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On one mixed problem for parabolic type equation with constant coefficients

Hikmat I. Ahmadov 0009-0003-5412-2628,
Diego Julio Cirilo-Lombardo* 0000-0002-3113-8390

Abstract. The problem of the parabolic equation with constant coefficients and time displacement is investigated under non-local and non-homogeneous boundary conditions. Under minimal conditions on the data, the uniqueness of the solution is demonstrated and an explicit analytical representation of it is obtained.

1. Introduction

Along the development of the differential equations, mixed problems for parabolic-type equations with different boundary conditions were studied by many authors (e.g. see refs [1–10]). Following the same line of research, mixed problems for partial differential equations, including parabolic ones, with deviating arguments that were studied in the works [11–15] and also problems in which functional conditions are set instead of boundary ones, were considered in works [11, 16, 17]. From the mathematical physics viewpoint, in work [18] mixed problems for the one-dimensional heat conduction equation with a partially determined boundary regime were studied, showing that, under certain conditions on the initial data, the existence of a solution of this problem can be proved, being the solution represented in the form of a contour integral. In papers [19, 20], mixed problems for the heat conduction equation with a more general type of time advance in the boundary conditions were considered. The existence and uniqueness of the solution were demonstrated, and the solutions were represented in the form of a boundary integral, as in the previous case. Also in [23], mixed problems for a parabolic equation with homogeneous boundary conditions were considered and the uniqueness of the solution to the considered mixed problems was

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*Corresponding author.

demonstrated. In the present work, unlike [23], mixed problems for a parabolic equation with non-homogeneous boundary conditions are investigated. In this paper we will look at a more general case: a mixed problem for a parabolic-type equation with constant coefficients where non-homogeneous and non-local boundary conditions are considered, with time advance in the boundary conditions.

2. Statement of the problem

Let us to consider a general differential operator of parabolic type:

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u\left(x,t\right)=au_{xx}\left(x,t\right)+bu_x(x,t)+cu\left(x,t\right)-u_t(x,t),$$

$$l_ju\left(x,t\right)\equiv u\left(x,\ t+\left(1-j\right)\omega\right)+\alpha_ju\left(1-x,\ t+j\omega\right),\quad j=0,1,$$

$$l_ju\left(x,t\right)=a_{j-2}u_x^{(j-2)}(x,t)+b_{j-2}u_x^{(j-2)}\left(1-x,t\right),\quad j=2,3,$$

where a, b, c, ω, α_j, a_j, b_j are real constants, $a>0$, $\omega>0$, $\alpha_0\alpha_1\neq 0$, $j=0,1$. We want to find a classical solution $u(x,t)$ of the equation

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u\left(x,t\right)=0, \tag{1}$$

is the semi-strip $\Pi=\{(x,t):0<x<1,t>0\}$ satisfying the initial condition

$$u\left(x,0\right)=\varphi\left(x\right),\quad 0<x<1, \tag{2}$$

and the boundary conditions

$$l_ju(x,t)\Big|_{x=0}=\psi_j(t),\quad j=0,\ t>0, \tag{3}$$

and

$$l_ju(x,t)\big|_{x=0}=\psi_j(t),\quad j=2,3,\ 0<t\leq\omega, \tag{4}$$

$\varphi(x)$, $\psi_k(t)$ $(k=0,1,2,3)$ are given functions.

The solution of the problem (1)–(4) mean that the function $u(x,t)$, satisfying the following conditions:

1) $u(x,t)\in C^{2,1}(\Pi)\cap C(0<x<1,\ t\geq 0)$;
$\int\limits_0^t u(x,\tau)d\tau\in C(0\leq x\leq 1,\ t\geq 0)$;
2) $l_ju(x,t)\in C(0\leq x\leq 1,\ t>0)$, $(j=0,1)$;
3) $l_ju(x,t)\in C(0\leq x<1,\ 0\leq t\leq\omega)$, $(j=2,\ 3)$;
4) $u(x,t)$ satisfies equalities (1)–(4) in a usual sense.

It is known that with the help of substitution

$$v(x,\ t) = v_0(t) + xv_1(t) \tag{5}$$

the boundary conditions (4) can be reduced to homogeneous boundary condition (e.g., see [5]). Indeed, by substituting (5) into (4) we easily get

$$v_0(t) = \frac{\psi_2(t)}{a_0+b_0} - \frac{b_0\psi_3(t)}{(a_0+b_0)(a_1+b_1)}, \tag{6}$$

$$v_1(t) = \frac{\psi_3(t)}{a_1+b_1}, \tag{7}$$

$$v(x,t) = \frac{\psi_2(t)}{a_0+b_0} - \frac{b_0\psi_3(t)}{(a_0+b_0)(a_1+b_1)} + \frac{x\psi_3(t)}{a_1+b_1}. \tag{8}$$

With the help of the substitution

$$u(x,t) = v(x,t) + w(x,t), \tag{9}$$

we obtain mixed problems of the following form

$$w_t = aw_{xx} + bw_x + cw + \phi(x,t), \quad 0 < x < 1,\ 0 < t < \omega, \tag{10}$$

$$w(x,0) = \widetilde{\varphi}(x), \quad 0 < x < 1, \tag{11}$$

$$l_j w\Big|_{x=0} = 0, \quad 0 < t \le \omega, \tag{12}$$

where

$$l_j w(x,t) = a_{j-2} w_x^{(j-2)}(x,t) + b_{j-2} w_x^{(j-2)}(1-x,t), \quad j = 2,3,$$

$$\begin{aligned} \phi(x,t) = {} & \frac{b\psi_3(t)}{a_1+b_1} + \frac{c\psi_2(t)}{a_0+b_0} - \frac{b_0 c\psi_3(t)}{(a_0+b_0)(a_1+b_1)} \\ & + \frac{cx\psi_3(t)}{a_1+b_1} - \frac{\psi_2'(t)}{a_0+b_0} + \frac{b_0\psi_3'(t)}{(a_0+b_0)(a_1+b_1)} - \frac{x\psi_3'(t)}{a_1+b_1} \end{aligned} \tag{13}$$

and

$$\widetilde{\varphi}(x) = \varphi(x) - \frac{\psi_2(0)}{a_0+b_0} + \frac{b_0\psi_3(0)}{(a_0+b_0)(a_1+b_1)} - \frac{x\psi_3(0)}{a_1+b_1}. \tag{14}$$

3. The uniqueness of the solution

To see the uniqueness of the solution, we proceed according to the general scheme of M.L. Rasulov's residue method [9]. From it, two problems with a complex parameter μ are compared with the mixed problem (10)–(12), namely:

A. Cauchy problem.

$$\frac{dz}{dt}-\mu^2 z=\phi(x,t),\quad 0<x<1,\ \ 0<t\le\omega, \tag{15}$$

$$z(x,0)=\widetilde{\varphi}(x),\quad 0<x<1. \tag{16}$$

B. Spectral problem.

$$L\left(\frac{d}{dx},\mu^2\right)y\,(x,\mu)=h(x,\,\mu),\quad 0<x<1, \tag{17}$$

$$l_j y(x,\mu)\Big|_{x=0}=0,\quad (j=2,3). \tag{18}$$

We denote the set of eigenvalues of the problem equations (17), (18) by $S=\{\mu_\nu,\ \nu=1,2,\dots\}$. It is known [21] that if $a_0b_1+a_1b_0\neq 0$, then for all complex values μ not belonging to the set S, there is a Green function $G_1(x,\xi,\mu)$ of the spectral problem (17), (18) (for null eigenvalues, e.g., $h(x,\mu)=0$), which is analytic for μ everywhere except the points of the set S, which are its poles and have the following asymptotic representation

$$\mu_\nu=\sqrt{a}\pi\nu i+\frac{(-1)^\nu(a_0a_1+b_0b_1)}{2(a_0b_1+a_1b_0)}+O\left(\frac{1}{\nu}\right),\quad \nu\to\infty.$$

If $(a_0^2-b_0^2)(a_1^2-b_1^2)=0$, then μ_ν, $\nu=1,2,\dots$, are simple or double poles. It's clear that $|\mu_\nu|\to\infty$ $(\nu\to\infty)$, there are $h>0$, $\delta>0$ such that

$$-h<\operatorname{Re}\mu_\nu<h,\quad |\mu_{\nu+1}-\mu_\nu|>2\delta,\quad \nu=1,2,\dots \tag{19}$$

For $|\mu-\mu_\nu|>\delta$ the inequality

$$\left|\frac{\partial^k G_1(x,\xi,\mu)}{\partial x^k}\right|\le c|\mu|^{k-1},\quad c>0,\ k=0,1,2, \tag{20}$$

holds for $f(x)\in C[0,1]$:

$$L\left(\frac{d}{dx},\mu^2\right)\int\limits_0^1 G_1(x,\xi,\mu)f(\xi)d\xi=-f(x),$$

(20$_1$) where $G_1(x,\xi,\mu)$ fulfills

$$l_jG_1(x,\xi,\mu)\Big|_{x=0}=0,\quad (j=2,3.$$

Consequently, for any function $f(x)$ from the domain of definition of the spectral problem (17), (18) (for $h(x,\mu)=0$), i.e., $f(x)\in C^2[0,1]$, $l_j f\big|_{x=0} = 0$, $j=2,3$, the equality

$$\int_0^1 G_1(x,\xi,\mu)f(\xi)d\xi = \frac{f(x)}{\mu^2} + \frac{1}{\mu^2}\int_0^1 G_1(x,\xi,\mu) \times \Big[af''(\xi)+bf'(\xi)+cf(\xi)\Big]d\xi \tag{21}$$

holds.

Now let's introduce some notations that we will use later on. Let $c>0$ and $r>0$ be some numbers, z is a complex argument, $L_c=\{z:\operatorname{Re} z^2 = c\}$ is a hyperbola with branches $L_c^{\pm} = \big\{z:\operatorname{Re} z^2 = c,\ \pm\operatorname{Re} z>0\big\}$, $\Omega_r = \{z:|z|=r\}$, $\Omega_r(\theta_1,\theta_2)$ is an arc of the circumference Ω_r bounded by the rays $z=\sigma e^{i\theta j}$ $(0\le\sigma<\infty,\ i=\sqrt{-1},\ j=1,2)$.

Note that in our notation the arcs $\{z:|z|=r,\ \operatorname{Re} z^2\ge c,\ \operatorname{Re} z>0\}$, $\{z:|z|=r,\ \operatorname{Re} z^2\le c,\ \operatorname{Im} z>0\}$, $\{z:|z|=r,\ \operatorname{Re} z^2\ge c,\ \operatorname{Re} z<0\}$ and $\{z:|z|=r,\ \operatorname{Re} z^2\le c,\ \operatorname{Im} z<0\}$, connecting the branches and the sides of the hyperbola L_c will be denoted by $\Omega_r\left(-\theta_{c,r},\theta_{c,r}\right)$, $\Omega_r\left(\theta_{c,r},-\theta_{c,r}+\pi\right)$, $\Omega_r\left(-\theta_{c,r}+\pi,\theta_{c,r}+\pi\right)$, $\Omega\left(\theta_{c,r}+\pi,-\theta_{c,r}+2\pi\right)$, respectively, where $\theta_{c,r} = \arctan\sqrt{\frac{r^2-c}{r^2+c}}$.

Let's also introduce the contours

$$\hat{L}_c = \hat{L}_c^{+}\cup\hat{L}_c^{-},$$

$$\begin{aligned}\hat{L}_c^{\pm} = &\left\{z:\ \ \pm z = \sigma\ e^{-\frac{3\pi}{8}i},\sigma\in\left(2c\sqrt{1+\sqrt{2}},\ \infty\right]\right\}\\ &\cup\left\{z:\pm z = c(1+i\eta),\ \eta\in\left[-1-\sqrt{2},\ 1+\sqrt{2}\right]\right\}\\ &\cup\left\{z:\ \pm z = \sigma e^{\frac{3\pi}{8}i},\sigma\in\left[2c\sqrt{1+\sqrt{2}},\ \infty\right]\right\}.\end{aligned}$$

We denote some contours as L_c, $L_c^{\pm}$, $\hat{L}_c$, $\hat{L}_c^{\pm}$, and the contained inside the circle Ω_r, respectively by $L_{c,r}$, $L_{c,r}^{\pm}$, $\hat{L}_{c,r}$, $\hat{L}_{c,r}^{\pm}$. Finally, we denote the closed contours $\Gamma_{c,r}$, $\Gamma_{c,r}^{\pm}$, $\widehat{\Gamma}_{c,r}$, $\widehat{\Gamma}_{c,r}^{\pm}$ for $r\ge 2c\sqrt{1+\frac{\sqrt{2}}{2}}$ by

$$\Gamma_{c,r} = \Omega_r\left(\theta_{c,r}+\pi,-\theta_{c,r}+2\pi\right)\cup L_{c,r}^{+}\cup\Omega_r\left(\theta_{c,r},-\theta_{c,r}+\pi\right)\cup L_{c,r}^{-},$$

$$\Gamma_{c,r}^{+} = L_{c,r}^{+}\cup\left(-\theta_{c,r},\theta_{c,r}\right),\widehat{\Gamma}_{c,r}^{+} = \hat{L}_{c,r}^{+}\cup\Omega_r\left(-\frac{3\pi}{8},\frac{3\pi}{8}\right),$$

$$\widehat{\Gamma}_{c,r} = \Omega_r\left(-\frac{5\pi}{8},-\frac{3\pi}{8}\right)\cup\widehat{\mathrm{L}}_{c,r}^{+}\cup\Omega_r\left(\frac{3\pi}{8},\frac{5\pi}{8}\right)\cup\hat{L}_{c,r}^{-}.$$

The positive direction on all these lines and the lines encountered later will be taken to be the counter-clockwise direction.

Let $\{r_n\}$ be a sequence of numbers such that

$$0 < r_1 < r_2 < \cdots < r_n < \cdots, \qquad \lim_{n\to\infty} r_n = \infty,$$

circumferences Ω_{r_n} don't intersect δ-neighborhoods (δ is a small enough, fixed) of the points $\mu_\nu \in S$. In view of the structure of S, the existence of such number δ and such sequence $\{r_n\}$ is beyond doubt. We denote the number of points μ_ν lying inside $\widehat{\Gamma}_{c,r_n}$ by m_n. From (21) it's clear that for any function $f(x) \in c^2[0,1]$, $l_j f\big|_{x=0} = 0$, $j = 2,3$ the following representation holds:

$$\frac{1}{2\pi i}\int\limits_{\widehat{\Gamma}_{h,r_n}} \mu d\mu \int\limits_0^1 G_1(x,\xi,\mu) f(\xi)\, d\xi = \frac{1}{2\pi i}\int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} f(x) d\mu +$$

$$+\frac{1}{2\pi i}\int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} d\mu \int\limits_0^1 G_1(x,\xi,\mu)\left[af"(\xi) + bf'(\xi) + cf(\xi)\right] d\xi,$$

where the following conditions are fulfilled

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} f(x) d\mu = \frac{f(x)}{2\pi i}\lim_{n\to\infty}\int\limits_{\Omega_{r_n}} \frac{d\mu}{\mu} =$$

$$= \frac{f(x)}{2\pi i}\lim_{n\to\infty}\int\limits_0^{2\pi}\frac{i r_n e^{i\varphi} d\varphi}{r_n e^{i\varphi}} = \frac{f(x)}{2\pi i}\cdot 2\pi i = f(x),$$

$$\lim_{n\to\infty}\int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} d\mu \int\limits_0^1 G_1(x,\xi,\mu) f(\xi) d\xi = \lim_{n\to\infty}\int\limits_{\Omega_{r_n}} \mu^{-1} d\mu \int\limits_0^1 G_1(x,\xi,\mu) f(\xi) d\xi,$$

$$\left|\int\limits_{\Omega_{r_n}} \mu^{-1} d\mu \int\limits_0^1 G_1(x,\xi,\mu) f(\xi) d\xi\right| \le \int\limits_0^{2\pi}\left|\frac{d\mu}{\mu}\right| \cdot \frac{c}{|\mu|}\cdot c_0 \le$$

$$\le c_1\int\limits_0^{2\pi}\left|\frac{d\mu}{\mu^2}\right| = c_1\int\limits_0^{2\pi}\left|\frac{i r_n e^{i\varphi} d\varphi}{r_n^2 e^{2i\varphi}}\right| = c_1\int\limits_0^{2\pi}\frac{1}{r_n} d\varphi \to 0, \quad n\to\infty.$$

We can prove in a similar way that

$$\lim_{n\to\infty}\int\limits_{\widehat{\Gamma}_{h,r_n}} \mu^{-1} d\mu \int\limits_0^1 G_1(x,\xi,\mu)\big(af''(\xi) + bf'(\xi)\big) d\xi = 0,$$

hence

$$f(x)=\lim_{n\to\infty}\frac{1}{2\pi i}\int\limits_{\widehat{\Gamma}_{h,r_n}}\mu d\mu\int\limits_0^1 G_1(x,\xi,\mu)f(\xi)d\xi=$$
$$=\sum_{\nu=1}^{\infty}\operatorname*{res}_{\mu\nu}\mu\int\limits_0^1 G_1(x,\xi,\mu)f(\xi)d\xi$$

is uniformly for $x\in[0,1]$.

The solution of the problem A has the following form:

$$z(x,t,\mu)=\widetilde{\varphi}(x)e^{\mu^{2t}}+\int\limits_0^t e^{\mu^2(t-\tau)}\phi(x,\tau)d\tau.$$

Using the scheme of the residue method [9] for the standard problem (without time shift) (10)–(12) we prove that if it has a solution then this solution is represented by the formula

$$w(x,t)=\sum_{\nu=1}^{\infty}\operatorname*{res}_{\mu_\nu}\mu\int\limits_0^1 G_1(x,\xi,\mu)z(\xi,t,\mu)d\xi.$$

By transforming and justifying this formula with regard to (9) for the problem (1), (2), (4) (in the rectangle $0<x<1$, $0<t\leq\omega$), the following theorem is proved.

Theorem 1. *Let* $a_0b_1+a_1b_0\neq 0$, $\varphi(x)\in c^2\,[0,1]$ *and* $l_j\varphi\big|_{x=0}=0$, $j=2,3$, $\psi_k(t)\in C^1[0,\omega]$, $k=2,3$, *and* $(a_0+b_0)(a_1+b_1)\neq 0$.

Then the problem (1), (2), (4) has a solution and this solution is represented by formula

$$\begin{aligned}u(x,t)={}&\frac{\psi_2(t)}{a_0+b_0}-\frac{b_0\psi_3\,(t)}{(a_0+b_0)\,(a_1+b_1)}+\frac{x\psi_3\,(t)}{a_1+b_1}+\sum_{\nu=1}^{\infty}\operatorname*{res}_{\mu_\nu}\mu e^{\mu^2 t}\times\\&\times\int\limits_0^1 G_1(x,\xi,\mu)\Bigg[\varphi(\xi)-\frac{\psi_2(0)}{a_0+b_0}+\frac{b_0\psi_3(0)}{(a_0+b_0)(a_1+b_1)}-\frac{\xi\psi_3(0)}{a_1+b_1}+\\&+\int\limits_0^t e^{-\mu^2\tau}\left(\frac{b\psi_3(\tau)}{a_1+b_1}+\frac{c\psi_2(\tau)}{a_0+b_0}-\frac{bc\psi_3(\tau)}{(a_0+b_0)(a_1+b_1)}+\right.\\&\left.+\frac{c\xi\psi_3(\tau)}{a_1+b_1}-\frac{\psi_2'(\tau)}{a_0+b_0}+\frac{b_0\psi_3'(\tau)}{(a_0+b_0)(a_1+b_1)}-\frac{\xi\psi_3'(\tau)}{a_1+b_1}\right)d\tau\Bigg]d\xi\end{aligned}\tag{22}$$

at $0\leq x\leq 1$, $0\leq t\leq\omega$.

Formula (22) allows us to draw a few more conclusions that will be needed later. Under the conditions of Theorem 1 from (22) taking (21) into account, we have

$$\sum_{\nu=1}^{\infty} \operatorname*{res}_{\mu_\nu} \mu e^{\mu^2 t} \int_0^1 G_1(x,\xi,\mu)\varphi(\xi)d\xi$$

$$= \lim_{n\to\infty} \frac{1}{2\pi i} \times \int_{\widehat{\Gamma}_{h,rn}} \mu e^{\mu^2 t} d\mu \int_0^1 G_1(x,\xi,\mu)\varphi(\xi)d\xi$$

$$= \varphi(x) + \lim_{n\to\infty} \frac{c}{2\pi i} \times \int_{\widehat{\Gamma}_{h,rn}} \mu^{-1} e^{\mu^2 t} d\mu \int_0^1 G_1(x,\xi,\mu)\varphi(\xi)d\xi$$

$$+ \lim_{n\to\infty} \frac{1}{2\pi i} \times \int_{\widehat{\Gamma}_{h,rn}} \mu^{-1} e^{\mu^2 t} \int_0^1 G_1(x,\xi,\mu)\big[a\varphi''(\xi) + b\varphi'(\xi)\big] d\xi d\mu.$$

From the estimate (20) and the fact that on the arcs

$$\Omega_{r_n}\left(-\tfrac{5\pi}{8} + j\pi, -\tfrac{3\pi}{8} + j\pi\right), \quad (j = 0,1;\ \operatorname{Re}\mu^2 \le -\tfrac{\sqrt{2}}{2}|\mu|^2)$$

it follows that

$$\lim_{n\to\infty} \int_{\Omega_{r_n}\left(-\frac{5\pi}{8}+j\pi, -\frac{3\pi}{8}+j\pi\right)} \mu^{-1} e^{\mu^2 t} d\mu \times$$

$$\times \int_0^1 G_1(x,\xi,\mu)\left[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)\right] d\xi = 0, \quad j = 0,1.$$

Therefore

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\widehat{\Gamma}_{h,rn}} \mu e^{\mu^2 t} d \int_0^1 G(x,\xi,\mu)\varphi(\xi)d\xi$$

$$= \varphi(x) + \frac{1}{2\pi i} \int_{L_h} \mu^{-1} e^{\mu^2 t} d\mu \int_0^1 G_1(x,\xi,\mu)\left[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)\right] d\xi.$$

Finally by using $G_1(x,\xi,\mu) = G_1(x,\xi,-\mu)$, we get

$$
\begin{aligned}
u(x,t) = {} & \frac{\psi_2(t)}{a_0+b_0} - \frac{b_0\psi_3(t)}{(a_0+b_0)(a_1+b_1)} + \frac{x\psi_3(t)}{a_1+b_1} + \varphi(x) - \\
& - \frac{\psi_2(0)}{a_0+b_0} + \frac{b_0\psi_3(0)}{(a_0+b_0)(a_1+b_1)} - \frac{x\psi_3(0)}{a_1+b_1} + \\
& + \frac{1}{\pi i}\int\limits_{\widehat{\mathrm{L}}_h^+} \mu^{-1} d\mu \int\limits_0^1 G_1(x,\xi,\mu)\Big[a\varphi'(\xi) + b\varphi'(\xi) + c\varphi(\xi)\Big] d\xi + \\
& + \sum_{\nu=1}^{\infty} \operatorname*{res}_{\mu_\nu} \mu e^{\mu^2 t} \int\limits_0^1 G_1(x,\xi,\mu)\Bigg[\int\limits_0^t e^{-\mu^2\tau}\Bigg(\frac{b\psi_3(\tau)}{a_1+b_1} + \\
& + \frac{c\psi_2(\tau)}{a_0+b_0} - \frac{bc\psi_3(\tau)}{(a_0+b_0)(a_1+b_1)} + \frac{c\xi\psi_3(\tau)}{a_1+b_1} - \\
& - \frac{\psi_2'(\tau)}{a_0+b_0} + \frac{b_0\psi_3'(\tau)}{(a_0+b_0)(a_1+b_1)} - \frac{\xi\psi_3'(\tau)}{a_1+b_1}\Bigg) d\tau\Bigg] d\xi,
\end{aligned}
\tag{23}
$$

for $0 \le x \le 1$, $0 \le t \le \omega$, where h is a number coming from (19). From this theorem the correctness of the next statement follows.

Theorem 2. *Under the conditions of Theorem 1 the problem* (1)–(4) *can't have more than one solution.*

Indeed, if the problem (1)–(4) had two solutions $u_1(x,t)$, $u_2(x,t)$, then their difference $\upsilon = u_1(x,t) - u_2(x,t)$ would be a solution of the homogeneous problem (1)–(4) with $\varphi(x) = 0$, $\psi_k(t) = 0$, $k = 0,1,2,3$, and moreover of the homogeneous problem (1), (2), (4) in $\{(x,t) : 0 \le x \le 1,\ 0 \le t \le \omega\}$. Then, because of (23), $\upsilon(x,t) \equiv 0$ at $0 \le x \le 1$, $0 \le t \le \omega$, and from conditions 2), it follows that $\upsilon(0,t) = \upsilon(1,t) = 0$ at $t \ge 0$.

Assuming the opposite, the uniqueness of the solution to the mixed problem (1)–(4) was shown. But is it possible to construct a function that is a solution to the corresponding homogeneous problem, and can the maximum principle be applied to determine the uniqueness of the solution? Because of this and the condition 1) it's easily seen that, considering the function

$$
w(x,t) = \int\limits_\omega^t \upsilon(x,\tau) d\tau,
$$

the solution of homogeneous problem $w_t = aw_{xx} + bw_x + cw$, $0 < x < 1$, $t \ge \omega$, $w(x,\omega) = 0$, $0 \le x \le 1$, with $w(0,t) = w(1,t) = 0$, $t > \omega$, it is continuous in $\{0 \le x \le 1,\ t \ge \omega\}$, where with regard to the maximum principle [22] we can conclude that $w(x,t) \equiv 0$, $0 \le x < 1$, $t \ge \omega$, hence $\upsilon(x,t) \equiv 0$, $0 \le x \le 1$, $t \ge 0$.

The following estimation is necessary for the function $u(x,t)$ expressed by formula (23) to satisfy the mixed problem (1), (2), (4).

Since on the distant parts $\left(|\lambda| > 2h\sqrt{1+\frac{\sqrt{2}}{2}}\right)$ of the contour $\widehat{L}_h^+$

$$\left|\frac{\partial^{k+m}}{\partial t^k \partial x^m}\mu^{-1}e^{\mu^2 t}\int_0^1 G_1(x,\xi,\mu)\times\Big[a\varphi''(\xi)+b\varphi'(\xi)+c\varphi(\xi)\Big]d\xi\right| \tag{23_1}$$
$$\leq C|\mu|^{2k+m-2}e^{-\frac{\sqrt{2}}{2}t|\lambda|^2}, \quad (2k+m\leq 2),$$

the operations $L\left(\frac{\partial}{\partial x}, L\frac{\partial}{\partial t}\right)$, $l_j u|_{x\to 0}$, $j=2,3$, at $0\leq x\leq 1$, $0<t\leq\omega$, can be moved under the integral sign (23) and then, with regard to (20_1) and the conditions of Theorem 1, we have

$$L\left(\frac{\partial}{\partial x},\ L\frac{\partial}{\partial t}\right)u(x,t) = -\big(a\varphi''(x)+b\varphi'(x)+c\varphi(x)\big)+$$
$$+\frac{a\varphi''(x)+b\varphi'(x)+c\varphi(x)}{\pi i}\int_{\widehat{L}_h^+}\mu^{-1}e^{\mu^2 t}d\mu+\sum_{\nu=1}^{\infty}\operatorname*{res}_{\mu_\nu}\mu e^{\mu^2 t}L\left(\frac{d}{dx},\mu^2\right)\times$$
$$\times\int_0^1 G_1(x,\xi,\mu)\left[\int_0^t e^{-\mu^2\tau}\left(\frac{b\psi_3(\tau)}{a_1+b_1}+\frac{c\psi_2(\tau)}{a_0+b_0}-\frac{bc\psi_3(\tau)}{(a_0+b_0)(a_1+b_1)}+\right.\right.$$
$$\left.\left.+\frac{c\xi\psi_3(\tau)}{a_1+b_1}-\frac{\psi_2'(\tau)}{a_0+b_0}+\frac{b_0\psi_3'(\tau)}{(a_0+b_0)(a_1+b_1)}-\frac{\xi\psi_3'(\tau)}{a_1+b_1}\right)d\tau\right]d\xi=$$
$$=-\big(a\varphi''(x)+b\varphi'(x)+c\varphi(x)\big)+a\varphi''(x)+b\varphi'(x)+c\varphi(x)+0=0,$$

with

$$l_2u(x,t)\Big|_{x\to 0}=l_2(a_0u(x,t)+b_0u(1-x,t))\Big|_{x\to 0}=\psi_2(t),$$
$$l_3u(x,t)\Big|_{x\to 0}=l_3(a_1u(x,t)+b_1u_x(1-x,t))\Big|_{x\to 0}=\psi_3(t).$$

Form the estimate (23_1) it's clear that the limit at $t\to 0$ can be also moved under the integral sign for all $x\in[0,1]$:

$$u(x,0)=\lim_{t\to 0}u(x,t)=\frac{\psi_2(0)}{a_0+b_0}-\frac{b_0\psi_3(0)}{(a_0+b_0)(a_1+b_1)}+\frac{x\psi_3(0)}{a_1+b_1}+$$
$$+\varphi(x)-\frac{\psi_2(0)}{a_0+b_0}+\frac{b_0\psi_3(0)}{(a_0+b_0)(a_1+b_1)}-\frac{x\psi_3(0)}{a_1+b_1}+$$
$$+\frac{1}{\pi i}\int_{\widehat{L}_h^+}\mu^{-1}d\mu\int_0^1 G_1(x,\xi,\lambda)\times\big[a\varphi''(\xi)+b\varphi'(\xi)+c\varphi(\xi)\big]d\xi+0=$$

$$= \varphi(x) + \frac{1}{\pi i} \lim_{r\to\infty} \int\limits_{\Omega_2(-\frac{3\pi}{8};\frac{3\pi}{8})} \mu^{-1} d\mu \times$$

$$\times \int\limits_0^1 G_1(x,\xi,\mu)\big[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)\big] d\xi = \varphi(x).$$

Thus the problem (1), (2), (4) in the domain $\{(x,t) : 0 < x < 1,\ 0 < t \le \omega\}$ has the unique solution which can be represented by formula (23). Further, form (23) we find

$$\begin{aligned} u(s,t) &= \frac{\psi_2(t)}{a_0+b_0} - \frac{b_0\psi_3(t)}{(a_0+b_0)(a_1+b_1)} + \frac{s\psi_3(t)}{a_1+b_1} + \varphi(s) - \\ &- \frac{\psi_2(0)}{a_0+b_0} + \frac{b_0\psi_3(0)}{(a_0+b_0)(a_1+b_1)} - \frac{s\psi_3(0)}{a_1+b_1} + \\ &+ \frac{1}{\pi i} \int\limits_{\widehat{L}_h^+} \mu^{-1} d\mu \times \int\limits_0^1 G_1(s,\xi,\mu)\Big[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)\Big] d\xi + \\ &+ \sum_{\nu=1}^{\infty} \operatorname*{res}_{\mu_\nu} \mu e^{\mu^2 t} \int\limits_0^1 G_1(s,\xi,\mu) \times \Bigg[\int\limits_0^t e^{-\mu^2\tau}\bigg(\frac{b\psi_3(\tau)}{a_1+b_1} + \\ &+ \frac{c\psi_2(\tau)}{a_0+b_0} - \frac{bc\psi_3(\tau)}{(a_0+b_0)(a_1+b_1)} + \frac{c\xi\psi_3(\tau)}{a_1+b_1} - \frac{\psi_2'(\tau)}{a_0+b_0} + \\ &+ \frac{b_0\psi_3'(\tau)}{(a_0+b_0)(a_1+b_1)} - \frac{\xi\psi_3'(\tau)}{a_1+b_1}\bigg) d\tau\Bigg] d\xi, \quad s = 0,1, \end{aligned} \tag{23_2}$$

and since the integral in (23_2) at $t \ge 0$ and integrals obtained from it by formal differentiation by t any number of times at $t \ge t_1 > 0$, where $t_1 > 0$ is arbitrary, converge uniformly.

4. Analysis of the existence of a solution

By applying the integral transform

$$A(f) = \int\limits_0^\infty e^{-\lambda^2 t} f(t) dt, \quad \text{(see [10])}$$

to the equation (1) and the boundary condition (3), we obtain the following spectral problem with complex parameter λ:

$$L\left(\tfrac{d}{dx},\ \lambda^2\right) z(x,\lambda) = -\varphi(x), \tag{24}$$

$$\begin{cases} e^{\lambda^2\omega} z(0,\lambda) + \alpha_0 z(1,\lambda) = A(\lambda), \\ z(0,\lambda) + \alpha_1 e^{\lambda^2\omega} z(1,\lambda) = B(\lambda), \end{cases} \tag{25}$$

where

$$L\left(\tfrac{d}{dx},\ \lambda^2\right) z(x,\lambda) = az''(x,\lambda) + bz'(x,\lambda) + (c-\lambda^2)z(x,\lambda),$$

$$z(x,\lambda) = \int\limits_0^\infty e^{-\lambda^2 t}u(x,t)dt,$$

$$A(\lambda) = \widetilde{\psi}_1(\lambda) + e^{\lambda^2\omega}\int\limits_0^\omega e^{-\lambda^2 t}u(0,t)dt,$$

$$B(\lambda) = \widetilde{\psi}_2(\lambda) + \alpha_1 e^{\lambda^2\omega}\int\limits_0^\omega e^{-\lambda^2 t}u(1,t)dt,$$

$$\widetilde{\psi}_k(\lambda) = \int_0^\infty e^{-\lambda^2 t}\psi_k(t)dt, \qquad k=0,1,$$

and $u(s,t)$, $s=0,1$, are boundary values of the solution on parts of the lateral boundary of the domain $\{(x;t): 0<x<1,\ t>0\}$ which can be defined from (23_2).

Considering (25) as the system of algebraic equations for $z(0,\lambda)$ and $z(1,\lambda)$ we can bring them to the following form:

$$z(0,\lambda) = m(\lambda), z(1,\lambda) = n(\lambda), \tag{26}$$

where

$$\begin{aligned} m(\lambda) = z_0(\lambda) &= \left[\alpha_1 e^{2\lambda^2\omega} - \alpha_0\right]^{-1}\Big(A(\lambda)\alpha_1 e^{2\lambda^2\omega} - \alpha_0 B(\lambda)\Big), \\ n(\lambda) = z_1(\lambda) &= \left[\alpha_1 e^{2\lambda^2\omega} - \alpha_0\right]^{-1}\Big(B(\lambda)e^{\lambda^2\omega} - A(\lambda)\Big). \end{aligned} \tag{27}$$

The solution of the problem (24)–(25) can be represented in the form of the sum of the solutions of the two problems:

$$\text{(A)} \qquad L\left(\frac{d}{dx},\lambda^2\right) z(x,\lambda) = 0, \qquad z(0,\lambda) = m(\lambda), \quad z(1,\lambda) = n(\lambda),$$

$$\text{(B)} \qquad L\left(\frac{d}{dx},\lambda^2\right) z(x,\lambda) = -\varphi(x), \quad z(0,\lambda) = 0, \qquad z(1,\lambda) = 0.$$

The solutions of the problem (A) is represented by formula

$$Q(x,\lambda,m,n) = \left[e^{-\left(\frac{b}{2a}+\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)} - e^{\left(\frac{b}{2a}-\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)}\right]^{-1}$$

$$\times\left[\left(m(\lambda)\, e^{-\left(\frac{b}{2a}+\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)} - n(\lambda)\right) e^{-\left(\frac{b}{2a}-\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)x} +\right.$$

$$+\left(n\left(\lambda\right)-m(\lambda)e^{-\left(\frac{b}{2a}-\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)}\right)e^{-\left(\frac{b}{2a}-\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)x}\Bigg],$$

where $m(\lambda)$ and $n(\lambda)$ are defined by formula (27).

If $m = z_0(\lambda)$, $n = z_1(\lambda)$, then the function $Q(x, \lambda, m, n)$, is analytic for λ everywhere except for the points

$$\lambda_\nu = \sqrt{a}\pi\nu i + O\Big(\frac{1}{\nu}\Big), \qquad \nu = 0, \pm 1, \pm 2, \dots$$

and points

$$\lambda_m^{\pm} = \pm\left[\frac{1}{2\omega}\ln\left|\frac{\alpha_0}{\alpha_1} + 2\pi m i\right|\right]^{1/2}, \qquad m = 0, \pm 1, \pm 2, \dots,$$

which are its poles.

It is obvious that in all points λ, where $Q(x, \lambda, m, n)$ exists, the following identities hold

$$\begin{aligned} L\left(\tfrac{d}{dx}, \lambda^2\right) Q(x, \lambda, m, n) &= 0, \\ Q(0, \lambda, m, n) &= m, \\ Q(1, \lambda, m, n) &= n. \end{aligned}$$

The solution of the problem (B) is constructed using its Green function. We denote the Green function of the problem B by $G_2(x, \xi, \lambda)$ and this function is analytic by λ everywhere except for the points $\lambda_\nu = \sqrt{a}\pi\nu i + O(\frac{1}{\nu})$, which are its simple poles.

Following [5,21] it's easy to prove that there exists such $\delta > 0$ that in the λ-plane outside the set $\bigcup_{\nu=1}^{\infty}\big\{\lambda : |\lambda - \lambda_\nu| < \delta\big\}$ fulfill

$$\left|\frac{\partial^k G_2(x, \xi, \lambda)}{\partial x^k}\right| \le C_0|\lambda|^{k-1}, \qquad C_0 > 0,\ k = 0, 1, 2, \tag{28}$$

for all $x, \xi \in [0, 1]$; at $\lambda \neq \lambda_\nu$, $\nu = 0, \pm 1, \pm 2, \dots$

$$\begin{aligned} L\left(\tfrac{d}{dx}, \lambda^2\right)\int_0^1 G_2(x, \xi, \lambda)\varphi(\xi)d\xi &= -\varphi(x), \\ G_2(0, \xi, \lambda) = G_2(1, \xi, \lambda) &= 0. \end{aligned} \tag{29}$$

The solution of the spectral problem (24), (25) is represented by the sum of two solutions (problem (A) and problem (B))

$$z(x, \lambda) = -\int_0^1 G_2(x, \xi, \lambda)F(\xi, \lambda)d\xi + Q\big(x, \lambda, m(\lambda), n(\lambda)\big). \tag{30}$$

For any function $\varphi(x)$ from the domain of definition of the operator of the spectral problem (B) the equality

$$\int_0^1 G_2t(x,\xi,\lambda)\varphi(\xi)d\xi = -\frac{\varphi(x)}{\lambda^2} + \frac{1}{\lambda^2}\int_0^1 G_2(x,\xi,\lambda)\times \tag{31}$$
$$\times\left[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)\right]d\xi + \frac{Q(x,\lambda,\varphi(0),\varphi(1))}{\lambda^2}$$

holds.

From (31) it is clear that the solution of the problem (B) can be represented in the following form:

$$z(x,\lambda) = \frac{\varphi(x)}{\lambda^2} - \frac{1}{\lambda^2}\int_0^1 G_2(x,\xi,\lambda)\left[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)\right]d\xi$$
$$- \frac{1}{\lambda^2}Q\big(x,\lambda,m(\lambda),n(\lambda)\big),$$

where the number $c_1 > \max\Big(0, \ln\left|\frac{\alpha_0}{\alpha_1}\right|\Big)$ is fixed.

Theorem 3. *Let $a_0b_1 + a_1b_0 \neq 0$, $\varphi(x) \in c^2[0,1]$, $l_j\varphi\big|_{x=0} = 0$ $(j = 0,1)$; $\psi_k(t) \in C^1[0,\infty)$, $k = 0,1$, and $(a_0+b_0)(a_1+b_1) \neq 0$.*

Then the problem (1)–(4) has the classical solution, which is represented by formula

$$u(x,t) = \varphi(x) - \frac{1}{\pi i}\int_{\hat{L}_{c_1}^+} \lambda^{-1}e^{\lambda^2 t}d\lambda \times \left[\int_0^1 G_2(x,\xi,\lambda)\Big(a\varphi''(\xi)\right.$$
$$\left. + b\varphi'(\xi) + c\varphi(\xi)\Big)d\xi - Q\big(x,\lambda,\varphi(0),\varphi(1)\big)\right] \tag{32}$$
$$+ \frac{1}{\pi i}\int_{L_{c_1}^+} \lambda e^{\lambda^2 t}Q\big(x,\lambda,m(\lambda),n(\lambda)\big)d\lambda.$$

It is clear that this solution is the sum of three integrals, which we denote by

$$u_1(x,t) = -\frac{1}{\pi i}\int_{\hat{L}_{c_1}^+} \lambda^{-1}e^{\lambda^2 t}d\lambda \int_0^1 G_2(x,\xi,\lambda)(a\varphi''(\xi) + b\varphi'(\xi) + c\varphi))d\xi,$$

$$u_2(x,t) = \frac{1}{\pi i}\int_{\hat{L}_{c_1}^+} \lambda^{-1}e^{\lambda^2 t}Q\big(x,\lambda,\varphi(0),\varphi(1)\big)d\lambda,$$

$$u_3(x,t)=\frac{1}{\pi i}\int\limits_{L_{c_1}^{+}}\lambda e^{\lambda^2 t}Q\big(x,\lambda,m(\lambda),n(\lambda)\big)d\lambda.$$

Proof. First we analyze $u_1(x,t)$:

$$u_1(x,t)=-\frac{1}{\pi i}\int\limits_{\hat{L}_{c_1}^{+}}\lambda^{-1}e^{\lambda^2 t}d\lambda\int\limits_0^1 G_2(x,\xi,\lambda)\big(a\varphi'(\xi)+b\varphi'(\xi)+c\varphi)\big)d\xi.$$

At $\operatorname{Re}\lambda\geq C_1$ for the Green function $G_2(x,\xi,\lambda)$ and its derivatives, the estimates of the form (28) hold on distant parts of the contour $\hat{L}_{c_1}^{+}$

$$\left|e^{\lambda^2 t}\right|=e^{t\operatorname{Re}\lambda^2}=e^{t|\lambda|^2\cos 2\arg\lambda}=e^{t|\lambda|^2\cos(\pm\frac{3\pi}{4})}=e^{-\frac{\sqrt{2}}{2}|\lambda|^2},\tag{33}$$

it is clear that

$$u_1(x,t)\in C^{2,1}(0\leq x\leq 1,\ t>0)\cap C(0\leq x\leq 1,\ t>0),$$

and hence, the differential expression $L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)$ can be moved under the integral sign $u_1(x,t)$

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u_1(x,t)=-\frac{1}{\pi i}\int\limits_{\hat{L}_{c_1}^{+}}\lambda^{-1}e^{\lambda^2 t}L\left(\frac{d}{dx},\lambda^2\right)\times$$

$$\times\int\limits_0^1 G_2(x,\xi,\lambda)(a\varphi''(\xi)+b\varphi'(\xi)+c\varphi(\xi))d\xi d\lambda=$$

$$=-\frac{1}{\pi i}\Big[a\varphi''(\xi)+b\varphi'(\xi)+c\varphi(\xi)\Big]\int\limits_{\hat{L}_{c_1}^{+}}\lambda^{-1}e^{\lambda^2 t}d\lambda=$$

$$=-\frac{1}{2\pi i}\Big[a\varphi''(\xi)+b\varphi'(\xi)+c\varphi(\xi)\Big]\left[\int\limits_{\hat{L}_{c_1}^{+}}\lambda^{-1}e^{\lambda^2 t}d\lambda+\int\limits_{\hat{L}_{c_1}^{-}}\lambda^{-1}e^{\lambda^2 t}d\lambda\right]=$$

$$=-\frac{1}{2\pi i}\Big[a\varphi''(\xi)+b\varphi'(\xi)+c\varphi(\xi)\Big]\lim_{r\to\infty}\left[\int\limits_{\hat{L}_{c_1,r}^{-}}\lambda^{-1}e^{\lambda^2 t}d\lambda+\right.$$

$$\left.+\int\limits_{\Omega_r(-\frac{5\pi}{8},-\frac{3\pi}{8})}\lambda^{-1}e^{\lambda^2 t}d\lambda++\int\limits_{\hat{L}_{c_1,r}^{+}}\lambda^{-1}e^{\lambda^2 t}d\lambda+\int\limits_{\Omega_r(\frac{3\pi}{8},\frac{5\pi}{8})}\lambda^{-1}e^{\lambda^2 t}d\lambda\right]=$$

$$=-\Big[a\varphi''(\xi)+b\varphi'(\xi)+c\varphi(\xi)\Big],$$

at $t>0$. Note that the estimate (33) is also correct on the arcs $\Omega_r\left(-\frac{5\pi}{8},-\frac{3\pi}{8}\right)$, $\Omega_r\left(\frac{3\pi}{8},\frac{5\pi}{8}\right)$, therefore the limits of corresponding integrals at $r\to\infty$, it goes to zero.

Next the operators at the limits $x \to 0$, $x \to 1$ and $t \to 0$ can also be moved under the integral sign of $u_1(x,t)$, namely

$$u_1(x,0) = -\frac{1}{\pi i} \int\limits_{\hat{L}^+_{c_1}} \lambda^{-1} d\lambda \int\limits_0^1 G_2(x,\xi,\lambda)\Big[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi\Big] d\xi$$

$$= -\frac{1}{\pi i} \int\limits_{\hat{L}^+_{c_1}} \lambda^{-1} d\lambda \int\limits_0^1 G_2(x,\xi,\lambda)\Big[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi\Big] d\xi +$$

$$+ \int\limits_{\Omega_r(-\frac{5\pi}{8},-\frac{3\pi}{8})} \lambda^{-1} d\lambda \int\limits_0^1 G_2(x,\xi,\lambda)\Big[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi\Big] d\xi = 0$$

and for $t > 0$, due estimates (28) and (33), we have

$$u_1(s,t) = -\frac{1}{\pi i} \int\limits_{\hat{L}^+_{c_1}} \lambda^{-1} e^{\lambda^2 t} d\lambda \int\limits_0^1 G_2(s,\xi,\lambda)$$
$$\times \Big[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi\Big] d\xi = 0, \quad s = 0,1.$$

Consequently, the function $Q\big(x,\lambda,\varphi(0),\varphi(1)\big)$ is analytic in the domain $\operatorname{Re}\lambda > c_1$, and the estimates

$$\left| \frac{\partial^k Q\Big(x,\lambda,\varphi(0),\varphi(1)\Big)}{\partial x^k} \right| \le c|\lambda|^k + \frac{c_0}{|\lambda|^k}, \qquad k = 0,1,2,$$

hold for all $x \in [0,1]$.

On distant parts of the contour $\hat{L}^+_{c_1}$ ($\operatorname{Re}\lambda > c_1$) and on the arcs $\Omega_r\left(-\frac{3\pi}{8}, \frac{3\pi}{8}\right)$, $r > 2c_1\sqrt{1+\sqrt{2}}$, the estimate

$$\left| Q\big(x,\lambda,\varphi(0),\varphi(1)\big) \right| \le c_1 e^{-\left|\frac{\lambda}{\sqrt{a}}\right|(1-x)\cos\frac{3\pi}{8}} + c_2 e^{-\left|\frac{\lambda}{\sqrt{a}}\right| x\cos\frac{3\pi}{8}} + \frac{c_3}{|\lambda|} \tag{34}$$

is correct. From (34) it follows that

$$u_2(x,t) \in c^{2,1}\,(0 \le x \le 1,\ t > 0)\,,$$

at $t > 0$ the operators $L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)$ for $x \to 0$, $x \to 1$, can be moved under the integral sign $u_2(x,t)$. Then with regard to (29) we get

$$L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u_2(x,t) = 0,$$

$$u_2(0,t) = \frac{\varphi(0)}{\pi i} \int\limits_{\hat{L}^+_{c_1}} \lambda^{-1} e^{\lambda^2 t} d\lambda = \frac{\varphi(0)}{2\pi i} \lim_{r\to\infty} \int\limits_{\hat{L}^+_{c_1,r}} \lambda^{-1} e^{\lambda^2 t} d\lambda = \frac{\varphi(0)}{2\pi i}\cdot 2\pi i = \varphi(0),$$

$$u_2(1,t) = \frac{\varphi(1)}{\pi i} \int\limits_{\hat{L}_{c_1}^+} \lambda^{-1} e^{\lambda^2 t} d\lambda = \frac{\varphi(1)}{2\pi\ i} \lim_{r\to\infty} \int\limits_{\hat{L}_{c_{1,r}}^+} \lambda^{-1} e^{\lambda^2 t} d\lambda =$$

$$= \frac{\varphi(1)}{2\pi i} \cdot 2\pi i = \varphi(1).$$

From the equality (34) it's clear that for x belonging to any segment $[x_1, x_2] \in (0,1)$, the integral $u_2(x,t)$ converges uniformly by $t \geq 0$. Then

$$u_2(x,t) \in C(0 < x < 1,\ t \geq 0),$$

and for $x \in [x_1, x_2]$

$$u_2(x,0) = \frac{1}{\pi i} \int\limits_{\hat{L}_{c_1}^+} \lambda^{-1} Q\Big(x, \lambda, \varphi(0), \varphi(1)\Big) d\lambda$$

$$= \frac{1}{\pi i} \lim_{r\to\infty} \Bigg[\int\limits_{\hat{L}_{c_{1,r}}^+} \lambda^{-1} Q\Big(x, \lambda, \varphi(0), \varphi(1)\Big) d\lambda$$

$$+ \int\limits_{\Omega_r\left(-\frac{3\pi}{8}, \frac{3\pi}{8}\right)} \lambda^{-1} Q\Big(x, \lambda, \varphi(0), \varphi(1)\Big) d\lambda \Bigg] = 0,$$

because of the analyticity of $Q\big(x, \lambda, \varphi(0), \varphi(1)\big)$ inside the closed contour $\widehat{\Gamma}_{c_{1,r}}^+$.

Note that he integral $u_3(x,t)$ can be analyzed in the same way. □

Combining the Theorems 1 and 3 we arrive at the following final statement.

Theorem 4. *Let all the conditions of Theorems 1 and 3 hold. Then the problem* (1)–(4) *has the unique solution represented by the formula* (32).

5. Conclusion

In this paper the problem of the parabolic equation with constant coefficients and time displacement is studied under non-local and non-homogeneous boundary conditions. Under minimal conditions on the data, the uniqueness of the solution is demonstrated and an explicit analytical representation of it is obtained. Under certain conditions on the boundary values and initial data, the existence and uniqueness of a solution to the mixed problems under consideration are proven, and the solution is expressed by contour integrals chosen in a particular order.

In the future, we plan to consider mixed problems for a parabolic equation with variable coefficients with boundary conditions of a differential character (e.g., containing time derivatives). Other possible points of exploration are biharmonic version of the heat equation, with fractional derivatives, and the blowing-up conditions in physical scenarios[24–26].

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Hikmat I. Ahmadov
Department of Equation of Mathematical Physics
Faculty of Applied Mathematics and Cybernetics
Baku State University
Z. Khalilov ave 23
AZ-1148, Baku
Azerbaijan
E-mail address: hikmatahmadov@yahoo.com

Diego Julio Cirilo-Lombardo
Universidad de Buenos Aires
Facultad de Ciencias Exactas y Naturales
Departamento de Física

CONICET - Universidad de Buenos Aires
Instituto de Física Interdisciplinaria y Aplicada (INFINA)
Av. Intendente Güiraldes 2160
C1428EGA Buenos Aires
Argentina

Laboratory of Information Technologies (LIT)
Joint Institute for Nuclear Research (JINR)
Joliot Curie 6
141980 Dubna (Moscow region)

Special Astrophysical Observatory (SAO)
Russian Academy of Sciences
Nizhnij Arkhyz, Zelenchukskiy region
369167 Karachai-Cherkessian Republic
Russian Federation
E-mail address: diego777jcl@gmail.com