorem 5. *Let $K \in \mathcal{B}(\mathcal{H}), T \in \mathcal{B}(\mathcal{H})$ and $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a tight K-d-frame for $\mathcal{H}$ with bound A, then $\{Tx_{ij}\}_{i,j\in\mathbb{N}}$ is also a tight TK-d-frame for $\mathcal{H}$ with the same bound A.*

Proof. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a tight K-d-frame, i.e., for all $x \in \mathcal{H}$

$$A\|K^*x\|^2 = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2. \tag{7}$$

Since, $T \in \mathcal{B}(\mathcal{H})$ implies $T^*x \in \mathcal{H}$. So,

$$\begin{aligned} A\|K^*T^*x\|^2 &= A\|(TK)^*x\|^2 = \\ \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle T^*x, x_{ij}\rangle|^2 &= \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, Tx_{ij}\rangle|^2. \end{aligned}$$

Hence, $\{Tx_{ij}\}_{i,j\in\mathbb{N}}$ is also a tight TK-d-frame for $\mathcal{H}$ with the same bound A. □

Taking linear bounded operator $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable Hilbert spaces, we obtain the following result for the operator perturbation of a K-d-frame.

Theorem 6. *Let $K_1 \in \mathcal{B}(\mathcal{H}_1)$ and let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K_1-d-frame for $\mathcal{H}_1$. Let $K_2 \in \mathcal{B}(\mathcal{H}_2)$ and let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with a closed range and $TK_1 = K_2T$. If $R({K_2}^*) \subset R(T)$, then $\{Tx_{ij}\}_{i,j\in\mathbb{N}}$ is a K_2-d-frame for $\mathcal{H}_2$.*

Proof. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K_1-d-frame, then

$$A\|K_1^*x\|^2 \le \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \le B\|x\|^2, \quad \text{for all } x \in \mathcal{H}_1.$$

For all $y \in \mathcal{H}_2$, we obtain

$$\begin{aligned} A\|K_1^*T^*y\|^2 \le \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle T^*y, x_{ij}\rangle|^2 &= \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle y, Tx_{ij}\rangle|^2 \\ &\le B\|T^*y\|^2 \le B\|T\|^2\|y\|^2. \end{aligned}$$

Since, $TK_1 = K_2T$. So, $K_1^*T^* = T^*K_2^*$.

We know that $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ has a closed range $R(K_2)^* \subset R(T)$, then from the Definition 4, T has the pseudo-inverse $T^\dagger$ such that $TT^\dagger = I$. This implies $(T^\dagger)^*T^* = I$.

Then, for all $x \in R(T)$

$$\|x\| = \|(T^\dagger)^*T^*x\| \le \|T^\dagger\|\|T^*x\|$$

implies

$$\|T^\dagger\|^{-1}\|x\| \le \|T^*\|\,\|x\|, x \in R(T).$$

Now

$$\begin{aligned} A\|K_1^*T^*y\|^2 &= A\|T^*K_2^*y\|^2 \\ &\geq A\|T^\dagger\|^{-2}\|K_2^*y\|^2. \end{aligned}$$

For all $y \in \mathcal{H}_2$,

$$\begin{aligned} A\|T^\dagger\|^{-2}\|K_2^*y\|^2 &\leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle y, Tx_{ij}\rangle|^2 \\ &\leq B\|T\|^2\|y\|^2, y \in \mathcal{H}_2. \end{aligned}$$

Hence, $\{Tx_{ij}\}_{i,j\in\mathbb{N}}$ is a K_2-d-frame for $\mathcal{H}_2$. □

Corollary 2. *Let $K \in \mathcal{B}(\mathcal{H})$ and $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-frame for $\mathcal{H}$. Let $T \in \mathcal{B}(\mathcal{H})$ has a closed range with $TK = KT$. If $R(K^*) \subset R(T)$, then $\{Tx_{ij}\}_{i,j\in\mathbb{N}}$ is a K-d-frame for $\mathcal{H}$.*

Corollary 3. *Let $K_1 \in \mathcal{B}(\mathcal{H}_1)$ and $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K_1-d-frame for $\mathcal{H}_1$. Let $K_2 \in \mathcal{B}(\mathcal{H}_2)$ and $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be surjective with $TK_1 = K_2T$. Then, $\{Tx_{ij}\}_{i,j\in\mathbb{N}}$ is a K_2-d-frame for $\mathcal{H}_2$.*

We give the following result for the perturbation of a linear bounded operator T.

Theorem 7. *Let $K \in \mathcal{B}(\mathcal{H}_1)$ with a closed range and $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-frame for $\mathcal{H}_1$. Let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, if $R(T^*) \subset R(K)$, then $\{Tx_{ij}\}_{i,j\in\mathbb{N}}$ is a T-d-frame for $\mathcal{H}_2$.*

Proof. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-frame for $\mathcal{H}_1$, i.e.,

$$A\|K^*x\|^2 \leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \leq B\|x\|^2, \quad \text{for all}\, x \in \mathcal{H}_1.$$

For all $y \in \mathcal{H}_2$ and $T^*y \in \mathcal{H}_1$, we obtain

$$\begin{aligned} A\|K^*T^*y\|^2 &\leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle T^*y, x_{ij}\rangle|^2 \leq \\ B\|T^*y\|^2 &\leq B\|T\|^2\|y\|^2, \quad \text{for all } T^*y \in \mathcal{H}_1. \end{aligned}$$

We know that K has a closed range and $R(T^*) \subset R(K)$ then from equation (6), we get

$$\|K^\dagger\|^{-2}\, \|T^*y\|^2 \leq A\|K^*T^*y\|^2.$$

So, we have

$$\begin{aligned} \|K^\dagger\|^{-2}\, \|T^*y\|^2 &\leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle T^*y, x_{ij}\rangle|^2 = \\ \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle y, Tx_{ij}\rangle|^2 &\leq B\|T\|^2\|y\|^2, \quad \text{for all } y \in \mathcal{H}_2. \end{aligned}$$

Hence, $\{Tx_{ij}\}_{i,j\in\mathbb{N}}$ is a T-d-frame for $\mathcal{H}_2$. ☐

Now we define the dual of K-d-frame and establish some results related to K-d-dual.

Dual of K-d-frame: Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-frame for a separable Hilbert space $\mathcal{H}$. A d- Bessel sequence $\{y_{ij}\}_{i,j\in\mathbb{N}}$ of $\mathcal{H}$ is called a K-d- dual of $\{x_{ij}\}_{i,j\in\mathbb{N}}$ if

$$Kx = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, y_{ij}\rangle x_{ij}, \quad \text{for all } x \in \mathcal{H}. \tag{8}$$

Theorem 8. *Every K-d-dual is K^*-d-frame.*

Proof. Let a d-Bessel sequence $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is K-d-dual of K-d-frame $\{x_{ij}\}_{i,j\in\mathbb{N}}$. By definition, we have

$$Kx = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, y_{ij}\rangle x_{ij}, \quad \text{for all } x \in \mathcal{H}.$$

$$\begin{aligned}
\|Kx\|^4 = |\langle Kx, Kx\rangle|^2 &= \left|\left\langle \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, y_{ij}\rangle x_{ij}, Kx\right\rangle\right|^2 \\
&\leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, y_{ij}\rangle|^2 \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle Kx, x_{ij}\rangle|^2 \\
&\leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, y_{ij}\rangle|^2 B\|Kx\|^2, \\
\|Kx\|^2 &\leq B \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, y_{ij}\rangle|^2, \\
\frac{1}{B}\|Kx\|^2 &\leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, y_{ij}\rangle|^2.
\end{aligned}$$

Hence, $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a K^*-d-frame. ☐

Theorem 9. *Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a tight K-d-frame for separable Hilbert space $\mathcal{H}$ and a d-Bessel sequence $\{y_{ij}\}_{i,j\in\mathbb{N}}$ for $\mathcal{H}$ be a K-d-dual of $\{x_{ij}\}_{i,j\in\mathbb{N}}$, then*

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \|y_{ij}\|^2 \geq \frac{1}{A}.$$

Proof. We know that

$$Kx = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, y_{ij}\rangle x_{ij}, \quad \text{for all}\, x \in \mathcal{H},$$

implies

$$K^*x = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, x_{ij}\rangle y_{ij}, \quad \text{for all}\, x \in \mathcal{H}.$$

Since, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a tight K-d-frame i.e.,

$$A\|K^*x\|^2 = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2.$$

Hence,

$$\begin{aligned}
&\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \\
= \quad & A\left\|\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, x_{ij}\rangle y_{ij}\right\|^2 \\
= \quad & A \sup_{\|y\|=1, y\in\mathcal{H}} \left\|\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, x_{ij}\rangle \langle y_{ij}, y\rangle\right\|^2 \\
\leq \quad & A \sup_{\|y\|=1, y\in\mathcal{H}} \left(\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle y_{ij}, y\rangle|^2\right) \\
= \quad & A \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \sup_{\|y\|=1, y\in\mathcal{H}} \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle y_{ij}, y\rangle|^2 \\
\leq \quad & A \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2 \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \|y_{ij}\|^2 \\
\Longrightarrow \quad & \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \|y_{ij}\|^2 \geq \frac{1}{A}.
\end{aligned}$$

□

Theorem 10. *Let* $K \in \mathcal{B}(\mathcal{H})$ *and* $\{x_{ij}\}_{i,j\in\mathbb{N}}$ *be a* K*-*d*-frame for* $\mathcal{H}$ *and* $\{y_{ij}\}_{i,j\in\mathbb{N}}$ *be a* K*-*d*-dual of* $\{x_{ij}\}_{i,j\in\mathbb{N}}$*, then for any* $L \subseteq \mathbb{N}$*,*

$$\begin{aligned}
&\sum_{i,j\in L} \langle x, y_{ij}\rangle \overline{\langle Kx, x_{ij}\rangle} - \left\|\sum_{i,j\in L} \langle x, y_{ij}\rangle x_{ij}\right\|^2 \\
= \quad & \left(\sum_{i,j\in L^C} \overline{\langle x, y_{ij}\rangle} \langle Kx, x_{ij}\rangle\right) - \left\|\sum_{i,j\in L^C} \langle x, y_{ij}\rangle x_{ij}\right\|^2, \quad \textit{for all}\ x \in \mathcal{H}.
\end{aligned}$$

Proof. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-frame for $\mathcal{H}, \{y_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-dual of $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and $L \subseteq \mathbb{N}$ and the operator

$$U_L x = \sum_{i,j\in L} \langle x, y_{ij}\rangle x_{ij}, \text{for all}\, x \in \mathcal{H}.$$

One can easily observe that U_L is well defined and bounded operator on $\mathcal{H}$. Furthermore, we have $U_L + U_{L^C} = K$, and

$$
\begin{aligned}
&\Big(\sum_{i,j\in L}\langle x, y_{ij}\rangle\overline{\langle Kx, x_{ij}\rangle} - \Big\|\sum_{i,j\in L}\langle x, y_{ij}\rangle x_{ij}\Big\|^2\Big)\\
= \;& \Big(\sum_{i,j\in L}\big\langle\langle x, y_{ij}\rangle x_{ij}, Kx\big\rangle\Big) - \|U_L x\|^2\\
= \;& \Big(\sum_{i,j\in L}\big\langle\langle x, y_{ij}\rangle x_{ij}, Kx\big\rangle\Big) - \langle U_L x, U_L x\rangle\\
= \;& \sum_{i,j\in L}\big\langle K^*\langle x, y_{ij}\rangle x_{ij}, x\big\rangle - \langle U_L^* U_L x, x\rangle\\
= \;& \langle K^* U_L x, x\rangle - \langle U_L^* U_L x, x\rangle\\
= \;& \langle (K^* - U_L^*) U_L x, x\rangle\\
= \;& \langle U_{L^C}^* U_L x, x\rangle\\
= \;& \langle U_{L^C}^* (K - U_{L^C}) x, x\rangle\\
= \;& \langle U_{L^C}^* K x, x\rangle - \langle U_{L^C}^* U_{L^C} x, x\rangle\\
= \;& \langle x, K^* U_{L^C} x\rangle - \|U_{L^C} x\|^2\\
= \;& \Big(\Big\langle Kx, \sum_{i,j\in L^C}\langle x, y_{ij}\rangle x_{ij}, x\Big\rangle\Big) - \Big\|\sum_{i,j\in L^C}\langle x, y_{ij}\rangle x_{ij}\Big\|^2\\
= \;& \Big(\sum_{i,j\in L^C}\overline{\langle x, y_{ij}\rangle}\langle Kx, x_{ij}\rangle\Big) - \Big\|\sum_{i,j\in L^C}\langle x, y_{ij}\rangle x_{ij}\Big\|^2. \qquad \square
\end{aligned}
$$

3. Conclusion

The paper gives a new concept of constructing frames using linear bounded operator K on d-frames. Further, the results which are true for the K-frames are extended and proved for the K-d-frames. The results and concept of K-d-frame can be further applied in the field of sampling theory or any other related field.

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