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# On the regularized trace formula for a Sturm–Liouville boundary value problem with two delays and two potentials

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Abstract. This paper is devoted to a Sturm–Liouville boundary value problem with two constant delays $\tau_1, \tau_2 \in [\pi/3, \pi)$ and two potentials $q_1, q_2 \in L_2[0, \pi]$. We derive an asymptotic expansion of the characteristic function and obtain asymptotic formulas for the eigenvalues. Using the Lidskii–Sadovnichii contour integration method, we establish an explicit formula for the first regularized trace of the corresponding operator.

## 1. Introduction

Trace and regularized trace formulas for differential operators constitute an important part of spectral theory and have been studied extensively (see: [1, 9, 15]). For classical Sturm–Liouville operators, trace formulas establish relations between the spectral characteristics of an operator and the coefficients of the corresponding differential equation, together with the boundary conditions. Such formulas provide valuable information on the distribution of eigenvalues and play an important role in inverse spectral problems (see: [4]). In recent years considerable attention has been devoted to differential operators with retarded arguments (see:[2, 3, 14, 16–20, 22]). Such operators naturally arise in various applications: control theory, economics, population dynamics, and related fields. The presence of delays significantly complicates the spectral analysis. In particular, the structure of the characteristic function and the asymptotic behavior of the eigenvalues differ essentially from those in the classical Sturm–Liouville case. Two main approaches are commonly used for deriving regularized trace formulas: Levitan's method and the contour integration method. Levitan's method is based on Hadamard's factorization theorem and has been applied in [5, 6, 10, 11]. The contour integration method is related to the argument principle and to the study

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of remainders in complex analysis (see: [12, 21]). In [7], a general method for calculating regularized sums of zeros of entire functions of class $K$ was developed. This method made it possible to obtain regularized trace formulas for a wide class of problems generated by ordinary differential equations on a finite interval. The application of this method, known as the Lidskii–Sadovnichii method, to differential equations with a delay was first presented in [13].

In this paper, we determine the first regularized trace of a Sturm–Liouville boundary value problem with two constant delays and two potentials by using the Lidskii–Sadovnichii method. Namely, we study the equation

$$-y''(x)+\sum_{k=1}^{2} q_k(x)y(x-\tau_k)=\lambda y(x), \quad x\in[0,\pi], \tag{1}$$

together with the boundary conditions

$$y'(0)-hy(0)=0, \tag{2}$$

and

$$y'(\pi)+Hy(\pi)=0, \tag{3}$$

where $\frac{\pi}{3}\le\tau_2<\frac{\pi}{2}<2\tau_2\le\tau_1<\pi$, $h,H\in R\setminus\{0\}$ and $\lambda=z^2$ is a spectral parametar. We assume that $q_1(x), q_2(x)$ are real-valued potential functions from $L_2[0,\pi]$, and such that $q_k(x)=0$, $x\in[0,\tau_k)$, $k=1,2$.

We first derive an asymptotic formula for the corresponding characteristic function $F_{\tau_1,\tau_2}$. On this basis, we obtain asymptotic representation formulas for the eigenvalues and establish the first regularized trace formula for problem (1)–(3).

## 2. Preliminaries

**Lemma 1.** *Let $y(x,z)$ be the solution of the differential equation*

$$-y''(x)+\sum_{k=1}^{2} q_k(x)y(x-\tau_k)=z^2 y(x),\ x\in[0,\pi], \tag{4}$$

*which satisfies the initial conditions $y(0,z)=1$, $y'(0,z)=h$ and for the functions $q_k(x)$, $k=1,2$, it holds $q_k(x)=0$, $x\in[0,\tau_k)$. Then $y(x,z)$ satisfies the integral equation*

$$y(x,z)=\cos zx+\frac{h}{z}\sin zx+\frac{1}{z}\sum_{k=1}^{2}\int_{\tau_k}^{x} q_k(t)\sin z(x-t)y(t-\tau_k,z)dt. \tag{5}$$

*Proof.* Equation (4) can be written in the form

$$-y''(x,z)+z^2y(x,z)=\sum_{k=1}^{2} q_k(x)y(x-\tau_k,z).$$

General solution of the homogeneous equation $-y''(x,z)+z^2y(x,z)=0$ with initial conditions $y(0,z)=1$, $y'(0,z)=h$ is

$$y_0(x,z)=\cos zx+\frac{h}{z}\sin zx.$$

By applying the method of variation of constants it is obtained

$$y(x,z)=\cos zx+\frac{h}{z}\sin zx+\frac{1}{z}\sum_{k=1}^{2}\int_0^x q_k(t)\sin z(x-t)y(t-\tau_k,z)dt. \tag{6}$$

Based on the assumption, $q_k(t)=0$ for $t<\tau_k$, equation (6) reduces to the integral equation (5), which should have been proven. □

By the method of steps (see: [10,13,14]), it can be easily verified that the solution of the integral equation (5) on the interval $(\tau_1,\pi]$ is

$$\begin{aligned} y(x,z)=\cos zx+\frac{h}{z}\sin zx+\frac{1}{z}\sum_{k=1}^{2}b_{sc}^{(k)}(x,z)+\frac{h}{z^2}\sum_{k=1}^{2}b_{s^2}^{(k)}(x,z)+\\ +\frac{1}{z^2}b_{s^2c}^{(2)}(x,z)+\frac{h}{z^3}b_{s^3}^{(2)}(x,z), \end{aligned} \tag{7}$$

where

$$\begin{aligned} b_{sc}^{(k)}(x,z)&=\int_{\tau_k}^{x} q_k(t)\sin z(x-t)\cos z(t-\tau_k)dt,\\ b_{s^2}^{(k)}(x,z)&=\int_{\tau_k}^{x} q_k(t)\sin z(x-t)\sin z(t-\tau_k)dt,\\ b_{s^3}^{(2)}(x,z)&=\int_{2\tau_2}^{x} q_2(t)\sin z(x-t)b_{s^2}^{(2)}(t-\tau_2,z)dt,\\ b_{s^2c}^{(2)}(x,z)&=\int_{2\tau_2}^{x} q_2(t)\sin z(x-t)b_{sc}^{(2)}(t-\tau_2,z)dt. \end{aligned}$$

Let denote

$$F_{\tau_1,\tau_2}(\pi,z)=y'(\pi,z)+Hy(\pi,z).$$

For $x=\pi$ from (7) we get the characteristic function

$$
\begin{aligned}
F_{\tau_1,\tau_2}(z) = &\left(-z+\frac{hH}{z}\right)\sin\pi z + (h+H)\cos\pi z + \sum_{k=1}^{2} b_{c^2}^{(k)}(z) + \\
(8)\qquad &+\frac{h}{z}\sum_{k=1}^{2} b_{cs}^{(k)}(z) + \frac{H}{z}\sum_{k=1}^{2} b_{sc}^{(k)}(z) + \frac{Hh}{z^2}\sum_{k=1}^{2} b_{s^2}^{(k)}(z) + \\
&+\frac{1}{z} b_{csc}^{(2)}(z) + \frac{h}{z^2} b_{cs^2}^{(2)}(z) + \frac{H}{z^2} b_{s^2c}^{(2)}(z) + \frac{Hh}{z^3} b_{s^3}^{(2)}(z),
\end{aligned}
$$

where

$$
\begin{aligned}
b_{cs}^{(k)}(z) &= \int_{\tau_k}^{\pi} q_k(t)\cos z(\pi-t)\sin z(t-\tau_k)dt, \\
b_{c^2}^{(k)}(z) &= \int_{\tau_k}^{\pi} q_k(t)\cos z(\pi-t)\cos z(t-\tau_k)dt, \\
(9)\qquad b_{s^2c}^{(2)}(z) &= \int_{2\tau_2}^{\pi} q_2(t)\sin z(\pi-t) b_{sc}^{(2)}(t-\tau_2,z)dt, \\
b_{csc}^{(2)}(z) &= \int_{2\tau_2}^{\pi} q_2(t)\cos z(\pi-t) b_{sc}^{(2)}(t-\tau_2,z)dt, \\
b_{cs^2}^{(2)}(z) &= \int_{2\tau_2}^{\pi} q_2(t)\cos z(\pi-t) b_{s^2}^{(2)}(t-\tau_2,z)dt.
\end{aligned}
$$

Here, to simplify the above-given equations for $F_{\tau_1,\tau_2}(z)$, we write $(z)$ as the argument of the functions instead of $(\pi,z)$. Applying the trigonometric identities in expressions (9) leads to the asymptotic form of the characteristic function $F_{\tau_1,\tau_2}(z)$ given in expression (10) for $|z|\to\infty$

$$
\begin{aligned}
F_{\tau_1,\tau_2}(z) = &\left(-z+\frac{hH}{z}\right)\sin\pi z + (h+H)\cos\pi z + \frac{1}{2}\sum_{k=1}^{2}\cos z(\pi-\tau_k)\int_{\tau_k}^{\pi} q_k(t)dt \\
(10)\qquad &+\frac{H+h}{2z}\sum_{k=1}^{2}\sin z(\pi-\tau_k)\int_{\tau_k}^{\pi} q_k(t)dt + \frac{1}{2}\sum_{k=1}^{2}\int_{\tau_k}^{\pi} q_k(t)\cos z(\pi-2t+\tau_k)dt \\
&+\frac{1}{4z}\sin z(\pi-2\tau_2)\int_{2\tau_2}^{\pi} q_2(t)\int_{\tau_2}^{t-\tau_2} q_2(t_1)dt_1dt + O\left(\frac{1}{|z|^2}\right).
\end{aligned}
$$

When we apply partial integration to integral $\int\limits_{\tau_k}^{\pi} q_k(t)\cos z(\pi-2t+\tau_k)dt$ and introducing notations

$$J_1^k=\int\limits_{\tau_k}^{\pi} q_k(t),\quad k=1,2;\qquad J_2^{(2)}=\int\limits_{2\tau_2}^{\pi} q_2(t)\int\limits_{\tau_2}^{t-\tau_2} q_2(t_1)dt_1dt,$$

the characteristic function given in (10) can be written in the following form

$$\begin{aligned}
F_{\tau_1,\tau_2}(z)&=\left(-z+\frac{hH}{z}\right)\sin\pi z+(h+H)\cos\pi z\\
&+\frac{1}{2}\sum_{k=1}^{2}J_1^{(k)}\cos z(\pi-\tau_k)+\frac{J_2^{(2)}}{4z}\sin z(\pi-2\tau_2)\\
&+\frac{H+h}{2z}\sum_{k=1}^{2}J_1^{(k)}\sin z(\pi-\tau_k)\\
&+\frac{1}{4z}\sum_{k=1}^{2}\Big(q_k(\pi)+q_k(\tau_k)\Big)\sin z(\pi-\tau_k)+O\left(\frac{1}{|z|^2}\right).
\end{aligned}\tag{11}$$

In order to be able to apply the Lidskii-Sadovnichii method, it is necessary to show that the function $F_{\tau_1,\tau_2}(z)$ is a function of class $K$.

**Definition 1** ([7])**.** Let $f(z)$ be an entire function which for every integer $h$ can be written in the form

$$f(z)=\sum_{k=0}^{N-1}e^{\alpha_k z}P_{k,h}(z),$$

where $\alpha_k$ are complex constants and

$$P_{k,h}(z)=z^{n_k}\sum_{\nu=0}^{h}\beta_\nu^{(k)}z^{-\nu}+o(z^{n_k-h}),\quad z\to\infty,$$

$n_k$ are integers and $\beta_0^{(k)}\neq 0,$ then the functions $f(z)$ are called functions of class $K$. The numbers $\alpha_k$ and $\beta_\nu^{(k)}$ are called parameters of the asymptotics of the function $f(z)$.

**Lemma 2.** *The characteristic function $F_{\tau_1,\tau_2}(z)$ of the considered Sturm–Liouville problem* (1)–(3) *belongs to the class $K$, and we can write it in the form*

$$\begin{aligned}
F_{\tau_1,\tau_2}(z)&=\sum_{k=0}^{2}e^{iz(\pi-k\tau_2)}P_k(z)+\sum_{k=1}^{2}e^{-iz(\pi-k\tau_2)}P_{k+3}(z)+e^{iz(\pi-\tau_1)}P_3(z)\\
&+e^{-iz(\pi-\tau_1)}P_6(z)+e^{-i\pi z}P_7(z)+P_8(z),\qquad |z|\to\infty,
\end{aligned}\tag{12}$$

*where*

$$(13)\quad P_0(z) = -\frac{z}{2i} + \frac{hH}{2iz} + \frac{h+H}{2},$$

$$(14)\quad P_1(z) = \frac{1}{4}J_1^{(2)} + \frac{1}{2i}\left[\frac{H+h}{2z}J_1^{(2)} + \frac{1}{4z}(q_2(\pi) + q_2(\tau_2))\right],$$

$$(15)\quad P_2(z) = \frac{J_2^{(2)}}{8iz},\ P_3(z) = \frac{1}{4}J_1^{(1)} + \frac{1}{2i}\left[\frac{H+h}{2z}J_1^{(1)} + \frac{1}{4z}(q_1(\pi) + q_1(\tau_1))\right],$$

$$(16)\quad P_4(z) = \frac{1}{4}J_1^{(2)} - \frac{1}{2i}\left[\frac{H+h}{2z}J_1^{(2)} + \frac{1}{4z}(q_2(\pi) + q_2(\tau_2))\right],$$

$$(17)\quad P_5(z) = -\frac{J_2^{(2)}}{8iz},$$

$$(18)\quad P_6(z) = \frac{1}{4}J_1^{(1)} - \frac{1}{2i}\left[\frac{H+h}{2z}J_1^{(1)} + \frac{1}{4z}(q_1(\pi) + q_1(\tau_1))\right],$$

$$(19)\quad P_7(z) = \frac{z}{2i} - \frac{hH}{2iz} + \frac{h+H}{2},$$

$$(20)\quad P_8(z) = 0.$$

*Proof.* As is known, trigonometric functions can be written in exponential form

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}, \qquad \cos \pi z = \frac{e^{i\pi z} + e^{-i\pi z}}{2},$$

$$\cos z(\pi - \tau_k) = \frac{e^{iz(\pi-\tau_k)} + e^{-iz(\pi-\tau_k)}}{2},$$

$$\sin z(\pi - k\tau_k) = \frac{e^{iz(\pi-k\tau_k)} - e^{-iz(\pi-k\tau_k)}}{2i}, \quad k = 1, 2.$$

Substituting these identities into the asymptotic formula of the characteristic function given in (11), we obtain

$$\begin{aligned} F_{\tau_1,\tau_2}(z) = &\left[-\frac{z}{2i} + \frac{hH}{2iz} + \frac{h+H}{2}\right]e^{i\pi z} + \left[\frac{z}{2i} - \frac{hH}{2iz} + \frac{h+H}{2}\right]e^{-i\pi z} \\ &+ \frac{J_2^{(2)}}{8iz}e^{iz(\pi-2\tau_2)} - \frac{J_2^{(2)}}{8iz}e^{-iz(\pi-2\tau_2)} + \frac{1}{4}\sum_{k=1}^{2} J_1^{(k)}e^{iz(\pi-\tau_k)} \\ &+ \frac{1}{2i}\sum_{k=1}^{2}\left[\frac{H+h}{2z}J_1^{(k)} + \frac{1}{4z}(q_k(\pi) + q_k(\tau_k))\right]e^{iz(\pi-\tau_k)} \qquad (21) \\ &- \frac{1}{2i}\sum_{k=1}^{2}\left[\frac{H+h}{2z}J_1^{(k)} + \frac{1}{4z}(q_k(\pi) + q_k(\tau_k))\right]e^{-iz(\pi-\tau_k)} \\ &+ \frac{1}{4}\sum_{k=1}^{2} J_1^{(k)}e^{-iz(\pi-\tau_k)} + O\left(\frac{1}{|z|^2}\right), \quad |z| \to \infty. \end{aligned}$$

When we insert the expressions given by (13)–(20) into equation (21), we obtain that the characteristic function can be written in the form (12). Thus, based on Definition 1, we see that the function $F_{\tau_1,\tau_2}(z)$ belongs to the class $K$. $\square$

## 3. Main results

In this section, we derive asymptotic formulas for the eigenvalues and establish the regularized trace formula for problem (1)–(3). It is known, see [9], that the spectrum of problem (1)–(3) is discrete and that $z_n \sim n + o(1)$, $n \to \infty$. Since the principal objective of this paper is to compute the first regularized trace of the Sturm–Liouville operator associated with problem (1)–(3), it is enough to obtain an asymptotic expansion of the zeros $z_n$ of the corresponding characteristic function in the form

$$z_n(\tau_k) = n + \frac{C_1(n,\tau_k)}{n} + \frac{C_2(n,\tau_k)}{n^2} + o\left(\frac{1}{n^2}\right), \quad k = 1,2;\ n \to \infty.$$

Using the analogous procedure shown in papers [10] and [11], we obtain that the zeros of the function $F_{\tau_1,\tau_2}(z)$ have an asymptotic shape

$$\begin{aligned} z_n(\tau_k) = n &+ \frac{c_{01} + \sum\limits_{k=1}^{2} c_{k1}\cos n\tau_k}{n} + \frac{\sum\limits_{k=1}^{2} c_{k2}\sin n\tau_k}{n^2} \\ &+ \frac{c_{42}\sin n(\tau_1+\tau_2) + c_{52}\sin n(\tau_1-\tau_2)}{n^2} \\ &+ \frac{c_{32}\sin 2n\tau_1 + c_{62}\sin 2n\tau_2}{n^2} + o\left(\frac{1}{n^2}\right), \end{aligned} \tag{22}$$

where

$$\begin{aligned} c_{01} &= \frac{h+H}{\pi}; \\ c_{k1} &= \frac{J_1^{(k)}}{2\pi}, \quad k = 1,2; \\ c_{k2} &= \frac{1}{2\pi^2}\left[-\tau_k J_1^{(k)}(h+H) - \frac{\pi}{2}(q_k(\pi) + q_k(\tau_k))\right], \quad k = 1,2; \\ c_{32} &= \frac{1}{8\pi^2}(J_1^{(1)})^2(\pi - \tau_1); \\ c_{42} &= \frac{1}{8\pi^2}J_1^{(1)}J_1^{(2)}(2\pi - (\tau_1+\tau_2)); \\ c_{52} &= -\frac{1}{8\pi^2}J_1^{(1)}J_1^{(2)}(\tau_1 - \tau_2); \\ c_{62} &= \frac{(J_1^{(2)})^2(\pi - \tau_2) - 2\pi J_2^{(2)}}{8\pi^2}. \end{aligned}$$

Since the eigenvalues $\lambda_n$ are given by $\lambda_n = z_n^2$, where $z_n$ are the zeros of the characteristic function, it follows from (22) that

$$\begin{aligned}\lambda_n(\tau_1,\tau_2) &= n^2 + 2c_{01} + 2\sum_{k=1}^{2} c_{k1}\cos n\tau_k + \frac{2}{n}\sum_{k=1}^{2} c_{k2}\sin n\tau_k \\ &\quad + \frac{2}{n}\left[c_{42}\sin n(\tau_1+\tau_2) + c_{52}\sin n(\tau_1-\tau_2)\right] \\ &\quad + \frac{2}{n}(c_{32}\sin 2n\tau_1 + c_{62}\sin 2n\tau_2) + o\left(\frac{1}{n}\right), \quad n\to\infty.\end{aligned} \tag{23}$$

The first regularized trace is defined as the sum obtained after subtracting from each eigenvalue the leading asymptotic terms responsible for the divergence of the usual eigenvalue series.

**Theorem 1.** *Let $\lambda_n(\tau_1,\tau_2)$, given in expression* (23), *denote the eigenvalues of problem* (1)–(3). *Then, the first regularized trace is given by*

$$S_1 = \sum_{n=1}^{\infty}\left\{\lambda_n(\tau_1,\tau_2) - n^2 - 2c_{01} - 2\sum_{k=1}^{2} c_{k1}\cos n\tau_k\right\} = \omega_3 - \Phi_2 \tag{24}$$

*where*

$$\omega_3 = \pi(h+H)^3 + h^2 + H^2, \quad \Phi_2 = -\frac{h+H}{\pi} - \frac{1}{2\pi}\sum_{k=1}^{2} J_1^{(k)}.$$

*Proof.* The coefficients $\omega_\nu$ can be determined by means of the following formula (see: [7, 8, 13])

$$\frac{F'(z,\tau)}{F(z,\tau)} \sim \sum_{\nu=0}^{\infty}\frac{\omega_\nu(\tau)}{z^\nu}, \quad z\to\infty. \tag{25}$$

Substituting $z = -iy$ into the asymptotic relation (12) and letting $y\to\infty$, we obtain

$$F_{\tau_1,\tau_2}(-iy) = \frac{e^{\pi y}}{2}y\left[1 + \frac{h+H}{y} + \frac{hH}{y^2} + O(e^{-\tau_2 y})\right].$$

It follows from (25) that

$$\frac{F'_{\tau_1,\tau_2}(-iy)}{F_{\tau_1,\tau_2}(-iy)} = \frac{\pi + \frac{\pi(h+H)+1}{y} + \frac{\pi hH}{y^2} - \frac{Hh}{y^3} + O(e^{-\tau_2 y})}{1 + \frac{h+H}{y} + \frac{hH}{y^2} + O(e^{-\tau_2 y})}. \tag{26}$$

Therefore, relation (26) implies

$$\frac{F'_{\tau_1,\tau_2}(-iy)}{F_{\tau_1,\tau_2}(-iy)} = \pi + \frac{1}{y} - \frac{h+H}{y^2} + \frac{\pi(h+H)^3 + h^2 + H^2}{y^3} + O(e^{-\tau_2 y}). \tag{27}$$

Using (27), we obtain the following coefficients:

$$\omega_0 = \pi, \quad \omega_1 = 1, \quad \omega_2 = -(h+H), \quad \omega_3 = \pi(h+H)^3 + h^2 + H^2.$$

The quantity $\Phi_2$ can be evaluated by means of the following formula ([13])

$$\Phi_2(-2,\tau_1,\tau_2) = \zeta(-2) + 2c_{01}\zeta(0) + 2c_{11}\zeta_{\cos}(0,\tau_1) + 2c_{21}\zeta_{\cos}(0,\tau_2),$$

where

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \qquad \zeta_{\cos}(z,\tau) = \sum_{n=1}^{\infty} \frac{\cos n\tau}{n^z}.$$

Accordingly

$$\zeta(-2) = 0, \qquad \zeta(0) = -\frac{1}{2}, \qquad \zeta_{\cos}(0,\tau_k) = -\frac{1}{2}, \quad k = 1, 2.$$

Substituting the previously obtained values into the formula for $\Phi_2$, we obtain

$$\Phi_2(-2,\tau_1,\tau_2) = -\frac{h+H}{\pi} - \frac{1}{2\pi}\sum_{k=1}^{2} J_1^{(k)}.$$

Consequently, the first regularized trace for problem (1)–(3) is given by

$$S_1 = \pi(h+H)^3 + h^2 + H^2 + \frac{h+H}{\pi} + \frac{1}{2\pi}\sum_{k=1}^{2} J_1^{(k)}.$$

This completes the proof. □

## 4. Conclusion

In this paper, we studied a Sturm–Liouville boundary value problem with two delayed arguments and two potential functions and derived its regularized trace formula by using the Lidskii–Sadovnichii method. The main contribution of the paper is the derivation of a regularized trace formula in which the influence of the boundary parameters, the delayed arguments, and the potential functions is explicitly represented. The obtained result generalizes the classical trace formulas for Sturm–Liouville operators and shows how delay terms modify the spectral structure of the problem. The proposed method provides a useful framework for further studies of spectral problems with delayed arguments, including higher regularized traces and inverse spectral problems for more general classes of differential operators.

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