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Viscosity approximations with φ-contractive mappings

Deepak Khantwal 0000-0002-1081-226X,
J. L. Mamabolo 0000-0002-2505-8597,
Rajendra Pant 0000-0001-9990-2298

Abstract. This paper investigates the viscosity approximation method with a general class of φ-contractive mappings. We demonstrate that Moudafi's viscosity approximation scheme can be derived from Browder and Halpern-type convergence theorems. Our results extend several existing convergence theorems. Finally, we present a series of deduced results based on these findings. We also present an application of the obtained results to split feasibility problems.

1. Introduction

The Banach contraction principle (BCP) is among the most notable findings in fixed point theory, asserting that a contraction mapping in complete metric spaces invariably possesses a unique fixed point. The literature contains numerous generalizations of the BCP (see [9, 10, 17]). Some significant generalizations among others have been presented in [2, 11, 15, 16]. Boyd and Wong [2] generalized the BCP by replacing the constant contraction coefficient with a control function having the property of upper semi-continuous from the right. Their main result was further refined by Song-il Ri [16], using a more general class of control functions φ, which need not be upper semi-continuous. The idea of such a control function was given in the paper of Boyd and Wong [2].

On the other hand, the theory of variational inequalities emerged in the 1960s as a robust framework to study a wide range of problems in mathematical analysis, especially those related to constrained optimization, equilibrium, and partial differential equations. The core concept of a variational inequality problem is to find a point within a given set that satisfies a specific inequality condition involving an operator or function. In 2000,

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Moudafi introduced the viscosity approximation method for approximating fixed points of nonexpansive mappings in Hilbert spaces. He demonstrated a strong convergence of the sequences generated by both implicit and explicit methods to the unique solution of some variational inequalities. In 2004, Xu [25] broadened Moudafi's viscosity approximation from Hilbert spaces to Banach spaces.

In 2007, Suzuki [22] provided a simple proofs for Xu's [25] results related to Moudafi's approximations. Additionally, he demonstrated that the convergence theorems of Browder and Halpern imply Moudafi's viscosity approximation with the Meir-Keeler contractions (MKC). In 2009, Song and Liu [20] demonstrated Browder's and Halpern's type convergence theorem using Moudafi's viscosity approximation with weak contraction (refer to [15]) and offered an estimated rate of convergence between Halpern's type iteration and Moudafi's viscosity approximation. Subsequently, Mishra et al. [12] expanded the findings of Song and Liu to a broader class of contractive mappings (cf. [12]), and investigated viscosity approximation with these mappings. It is essential to mention that the control functions employed in [12, 20] are continuous and non-decreasing.

This study intends to explore the Moudafi approximation method with a more general class of φ-contractive mappings. We show that Moudafi's viscosity approximation method with this φ-contractive mapping can be obtained from Browder- and Halpern-type convergence results. Our findings expand upon numerous established convergence theorems, such as the strong convergence theorem by Song and Liu [20], Mishra et al. [12], among others. Finally, we present a series of deduced results based on these findings and an application to split feasibility problems.

2. Preliminaries

Let $(\mathcal{W}, \rho)$ be a metric space. A mapping $\Gamma : \mathcal{W} \to \mathcal{W}$ is said to be contraction with contraction coefficient $k \in (0, 1)$ if for all $\omega, \upsilon \in \mathcal{W}$, $\rho(\Gamma(\omega), \Gamma(\upsilon)) \leq k\rho(\omega, \upsilon)$. In 1969, Boyd and Wong [2] expanded the BCP by establishing the following noteworthy fixed point theorem.

Theorem 1. *Assume that $(\mathcal{W}, \rho)$ is a complete metric space and $\Phi : \mathcal{W} \to \mathcal{W}$ satisfy*

$$d(\Gamma(\omega), \Gamma(\upsilon)) \leq \psi(d(\omega, \upsilon)) \quad \textit{for all } \omega, \upsilon \in X, \ \textit{where } \omega \neq \upsilon,$$

where $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right and satisfies $\psi(t) < t$ for all $t > 0$. Then, Γ has a unique fixed point ω_0, and $\Gamma^n\omega \to \omega_0$ for each $\omega \in \mathcal{W}$.

In 2016, Song-il Ri [16] proposed a refinement of the result of Boyd-Wong by replacing a control function ψ with a more general control function φ as defined in the following result.

Theorem 2. *Let Φ be a self-mapping on a complete metric space $(\mathcal{W}, \rho)$. If there exists a function $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

$$\varphi(t) < t \quad \textit{with} \quad \limsup_{s \to t^+} \varphi(s) < t \quad \textit{for } t > 0,$$

and

$$\rho(\Phi(\omega), \Phi(\upsilon)) \leq \varphi(\rho(\omega, \upsilon))$$

holds for all $\omega, \upsilon \in \mathcal{W}$, then Φ has only one fixed point in $\mathcal{W}$.

It is important to mention that the control function φ, as defined in Theorem 2, is continuous at $\omega = 0$ and additionally fulfills $\varphi(0) = 0$. To verify continuity at $\omega = 0$, consider a sequence $\{\omega_n\}$ of positive real numbers that converges 0. Then by virtue of φ, we have $\varphi(\omega_n) < \omega_n$ for n. Making $n \to \infty$, it follows that $\varphi(\omega_n) \to 0 = \varphi(0)$.

In this paper, we represent the collection of all positive integers by $\mathbb{N}$. Let $\mathcal{B}$ represent a Banach space and $\mathcal{B}^*$ indicate the dual space of $\mathcal{B}$. The space $\mathcal{B}$ is referred to as *smooth* or having a *Gâteaux differentiable norm* if the limit $\lim_{t \to 0} \frac{\|\omega + t\upsilon\| - \|\omega\|}{t}$ exists for every $\omega, \upsilon \in \mathcal{B}$ where $\|\omega\| = \|\upsilon\| = 1$. The space $\mathcal{B}$ is called to have a *uniformly Gâteaux differentiable norm* if for every $\upsilon \in \mathcal{B}$ with $\|\upsilon\| = 1$, the limit ω is achieved uniformly in $\mathcal{B}$ where $\|\omega\| = 1$. If the limit is attained uniformly in $\omega, \upsilon \in \mathcal{B}$ with $\|\omega\| = \|\upsilon\| = 1$, then the space $\mathcal{B}$ is referred to as *uniformly smooth* or having a uniformly Fréchet differentiable norm. A mapping $\mathcal{P}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is called as *sunny* if $\mathcal{P}_{\mathcal{C}}(\mathcal{P}_{\mathcal{C}}(\omega) + t(\omega - \mathcal{P}_{\mathcal{C}}(\omega))) = \mathcal{P}_{\mathcal{C}}(\omega)$ holds for $\omega \in \mathcal{C}$, where $\mathcal{P}_{\mathcal{C}}(\omega) + t(\omega - \mathcal{P}_{\mathcal{C}}(\omega)) \in \mathcal{C}$ and $t \geq 0$. A mapping Γ on $\mathcal{C}$ is referred to as a *nonexpansive mapping* if $\|\Gamma(\omega) - \Gamma(\upsilon)\| \leq \|\omega - \upsilon\|$ holds for every $\omega, \upsilon \in \mathcal{C}$. We denote $F(\Gamma)$ as the collection of fixed points of Γ.

Let $\mathcal{B}$ be a smooth Banach space and $\mathcal{C} \subseteq \mathcal{B}$ a nonempty, closed, convex subset. If $\Gamma : \mathcal{C} \to \mathcal{C}$ represents a specified operator, then the classical variational inequality problem, referred to as $\mathrm{VI}(\mathcal{C}, \Gamma)$, involves locating $\omega^* \in \mathcal{C}$ such that

$$\langle (I - \Gamma)\omega^*, \omega - \omega^* \rangle \geq 0 \quad \text{for } \omega \in \mathcal{C}. \tag{1}$$

The collection of solutions for the variational inequality problem $VI(\mathcal{C}, \Gamma)$ is referred to as Ω. The variational inequality (1) possesses a solution if and only if the operator Γ has a fixed point $\omega^* \in \mathcal{C}$. The proof for the following can be found in [14].

Lemma 1. *Assume that $\mathcal{B}$ is a smooth Banach space and $\mathcal{J} : \mathcal{B} \to \mathcal{B}^*$ is the duality mapping, meaning that for every $\omega \in \mathcal{B}$, it holds that $\langle \omega, \mathcal{J}(\omega) \rangle = \|\omega\|^2 = \|\mathcal{J}(\omega)\|^2$. If $\mathcal{K} \subseteq \mathcal{C}$, with $\mathcal{C}$ being a convex subset of $\mathcal{B}$, and $\mathcal{P}_{\mathcal{C}} : \mathcal{C} \to \mathcal{K}$ acts as a retraction, then the subsequent statements are equivalent:*

(i). for all $\omega \in \mathcal{C}$ and $\upsilon \in \mathcal{K}$, $\langle \omega - \mathcal{P}_{\mathcal{C}}(\omega), \mathcal{J}(\mathcal{P}_{\mathcal{C}}(\omega) - \upsilon) \rangle \geq 0$;

(ii). $\mathcal{P}_{\mathcal{C}}$ is both sunny and nonexpansive.

Thus, there is at most one sunny nonexpansive retraction from $\mathcal{C}$ to $\mathcal{K}$. It is noteworthy that if $\mathcal{B}$ is a Hilbert space and $\mathcal{K}$ is both closed and convex, then the metric projection and the sunny nonexpansive retraction from $\mathcal{C}$ to $\mathcal{K}$ are the same. The subsequent result is widely recognized, and its proof can be easily found in [12, 18, 23].

Proposition 1. *Let $\mathcal{C}$ be a closed convex subset of a smooth Banach space $\mathcal{B}$. Let $\mathcal{K}$ be a subset of $\mathcal{C}$ and let $\mathcal{P}_{\mathcal{C}}$ be the unique sunny nonexpansive retraction from $\mathcal{C}$ onto $\mathcal{K}$. Let Φ be a mapping on $\mathcal{C}$ and $z \in \mathcal{K}$. Then the following are equivalent:*

(1) *z is a fixed point of $\mathcal{P}_{\mathcal{C}} \circ \Phi$;*
(2) *z is a solution of a variational inequality $\langle \Phi(z) - z, \mathcal{J}(z - \upsilon) \rangle \geq 0$ for all $\upsilon \in \mathcal{K}$.*

In 1967, Browder [3] and Halpern [8] established strong convergence theorems in Hilbert spaces for both implicit and explicit iteration, respectively. These theorems have been expanded in various ways. Reich [14] expanded Browder's result to uniformly smooth Banach spaces.

Theorem 3. [14]. *Consider $\mathcal{C}$ as a closed and bounded convex subset of a uniformly smooth Banach space $\mathcal{B}$, and let Γ be a nonexpansive mapping defined on $\mathcal{C}$. Fix $u \in \mathcal{C}$ and define a net $\{\upsilon_\alpha\}$ in $\mathcal{C}$ by*

$$\upsilon_\alpha = (1-\alpha)\Gamma(\upsilon_\alpha) + \alpha u \tag{2}$$

for $\alpha \in (0,1)$. Thus, $\{\upsilon_\alpha\}$ converges strongly to $\mathcal{P}_{\mathcal{C}}(u)$ as α approaches $+0$, where $\mathcal{P}_{\mathcal{C}}$ being the unique sunny retraction from $\mathcal{C}$ onto $F(\Gamma)$.

Xu [24] expanded Halpern's findings to uniformly smooth Banach spaces.

Theorem 4. *Assume $\mathcal{B}, \mathcal{C}, \Gamma, \mathcal{P}_{\mathcal{C}}$ and u are same as in Theorem 3. Define a sequence $\{\upsilon_n\}$ in $\mathcal{C}$ with υ_1 and*

$$\upsilon_{n+1} = (1-\alpha_n)\Gamma(\upsilon_n) + \alpha_n u$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence of real numbers in the interval $(0,1)$ that fulfills

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$,
(C2) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,
(C3) $\lim\limits_{n\to\infty} \dfrac{\alpha_{n+1}}{\alpha_n} = 1$.

Then $\{\upsilon_n\}$ converges to $\mathcal{P}_{\mathcal{C}}(u)$.

Corollary 1. *Let $\mathcal{C}$ denote a closed convex subset of a uniformly smooth Banach space $\mathcal{B}$, and let Γ represent a nonexpansive mapping on $\mathcal{C}$ that has a fixed point. If $\mathcal{B}, \mathcal{P}_{\mathcal{C}}, u, \{\upsilon_n\}$ are as defined in Theorem 4, then the sequence $\{\upsilon_n\}$ converges strongly to a fixed point of Γ.*

In 2000, Moudafi [13] extended the results of Browder and Halpern differently. He established strong convergence results for explicit and implicit type iterative schemes involving a contraction mapping. Moudafi's generalizations are referred to as viscosity approximations.

Theorem 5. *Let $\mathcal{C}$ be a closed and convex subset of a real Hilbert space $\mathcal{H}$, and let $\Gamma, \Phi : \mathcal{C} \to \mathcal{C}$ be, respectively, nonexpansive and contraction mappings. Then the sequence $\{\upsilon_n\}$ defined by $\upsilon_0 \in \mathcal{C}$,*

$$\upsilon_n = \frac{1}{1+\alpha_n}\Gamma(\upsilon_n) + \frac{\alpha_n}{1+\alpha_n}\Phi(\upsilon_n) \tag{3}$$

where $\alpha_n \in (0,1)$ and satisfies $(C1)$, converges strongly to a unique solution of VIP($\mathcal{C}, \Phi$).

Theorem 6. *Let $\mathcal{C}, \mathcal{H}, \Gamma, \Phi$ are same as in Theorem 5, in addition that $\{\alpha_n\}$ satisfy $C2$ and $C3$. Then the sequence $\{\upsilon_n\}$ defined by $\upsilon_0 \in \mathcal{C}$,*

$$\upsilon_{n+1} = \frac{1}{1+\alpha_n}\Gamma(\upsilon_n) + \frac{\alpha_n}{1+\alpha_n}\Phi(\upsilon_n) \tag{4}$$

converges strongly to a unique solution of VIP($\mathcal{C}, \Phi$).

Xu [25] generalized Moudafi's result to uniformly smooth Banach spaces (refer to Theorem 7 and 8).

The subsequent lemma is proved in [1] and is vital for our findings.

Lemma 2. [1]. *Let $\{\mu_k\}, \{\alpha_k\}, \{\beta_k\}$, and $\{\gamma_k\}$ denote sequences of non-negative real numbers that fulfil the recurrence inequality.*

$$\mu_{k+1} \leq \mu_k - \alpha_k\beta_k + \gamma_k. \tag{5}$$

Suppose there is a constant $c > 0$ such that $\alpha_k \leq c$ for every k, and that $\sum\limits_{k=1}^{\infty} \alpha_k = \infty$, with $\lim\limits_{k\to\infty} \frac{\gamma_k}{\alpha_k} = 0$. Then

(i) *there exists a sequence of indices $\{\ell_k\} \subset \mathbb{N}$ such that*

$$\beta_{\ell_k} \leq \frac{1}{\sum\limits_{j=1}^{\ell_k} \alpha_j} + \frac{\gamma_{\ell_k}}{\alpha_{\ell_k}}$$

and consequently, $\lim\limits_{k\to\infty} \beta_{\ell_k} = 0$,

(ii) *if $\lim\limits_{k\to\infty} \mu_{l_k} = 0$ then $\lim\limits_{k\to\infty} \mu_k = 0$,*

(iii) *if $\{\beta_k\}$ converges then $\lim\limits_{k\to\infty} \beta_k = 0$.*

Remark 1 ([1]). Assume β_k from Lemma 2 is expressed as $\beta_k = \psi(\mu_k)$, with $\psi : [0,\infty) \to [0,\infty)$ being a function defined such that $\psi(0) = 0$, $\psi(t) > 0$ for $t > 0$, and ψ is continuous at $\omega = 0$. Thus, the inequality (5) takes the form

$$\mu_{k+1} \leq \mu_k - \alpha_k\psi(\mu_k) + \gamma_k \tag{6}$$

with

$$\alpha_k \leq c, \ \sum_{k=1}^{\infty} \alpha_k = \infty, \ \lim_{k\to\infty} \frac{\gamma_k}{\alpha_k} = 0.$$

In this case, Lemma 2 asserts that

$$\psi(\mu_{\ell_k}) \leq \frac{1}{\sum\limits_{j=1}^{\ell_k} \alpha_j} + \frac{\gamma_{\ell_k}}{\alpha_{\ell_k}}$$

and $\lim\limits_{k\to\infty} \psi(\mu_{\ell_k}) = 0 = \psi(0)$. It follows by continuity of ψ at $\omega = 0$ that $\lim\limits_{k\to\infty} \mu_{\ell_k} = 0$ which from (ii) of Lemma 2 further implies $\lim\limits_{k\to\infty} \mu_k = 0$.

3. MAIN RESULTS

In this section, following the proof technique used in [21], we generalize the result of Xu [25] under φ-contractive mappings.

Proposition 2. *Let $\mathcal{C}$ be a convex subset of a Banach space $\mathcal{B}$. If Γ is a nonexpansive mapping and Φ is a φ-contractive self-mapping on $\mathcal{C}$, then the following hold:*

(1) *$\Gamma \circ \Phi$ is a φ-contractive self-mapping on $\mathcal{C}$.*
(2) *For each $\alpha \in (0,1)$, the mapping $\omega \mapsto (1-\alpha)\Gamma(\omega) + \alpha\Phi(\omega)$ is an φ-contractive mapping on $\mathcal{C}$.*

Proof. (1) For any $\omega, \upsilon \in \mathcal{C}$, we have

$$\|\Gamma \circ \Phi(\omega) - \Gamma \circ \Phi(\upsilon)\| \leq \|\Phi(\omega) - \Phi(\upsilon)\| \leq \varphi(\|\omega - \upsilon\|).$$

So, $\Gamma \circ \Phi$ is a φ-contractive mapping on $\mathcal{C}$.

(2) For $\omega, \upsilon \in \mathcal{C}$ and fixed $\alpha \in (0,1)$, we have

$$\begin{aligned}
&\|(1-\alpha)\Gamma(\omega) + \alpha\Phi(\omega) - (1-\alpha)\Gamma(\upsilon) - \alpha\Phi(\upsilon)\| \\
&\leq (1-\alpha)\|\Gamma(\omega) - \Gamma(\upsilon)\| + \alpha\|\Phi(\omega) - \Phi(\upsilon)\| \\
&\leq (1-\alpha)\|\omega - \upsilon\| + \alpha\varphi(\|\omega - \upsilon\|) \\
&= ((1-\alpha)\mathcal{I} + \alpha\varphi)\,\|\omega - \upsilon\|,
\end{aligned}$$

where $\mathcal{I}$ is an identity mapping. We define $\psi := (1-\alpha)\mathcal{I} + \alpha\varphi$ and observe that for $t > 0$, $\psi(t) < t$ and $\limsup\limits_{t\in(\varepsilon,\varepsilon+\delta)} \psi(t) < \varepsilon$. Then it follows from the above,

$$\|(1-\alpha)\Gamma(\omega) + \alpha\Phi(\omega) - (1-\alpha)\Gamma(\upsilon) - \alpha\Phi(\upsilon)\| \leq \psi(\|\omega - \upsilon\|).$$

Hence $\omega \mapsto (1-\alpha)\Gamma(\omega) + \alpha\varphi(\omega)$ is a contractive mapping on $\mathcal{C}$ with control function ψ. □

Theorem 7. *Consider $\mathcal{C}$ as a closed, bounded, convex subset of a uniformly smooth Banach space $\mathcal{B}$. Let Γ be a nonexpansive mapping on $\mathcal{C}$. If $\mathcal{P}_{\mathcal{C}}$ denotes the sunny nonexpansive retraction from $\mathcal{C}$ to $F(\Gamma)$ and Φ is a φ-contractive on $\mathcal{C}$, then for a net $\{\omega_\alpha\}$ in $\mathcal{C}$, we have*

$$\omega_\alpha = (1-\alpha)\Gamma(\omega_\alpha) + \alpha\Phi(\omega_\alpha)$$

for $\alpha \in (0,1)$. As α approaches $+0$, the sequence $\{\omega_\alpha\}$ converges strongly to the unique point $z \in \mathcal{C}$.

Proof. Define a net $\{\upsilon_\alpha\}$ in $\mathcal{C}$ by $\upsilon_\alpha = (1-\alpha)\Gamma(\upsilon_\alpha) + \alpha\Phi(z)$ for $\alpha \in (0,1)$. Then by Theorem 3 and Propostion 2, $\{\upsilon_\alpha\}$ converges strongly to unique $z = \mathcal{P}_{\mathcal{C}} \circ \Phi(z)$.
For every $\alpha \in (0,1)$, we have

$$\begin{aligned}\|\omega_\alpha - \upsilon_\alpha\| &\le (1-\alpha)\|\Gamma(\omega_\alpha) - \Gamma(\upsilon_\alpha)\| + \alpha\|\Phi(\omega_\alpha) - \Phi(z)\| \\ &\le (1-\alpha)\|\omega_\alpha - \upsilon_\alpha\| + \alpha\varphi(\|\omega_\alpha - z\|) \\ &\le \varphi(\|\omega_\alpha - z\|).\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}\|\omega_\alpha - z\| &\le \|\omega_\alpha - \upsilon_\alpha\| + \|\upsilon_\alpha - z\| \\ &\le \varphi(\|\omega_\alpha - z\|) + \|\upsilon_\alpha - z\|.\end{aligned}$$

It follows that

$$(\mathcal{I} - \varphi)\|\omega_\alpha - z\| \le \|\upsilon_\alpha - z\|,$$

where $\mathcal{I}$ is an identity mapping. Define $\psi := \mathcal{I} - \varphi$, we note that $\psi(0) = 0$, $\psi(t) > 0$ and $\psi(t) < t$ for $t > 0$, and ψ is continuous at $\omega = 0$. Then,

$$\psi(\|\omega_\alpha - z\|) \le \|\upsilon_\alpha - z\|.$$

Since $\{\upsilon_\alpha\}$ converges strongly to z implies $\lim\limits_{\alpha\to 0+} \psi(\|\omega_\alpha - z\|) = 0$. By continuity of φ at $\omega = 0$ implies $\lim\limits_{\alpha\to 0+} \|\omega_\alpha - z\| = 0$. Hence $\{\omega_\alpha\}$ converges strongly to z □

Theorem 8. *Consider $\mathcal{B}, \mathcal{C}, \Gamma, \mathcal{P}_{\mathcal{C}}, \Phi$ and z as defined in Theorem 7. Define a sequence $\{\omega_n\}$ in $\mathcal{C}$ such that $\omega_1 \in \mathcal{C}$ and*

$$\omega_{n+1} = (1-\alpha_n)\Gamma(\omega_n) + \alpha_n\Phi(\omega_n)$$

for $n \in \mathbb{N}$, where the sequence $\{\alpha_n\} \in (0,1)$ and satisfies the conditions (C1), (C2), *and* (C3). *Then, $\{\omega_n\}$ converges strongly to z.*

Proof. Define a sequence $\{\upsilon_n\}$ in $\mathcal{C}$ by $\upsilon_{n+1} = (1-\alpha_n)\Gamma(\upsilon_n) + \alpha_n\Phi(z)$ for $n \in \mathbb{N}$. Then by Theorem 4, $\{\upsilon_n\}$ converges strongly to a unique $z \in \mathcal{C}$ such that $\mathcal{P}_{\mathcal{C}} \circ \Phi(z) = z$. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}\|\omega_{n+1} - \upsilon_{n+1}\| &\le (1-\alpha_n)\|\Gamma(\omega_n) - \Gamma(\upsilon_n)\| + \alpha_n\|\Phi(\omega_n) - \Phi(\upsilon_n)\| \\ &\quad + \alpha_n\|\Phi(\upsilon_n) - \Phi(z)\| \\ &\le (1-\alpha_n)\|\omega_n - \upsilon_n\| + \alpha_n\varphi(\|\omega_n - \upsilon_n\|) + \alpha_n\varphi(\|\upsilon_n - z\|)\end{aligned}$$

$$= \|\omega_n - \upsilon_n\| - \alpha_n(\mathcal{I} - \varphi)\|\omega_n - \upsilon_n\| + \alpha_n\varphi(\|\upsilon_n - z\|),$$

where $\mathcal{I}$ stands for an Identity mapping. Set $\psi := \mathcal{I} - \varphi$, we observe that $\psi(t) < t$ for $t > 0$, $\psi(0) = 0$ and ψ is continuous at $\omega = 0$.

$$\begin{aligned}\|\omega_{n+1} - \upsilon_{n+1}\| &\leq \|\omega_n - \upsilon_n\| - \alpha_n\psi(\|\omega_n - \upsilon_n\|) + \alpha_n\varphi(\|\upsilon_n - z\|)\\ &\leq \|\omega_n - \upsilon_n\| - \alpha_n\psi(\|\omega_n - \upsilon_n\|) + \alpha_n\|\upsilon_n - z\|.\end{aligned}$$

Assuming that $\mu_n = \|\omega_n - \upsilon_n\|$ and $\gamma_n = \alpha_n\|\upsilon_n - z\|$, we have

$$\mu_{n+1} \leq \mu_n - \alpha_n\psi(\mu_n) + \gamma_n.$$

Then from Lemma 2 and Remark 1, we have $\lim\limits_{n\to\infty} \mu_n = 0$. □

Remark 2. It is not difficult to verify that the point z in Theorem 7 and 8 is a fixed point of the mapping Γ.

4. HALPERN'S AND BROWDER'S TYPES CONVERGENCE RESULTS

This section shows that Halpern's and Browder's type convergence theorems imply Moudafi's viscosity approximation with φ-contractive mappings.

Consider $\{\mathcal{S}_n\}$ as a sequence of nonexpansive functions defined on a closed convex subset $\mathcal{C}$ within a Banach space and denote $\{\alpha_n\}$ as a sequence in $(0, 1]$ that fulfills (C1). Then $(\mathcal{B}, \mathcal{C}, \{\mathcal{S}_n\}, \{\alpha_n\})$ is referred to as having Browder's property if, for every $u \in \mathcal{C}$, a sequence $\{\upsilon_n\}$ is defined by

$$\upsilon_n = (1 - \alpha_n)S_n(\upsilon_n) + \alpha_n u \tag{7}$$

for $n \in \mathbb{N}$ converges strongly.

Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ that meets the conditions (C1) and (C2). Then $(\mathcal{B}, \mathcal{C}, \{\mathcal{S}_n\}, \{\alpha_n\})$ is considered to possess Halpern's property if for every $u \in \mathcal{C}$, a sequence $\{\upsilon_n\}$ defined by

$$\upsilon_{n+1} = (1 - \alpha_n)S_n(\upsilon_n) + \alpha_n u \tag{8}$$

for $n \in \mathbb{N}$ converges robustly.

We first discuss Browder's type convergence.

Theorem 9. *Assume that $(\mathcal{B}, \mathcal{C}, \{\Gamma_n\}, \{\alpha_n\})$ have Browder's property. For each $u \in \mathcal{C}$, put $\mathcal{P}_\mathcal{C}(u) = \lim\limits_{n\to\infty} \upsilon_n$, where $\{\upsilon_n\}$ is a sequence in $\mathcal{C}$ defined by (7). If $\Phi : \mathcal{C} \to \mathcal{C}$ is a φ-contractive mapping and $\{\omega_n\}$ is a sequence in $\mathcal{C}$ defined by*

$$\omega_n = (1 - \alpha_n)\Gamma_n(\omega_n) + \alpha_n\Phi(\omega_n)$$

for $n \in \mathbb{N}$. Then $\{\omega_n\}$ converges strongly to the unique point $z \in \mathcal{C}$ satisfying $\mathcal{P}_\mathcal{C} \circ \Phi(z) = z$.

Proof. Construct a sequence $\{\upsilon_n\}$ in $\mathcal{C}$ as $\upsilon_n = (1-\alpha_n)\Gamma_n(\upsilon_n) + \alpha_n\Phi(z)$ for $n \in \mathbb{N}$. Then by assumption, $\{\upsilon_n\}$ converges strongly to $\mathcal{P}_\mathcal{C} \circ \Phi(z) = z$. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}\|\omega_n - \upsilon_n\| &\le (1-\alpha_n)\|\Gamma_n(\omega_n) - \Gamma_n(\upsilon_n)\| + \alpha_n\|\Phi(\omega_n) - \Phi(z)\| \\ &\le (1-\alpha_n)\|\omega_n - \upsilon_n\| + \alpha_n\varphi(\|\omega_n - z\|) \\ &\le \varphi(\|\omega_n - z\|).\end{aligned}$$

By the triangular inequality,

$$\begin{aligned}\|\omega_n - z\| &\le \|\omega_n - \upsilon_n\| + \|\upsilon_n - z\| \\ &\le \varphi(\|\omega_n - z\|) + \|\upsilon_n - z\|.\end{aligned}$$

Let $\psi := \mathcal{I} - \varphi$. Then $\psi(0) = 0$, $\psi(t) > 0$ and $\psi(t) < t$ for $t > 0$, and ψ is continuous at $\omega = 0$. It follows from the above that

$$\psi(\|\omega_n - z\|) \le \|\upsilon_n - z\|.$$

Since $\{\upsilon_n\}$ converges strongly to z implies $\lim_{n\to\infty} \psi(\|\omega_n - z\|) = 0$. By continuity of φ from right at $\omega = 0$ implies $\lim_{n\to\infty} \|\omega_n - z\| = 0$. Hence $\{\omega_n\}$ converges strongly to z. □

We now investigate Moudafi's viscosity method under φ-contractive mappings using Halpern's property.

Theorem 10. *Assume that $(\mathcal{B}, \mathcal{C}, \{\Gamma_n\}, \{\alpha_n\})$ have Halpern's property. For each $u \in \mathcal{C}$, put $\mathcal{P}_\mathcal{C}(u) = \lim_{n\to\infty} \upsilon_n$, where $\{\upsilon_n\}$ is a sequence in $\mathcal{C}$ defined by* (8)*. If $\Phi : \mathcal{C} \to \mathcal{C}$ is a φ-contractive and $\{\omega_n\}$ is a sequence in $\mathcal{C}$ defined by $\omega_1 \in \mathcal{C}$ and*

$$\omega_{n+1} = (1-\alpha_n)\Gamma_n(\omega_n) + \alpha_n\Phi(\omega_n)$$

for $n \in \mathbb{N}$, then $\{\omega_n\}$ converges strongly to the unique point $z \in \mathcal{C}$ satisfying $\Gamma \circ \Phi(z) = z$.

Proof. Define a sequence $\{\upsilon_n\}$ in $\mathcal{C}$ by $\upsilon_{n+1} = (1-\alpha_n)\Gamma_n(\upsilon_n) + \alpha_n\Phi(z)$ for $n \in \mathbb{N}$. Then by assumption, $\{\upsilon_n\}$ converges strongly to a unique $z \in \mathcal{C}$ such that $\mathcal{P}_\mathcal{C} \circ \Phi(z) = z$. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}\|\omega_{n+1} - \upsilon_{n+1}\| &\le (1-\alpha_n)\|\Gamma(\omega_n) - \Gamma(\upsilon_n)\| + \alpha_n\|\Phi(\omega_n) - \Phi(\upsilon_n)\| \\ &\quad + \alpha_n\|\Phi(\upsilon_n) - \Phi(z)\| \\ &\le (1-\alpha_n)\|\omega_n - \upsilon_n\| + \alpha_n\varphi(\|\omega_n - \upsilon_n\|) + \alpha_n\varphi(\|\upsilon_n - z\|) \\ &= \|\omega_n - \upsilon_n\| - \alpha_n(\mathcal{I} - \varphi)\|\omega_n - \upsilon_n\| + \alpha_n\varphi(\|\upsilon_n - z\|).\end{aligned}$$

Let $\psi := \mathcal{I} - \varphi$. Then, $\psi(t) < t$ for $t > 0$, $\psi(0) = 0$ and ψ is continuous at $\omega = 0$.

$$\begin{aligned}\|\omega_{n+1} - \upsilon_{n+1}\| &\le \|\omega_n - \upsilon_n\| - \alpha_n\psi(\|\omega_n - \upsilon_n\|) + \alpha_n\varphi(\|\upsilon_n - z\|) \\ &\le \|\omega_n - \upsilon_n\| - \alpha_n\psi(\|\omega_n - \upsilon_n\|) + \alpha_n\|\upsilon_n - z\|.\end{aligned}$$

Assuming that $\mu_n = \|\omega_n - \upsilon_n\|$ and $\gamma_n = \alpha_n\|\upsilon_n - z\|$, we have

$$\mu_{n+1} \le \mu_n - \alpha_n\psi(\mu_n) + \gamma_n.$$

Then from Lemma 2 and Remark 1, we have $\lim_{n\to\infty} \mu_n = 0$. □

Now, we present an example in support of Theorem 9.

Example 1. Let $(\mathcal{B}, \|.\|_2) = (\ell^2, \|.\|_2)$ be a Banach space and $\mathcal{C} = \{\omega := (\omega_1, \omega_2, \omega_3, \dots) \in \ell^2 : \omega_{2n} = 0,\ n \in \mathbb{N}\}$, a closed and convex set. Define $\Gamma_n : \mathcal{C} \to \mathcal{C}$ such that

$$\Gamma_n(\omega_1, \omega_2, \omega_3, \dots) = \left(2\omega_1 - 1 - \frac{1}{n},\ 0,\ 2\omega_3 - 1 - \frac{1}{n}, \dots\right)$$

and $\Phi : \mathcal{C} \to \mathcal{C}$ such that

$$\Phi(\omega_1, \omega_2, \omega_3, \dots) = \left(\frac{1}{1+\omega_1^2}, \frac{1}{1+\omega_2^2},\ \frac{1}{1+\omega_3^2},\ \dots\right).$$

If we take $\varphi \in \Omega$, defined as

$$\phi(t) = \begin{cases} \dfrac{t}{2}, & \text{if } 0 \le t < 1, \\ \dfrac{1}{3}, & \text{if } t = 1, \\ \dfrac{t^2}{1+t}, & \text{if } t > 1, \end{cases}$$

then one can easily see that φ is neither upper semi-continuous nor continuous form the right at $\omega = 1$ and we leave it to readers to verify Γ_n is nonexpansive and Φ is a φ-contractive mapping.

Now, suppose $\alpha_n = \dfrac{1}{n}$ then the quadruple $(\mathcal{B}, \mathcal{C}, \{\Gamma_n\}, \{\alpha_n\})$ satisfies the Browder property. For each $\omega \in \mathcal{C}$, we have

$$\omega_n = \alpha_n u + (1-\alpha_n)\Gamma_n(\omega_n) = \frac{u}{n} + \left(1 - \frac{1}{n}\right)\Gamma_n(\omega_n).$$

Making $n \to \infty$, we get $P_{\mathcal{C}}(u) = \lim_{n\to\infty} \omega_n = (2\omega_1 - 1,\ 0,\ 2\omega_3 - 1, \dots)$. Then by Theorem 9, $\omega_n = \alpha_n\Phi(\omega_n) + (1-\alpha_n)\Gamma_n(\omega_n)$ converges strongly to a unique fixed point of $P_{\mathcal{C}}(u)$ that is $(1, 0, 1, \dots) \in \ell^2$.

5. DEDUCED RESULTS

In this section, we deduce many convergence theorems using Theorem 7, 8, 9, and 10.

Theorem 11. *Let $\mathcal{B}$ be a Banach space with a norm that is uniformly Gâteaux differentiable. Let Γ be a nonexpansive mapping defined on a closed convex subset $\mathcal{C}$ of $\mathcal{B}$, and let Φ be a $\varphi-$contractive mapping on $\mathcal{C}$. Define a sequence $\{\omega_n\}$ in $\mathcal{C}$ by*

$$\omega_{n+1} = (1-\alpha_n)(\lambda\Gamma(\omega_n) + (1-\lambda)\omega_n) + \alpha_n\Phi(\omega_n) \tag{9}$$

for $n \in \mathbb{N}$, where $\lambda \in (0,1)$ and $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying (C1) *and* (C2). *Suppose that $(\mathcal{B}, \mathcal{C}, \{\Gamma\}, \frac{1}{n})$ possesses a Browder property, and denote $\mathcal{P}_{\mathcal{C}}$ as stated in Theorem 7. Thus, $\{\omega_n\}$ converges strongly to the only point $z \in \mathcal{C}$ that fulfills $\mathcal{P}_{\mathcal{C}} \circ \Phi(z) = z$.*

Proof. Given that $(\mathcal{B}, \mathcal{C}, \{\Gamma\}, \frac{1}{n})$ possesses Browder's property, it follows that for every $u \in \mathcal{C}$,

$$\omega_n = \left(1 - \frac{1}{n}\right) \Gamma(\omega_n) + \frac{1}{n} u \tag{10}$$

converges strongly to some points $z \in F(\Gamma)$. Then from [23, Theorem 3], we have the iteration,

$$\omega_{n+1} = (1 - \alpha_n)(\lambda\Gamma(\omega_n) + (1 - \lambda)\omega_n) + \alpha_n u$$

converges strongly to the z. Considering $G(\omega) = \lambda\Gamma(\omega) + (1 - \lambda)\omega$ for all $\omega \in \mathcal{C}$ and fix $\lambda \in (0, 1)$ in the iteration scheme (10), we get

$$\omega_{n+1} = (1 - \alpha_n)G(\omega_n) + \alpha_n u.$$

Then, it is not difficult to prove that G is a nonexpansive mapping and $F(G) = F(\Gamma)$. Now, following the proof of Theorem 8, we prove that iteration scheme (9) or

$$\omega_{n+1} = (1 - \alpha_n)G(\omega_n) + \alpha_n \Phi(\omega_n)$$

converges strongly to the $z \in F(\Gamma)$. □

Remark 3. If $\mathcal{B}$ is a Hilbert space, $\mathcal{C}$ is bounded, $\{\mathcal{S}_n\}$ is a constant sequence, then $(\mathcal{B}, \mathcal{C}, \{\mathcal{S}_n\}, \{1/n\})$ has both Browder's and Halpern's property.

We obtain the following as a direct consequence of Theorem 11.

Corollary 2. *Let $\mathcal{C}$ be a closed, convex set that is bounded within a Hilbert space $\mathcal{B}$. Let $\Gamma, \Phi, \alpha_n, \lambda$, and $\{\omega_n\}$ be defined as in Theorem 11. Thus, the sequence $\{\omega_n\}$ converges strongly to the distinct fixed point $z \in \mathcal{C}$ that fulfills $\mathcal{P}_{\mathcal{C}} \circ \Phi(z) = z$, with $\mathcal{P}_{\mathcal{C}}$ being the metric projection from $\mathcal{C}$ onto $F(\Gamma)$.*

Subsequently, we will examine convergence theorems for collections of nonexpansive mappings. Although $\{\mathcal{S}_n\}$ represents a constant sequence in Theorem 11, it does not serve as a constant in the subsequent theorems.

Theorem 12. *Let $\mathcal{C}$ be a closed, convex set that is bounded in a Hilbert space $\mathcal{B}$. Let $\{\mathcal{S}_k : k \in \mathbb{N}\}$ denote an infinite collection of commuting nonexpansive mappings on $\mathcal{C}$, and let Φ represent a $\varphi-$contractive mapping on $\mathcal{C}$. Let $\{\alpha_n\}$ and $\{t_n\}$ be a sequence in $\left(0, \frac{1}{2}\right)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \frac{\alpha_n}{t_n^{\ell}} = 0$ for $\ell \in \mathbb{N}$. Define a sequence $\{\omega_n\}$ within $\mathcal{C}$ by*

$$\omega_n = (1 - \alpha_n)\left(\frac{1 - 2t_n}{1 - t_n}\mathcal{S}_1(\omega_n) + \sum_{k=1}^{\infty} t_n^k \mathcal{S}_{k+1}(\omega_n)\right) + \alpha_n \Phi(\omega_n) \tag{11}$$

for $n \in \mathbb{N}$. Thus, the sequence $\{\omega_n\}$ converges strongly to a unique fixed point $z \in \mathcal{C}$ that fulfills $\mathcal{P}_{\mathcal{C}} \circ \Phi(z) = z$, where $\mathcal{P}_{\mathcal{C}}$ denotes the metric projection from $\mathcal{C}$ onto $\bigcap_{n=1}^{\infty} F(\mathcal{S}_n)$.

Proof. We define a family of mappings $\Gamma_n : \mathcal{C} \to \mathcal{B}$ such as

$$\Gamma_n(\omega) = \left(1 - \sum_{k=1}^{n} t_n^k\right) \mathcal{S}_1(\omega) + \sum_{k=1}^{\infty} t_n^k \mathcal{S}_{k+1}(\omega).$$

Then, one can easily verify that for any $\omega, \upsilon \in \mathcal{C}$, the mappings Γ_n satisfies

$$\begin{aligned}\|\Gamma_n(\omega) - \Gamma_n(\upsilon)\| &\leq \left(\frac{1-2t_n}{1-t_n}\right) \|\mathcal{S}_1(\omega) - \mathcal{S}_1(\upsilon)\| + \sum_{k=1}^{\infty} t_n^k \|\mathcal{S}_{k+1}(\omega) - \mathcal{S}_{k+1}(\upsilon)\| \\ &\leq \left(\frac{1-2t_n}{1-t_n}\right) \|\omega - \upsilon\| + \sum_{k=1}^{\infty} t_n^k \|\omega - \upsilon\| \\ &\leq \|\omega - \upsilon\|.\end{aligned}$$

Thus $\{\Gamma_n : n \in \mathbb{N}\}$ is a family of nonexpansive mappings and $F(\Gamma_n) = F(S_n)$. Then, we can rewrite the iteration scheme (11) as:

$$\omega_n = (1 - \alpha_n)\Gamma_n(\omega_n) + \alpha_n \Phi(\omega_n),$$

and from [21, Theorem 3.2], we obtain $F(\Gamma_n) \neq \emptyset$. From [21, Theorem 5.1], we can see that $(\mathcal{B}, \mathcal{C}, \{\Gamma_n\}, \{\alpha_n\})$ has Browder's property which follows from Theorem 9 that the iteration scheme (11) converges strongly to a unique fixed point $z \in \mathcal{C}$ satisfyiny $\mathcal{P}_{\mathcal{C}} \circ \Phi(z) = z$. $\square$

Now, we examine convergence theorems related to families of nonexpansive mappings. Let $\mathcal{C}$ be a nonempty closed convex subset in a Banach space $\mathcal{B}$. A *one-parameter nonexpansive semigroup* refers to a collection $\mathcal{F} = \{\Gamma(t) : t > 0\}$ of self-mappings of $\mathcal{C}$ with the property that:

(i) $\Gamma(0)(\omega) = \omega$ for all $\omega \in \mathcal{C}$;
(ii) $\Gamma(t + s)(\omega) = \Gamma(t) \circ \Gamma(s)(\omega)$ for all $t, s > 0$ and $\omega \in \mathcal{C}$;
(iii) $\lim_{t\to 0} \Gamma(t)(\omega) = \omega$ for all $\omega \in \mathcal{C}$;
(iv) For each $t > 0$, $\Gamma(t)$ is nonexpansive, that is,

$$\|\Gamma(t)(\omega) - \Gamma(t)(\upsilon)\| \leq \|\omega - \upsilon\|, \quad \text{for all } \omega, \upsilon \in \mathcal{C}.$$

We define $F(\mathcal{F}) := \{\omega \in \mathcal{C} : \Gamma(t)(\omega) = \omega \text{ for } t > 0\} = \bigcap_{t>0} F(\Gamma(t))$.

A continuous operator semigroup $\mathcal{F}$ is termed *uniformly asymptotically regular* (u.a.r.) [7, 19] on $\mathcal{K}$ if for every $h \geq 0$ and for any bounded subset $\mathcal{C}$ of $\mathcal{K}$,

$$\lim_{t\to\infty} \sup_{\omega\in\mathcal{C}} \|\Gamma(h) \circ \Gamma(t)(\omega) - \Gamma(t)(\omega)\| = 0.$$

Let $\{t_n\}$ be a sequence of positive real numbers that diverges to ∞. For every $t > 0$ and $\omega \in \mathcal{C}$, let us define the average operator as follows:

$$\mathcal{S}(t)(\omega) = \frac{1}{t} \int_0^t \Gamma(s)(\omega)\, ds.$$

In 2007, Chen and Song [7] demonstrated that $(\mathcal{B}, \mathcal{C}, \{\mathcal{S}(t_n)\}, \{\alpha_n\})$ possess both Browder's and Halpern's property in a uniformly convex Banach space when the norm is uniformly Gâteaux differentiable, provided that $\|t_n\| \to \infty$ as $n \to \infty$. We derive the following outcome as a direct result of Theorems 9 and 10.

Theorem 13. *Let $\mathcal{B}$ denote a uniformly convex Banach space possessing a uniformly Gâteaux differentiable norm, $\mathcal{C}$ a nonempty closed convex subset of $\mathcal{B}$, and $\{\Gamma(t)\}$ a u.a.r nonexpansive semigroup from $\mathcal{C}$ into $\mathcal{C}$ such that $F(\mathcal{F}) \neq \emptyset$, and $\Phi : \mathcal{C} \to \mathcal{C}$ is a φ-contractive mapping. Assume that the collection $\{\Gamma(t)\}$ forms a nonexpansive semigroup from $\mathcal{C}$ to itself with the property that $F(\mathcal{F}) \neq \emptyset$. Let $\{\upsilon_n\}$ and $\{\omega_n\}$ be defined by*

$$\upsilon_n = \beta_n \Phi(\upsilon_n) + (1 - \beta_n)\mathcal{S}(t_n)(\upsilon_n), \quad n \in \mathbb{N},$$

$$\omega_{n+1} = \alpha_n \Phi(\omega_n) + (1 - \alpha_n)\mathcal{S}(t_n)(\omega_n), \quad n \in \mathbb{N},$$

where $t_n \to \infty$, $\beta_n \in (0, 1)$ meets condition (C1), *and $\alpha_n \in (0, 1)$ fulfills conditions* (C1) *and* (C2). *Subsequently, both $\{\upsilon_n\}$ and $\{\omega_n\}$ converge strongly to $z = \mathcal{P}_{\mathcal{C}} \circ \Phi(z)$.*

In 2008, Song and Xu [19] demonstrated that $(\mathcal{B}, \mathcal{C}, \{\Gamma(t_n)\}, \{\alpha_n\})$ possess both Browder's and Halpern's properties in a strictly convex reflexive Banach space with a uniformly Gâteaux differentiable norm, provided that $\|t_n\| \to \infty$ as $n \to \infty$. We also have the following outcome.

Theorem 14. *Let $\mathcal{B}$ denote a strictly convex reflexive Banach space that has a norm which is uniformly Gâteaux differentiable, and let $\mathcal{C}$ and Φ be defined as stated in Theorem 14. Assume that $t_n \to \infty$, $\beta_n \in (0, 1)$ meets condition* (C1), *and $\alpha_n \in (0, 1)$ fulfills conditions* (C1) *and* (C2). *If the sequences $\{\upsilon_n\}$ and $\{\omega_n\}$ are defined by*

$$\upsilon_n = \beta_n \Phi(\upsilon_n) + (1 - \beta_n)\Gamma(t_n)(\upsilon_n), \quad n \in \mathbb{N},$$

$$\omega_{n+1} = \alpha_n \Phi(\omega_n) + (1 - \alpha_n)\Gamma(t_n)(\omega_n), \quad n \geq 1,$$

then as $n \to \infty$, both $\{\upsilon_n\}$ and $\{\omega_n\}$ strongly converge to $z = \mathcal{P}_{\mathcal{C}} \circ \Phi(z)$.

6. Application to split feasibility problems

The Split Feasibility Problem (SFP) is an important class of feasibility problems that arises in various fields such as signal processing, image reconstruction, inverse problems, and optimization. Censor and Elfving [6] introduced SFP in 1994, which is useful when the constraints are split between two (or more) spaces and connected by a linear operator. To begin with, let us recall that the *split feasibility problem (SFP)* is to find a point.

$$\omega \in \mathcal{C} \quad \text{such that} \quad \mathcal{A}(\omega) \in \mathcal{Q}, \tag{12}$$

where $\mathcal{C}$ is a closed convex subset of a Hilbert space $\mathcal{H}_1$, $\mathcal{Q}$ is a closed convex subset of a Hilbert space $\mathcal{H}_2$, and $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. Iterative methods can be used to solve the SFP (12).

Byrne [4] was, among others, the first to propose the so-called CQ algorithm, which generates a sequence $\{\omega_n\}$ by the recursive procedure,

$$\omega_{n+1} = \mathcal{P}_{\mathcal{C}}\left(\mathcal{I} - \gamma\mathcal{A}^*(\mathcal{I} - \mathcal{P}_{\mathcal{Q}})\mathcal{A}\right)(\omega_n), \quad n \in \mathbb{N}, \tag{13}$$

where $\mathcal{P}_{\mathcal{C}}$ and $\mathcal{P}_{\mathcal{Q}}$ represent the (orthogonal) projections onto $\mathcal{C}$ and $\mathcal{Q}$, respectively, $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$, and $\gamma \in \left(0, \frac{2}{\lambda}\right)$, with λ being the spectral radius of the operator $\mathcal{A}^*\mathcal{A}$. Subsequently, Byrne [5] applied the Krasnosel'skii-Mann (KM) iteration to the CQ algorithm, and Zhao and Yang [26] applied the KM iteration to the perturbed CQ algorithm to solve the SFP. It is well known that the CQ algorithm and the KM iteration for the SFP do not necessarily converge strongly in infinite-dimensional Hilbert spaces.

In what follows, we investigate the strong convergence behaviour of the operator $\Gamma = \mathcal{P}_{\mathcal{C}}(\mathcal{I} - \gamma\mathcal{A}^*(\mathcal{I} - \mathcal{P}_{\mathcal{Q}})\mathcal{A})$, in infinite dimensional space, for the KM-algorithm with a ϕ-contractive mapping as defined below:

Let $\omega_0 \in \mathcal{H}_1$ be arbitrary and define

$$\omega_{n+1} = (1 - \alpha_n)\Phi(\omega_n) + \alpha_n\Gamma(\omega_n), \quad n \in \mathbb{N}, \tag{14}$$

where $\Gamma := \mathcal{P}_{\mathcal{C}}\left(\mathcal{I} - \gamma\mathcal{A}^*(\mathcal{I} - \mathcal{P}_{\mathcal{Q}})\mathcal{A}\right)$, $\alpha_n \in (0,1)$, $\gamma \in \left(0, \dfrac{1}{\lambda}\right)$, with λ defined as the spectral radius of the operator $\mathcal{A}^*\mathcal{A}$, and Φ represents a φ-contractive mapping.

Note that a SFP is consistent if it has at least one solution.

Theorem 15. *Assume that the SFP* (14) *is consistent and that α_n meets* (C1), (C2), *and* (C3). *Thus, $\{\omega_n\}$ strongly converges to a solution of SFP* (14) *or to the solution of a variational inequality problem:*

$$\langle(\mathcal{I} - \Gamma)\omega^*, v - \omega^*\rangle \geq 0 \quad \text{for all } v \in F(\Gamma). \tag{15}$$

Proof. Since the SFP (14) is consistent, it follows that $F(\Gamma) \neq \emptyset$. Moreover, verifying that the projection mapping $\Gamma : \mathcal{C} \to \mathcal{C}$ is nonexpansive is not difficult. Then, by Corollary 1, the quadruple $(\mathcal{H}_1, \mathcal{C}, \{\Gamma\}, \mathcal{P}_{\mathcal{C}})$ possesses Halpern's property. Following the proof of Theorem 8, we conclude that the sequence $\{\omega_n\}$ converges strongly to a solution of the variational inequality problem (15). □

7. Conflict of Interest

The authors declare that they have no conflict of interest.

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DEEPAK KHANTWAL
DEPARTMENT OF MATHEMATICS
AND APPLIED MATHEMATICS
UNIVERSITY OF JOHANNESBURG
KINGSWAY CAMPUS
AUCKLAND PARK 2006
SOUTH AFRICA
E-mail address: deepakkhantwal15@gmail.com
dkhantwal@uj.ac.za

J. L. MAMABOLO
DEPARTMENT OF MATHEMATICS
AND APPLIED MATHEMATICS
UNIVERSITY OF JOHANNESBURG
KINGSWAY CAMPUS
AUCKLAND PARK 2006
SOUTH AFRICA
E-mail address: mamabolojack19910727@gmail.com

RAJENDRA PANT
DEPARTMENT OF MATHEMATICS
AND APPLIED MATHEMATICS
UNIVERSITY OF JOHANNESBURG
KINGSWAY CAMPUS
AUCKLAND PARK 2006
SOUTH AFRICA

NATIONAL INSTITUTE FOR THEORETICAL
AND COMPUTATIONAL SCIENCES (NITHECS)
SOUTH AFRICA
E-mail address: pant.rajendra@gmail.com
rpant@uj.ac.za