

# NOTES ON THE MOORE-PENROSE INVERSE OF A LINEAR COMBINATION OF COMMUTING GENERALIZED AND HYPERGENERALIZED PROJECTORS

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## ABSTRACT

The aim of this paper is to give alternate representations of the Moore-Penrose inverse of a linear combination of generalized and hypergeneralized projectors and to provide alternate proofs of the invertibility of some linear combination of commuting generalized and hypergeneralized projectors.

**Keywords:** Idempotent, Projector, Generalized projector, Hypergeneralized projector, Moore-Penrose inverse.

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## INTRODUCTION

Let  $C^{n \times m}$  denote the set of all  $n \times m$  complex matrices. The symbols  $A^*$ ,  $R(A)$ ,  $N(A)$  and  $r(A)$  will denote the conjugate transpose, the range (column space), the null space and the rank of a matrix  $A \in C^{n \times m}$ , respectively. The Moore-Penrose inverse of  $A$ , is the unique matrix satisfying the equations.

$$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, AA^\dagger = (AA^\dagger)^*, A^\dagger A = (A^\dagger A)^*.$$

More about the Moore-Penrose inverse can be found in (Ben-Israel & Greville, 1974).

$I_n$  will denote the identity matrix of order  $n$ .  $P_S$  denotes the orthogonal projector onto subspace  $S$ . We use the notations  $C_n^P, C_n^{OP}, C_n^{EP}, C_n^{GP}$  and  $C_n^{HGP}$  for the subsets of  $C^{n \times n}$  consisting of projectors (idempotent matrices), orthogonal projectors (Hermitian idempotent matrices), EP (range-Hermitian) matrices, generalized and hypergeneralized projectors, respectively, i.e.

$$C_n^P = \{A \in C^{n \times n} : A^2 = A\},$$

$$C_n^{OP} = \{A \in C^{n \times n} : A^2 = A = A^*\},$$

$$C_n^{EP} = \{A \in C^{n \times n} : R(A) = R(A^*)\} = \{A \in C^{n \times n} : AA^\dagger = A^\dagger A\},$$

$$C_n^{GP} = \{A \in C^{n \times n} : A^2 = A^*\},$$

$$C_n^{HGP} = \{A \in C^{n \times n} : A^2 = A^\dagger\}.$$

The concepts of generalized and hypergeneralized projectors were introduced by Groß & Trenkler (1997), who presented very interesting properties of the classes of generalized and hypergeneralized projectors. Interesting results concerning generalized and hypergeneralized projectors can be found in the papers (Baksalary, 2009; Baksalary et al., 2008; Radosavljević & Djordjević, 2013; Stewart, 2006).

In this paper we give alternate representations of the Moore-Penrose inverse of a linear combination of generalized and hypergeneralized projectors and to provide alternate proofs of the invertibility of some linear combination of commuting generalized and hypergeneralized projectors of the paper (Tošić et al., 2011). We provide alternate proofs of the nonsingularity of  $\alpha_1 I_n + \alpha_2 A^s + \alpha_3 B^k$  and  $\alpha_1 A^s + \alpha_2 B^k + \alpha_3 C^l$  where  $s, k, l \in \mathbb{N}$ ,  $\alpha_1, \alpha_2, \alpha_3 \in C$  and  $A, B$  and  $C$  are commuting generalized or hypergeneralized projectors under various conditions. Also, we give the alternate form of Moore-Penrose inverse of a linear combination  $\alpha_1 A^s + \alpha_2 B^k$  of two generalized or hypergeneralized matrices.

## THE INVERSES OF GENERALIZED PROJECTORS OR HYPERGENERALIZED PROJECTIONS

In (Baksalary et al., 2008) authors proved that the generalized projector  $A \in C_r^{n \times n}$  can be represented by

$$A = U \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{1}$$

where,  $U \in C^{n \times n}$  is unitary,  $\Sigma = \text{diag}(\sigma_1 I_{r_1}, \dots, \sigma_t I_{r_t})$  is a diagonal matrix of singular values of  $A$ ,

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$\sigma_1 > \sigma_2 > \dots > \sigma_r > 0$ ,  $r_1 + r_2 + \dots + r_t = r$  and  $K \in C^{r \times r}$  satisfies  $(\Sigma K)^3 = I_r$  and  $KK^* = I_r$ .

In the following theorem, we use the star-orthogonality. The notion of star-orthogonality introduced by Hestenes (1961). Let us recall that matrices  $A, B \in C^{n \times m}$  are star-orthogonal, denoted by  $A \perp B$ , if  $AB^* = 0$  and  $A^*B = 0$ . It is well-known that for  $A, B \in C_n^{EP}$ ,

$$A \perp B \Leftrightarrow AB = 0 \Leftrightarrow BA = 0.$$

If  $A, B$  are hypergeneralized projectors, then  $A \perp B$  or  $AB = BA = 0$  is sufficient for  $A+B$  to be a hypergeneralized projector (see (Groß & Trenkler, 1997)).

If we consider a finite commuting family where all of the members are generalized or hypergeneralized projectors, then  $\prod_{i=1}^m A_i^{k_i}$ , where  $m, k_1, \dots, k_m \in \mathbb{N}$  is also a generalized or hypergeneralized projector. Hence, the following theorem is equivalent to Theorem 2.11 (Tošić et al., 2011) and shows that  $\alpha_1 I_n + \alpha_2 A^s + \alpha_3 B^k$  is nonsingular, in the case when  $A, B$  are generalized projectors such that  $A+B \in C_n^{GP}$  or when  $A, B$  are hypergeneralized projectors such that  $A \perp B$ .

First, we will state an auxiliary result.

**Lemma 2.1** (Tošić & Cvetković-Ilić, 2013). Let  $K \in C^{r \times r}$  be such that  $K^3 = I_r$  and let  $\alpha_1, \alpha_2 \in \mathbb{C}$ . If  $\alpha_1^3 + \alpha_2^3 \neq 0$ , then  $\alpha_1 K + \alpha_2 I_r$  and  $\alpha_1 K^2 + \alpha_2 I_r$  are nonsingular and

$$(\alpha_1 K + \alpha_2 I_r)^{-1} = \frac{1}{\alpha_1 + \alpha_2} (\alpha_1^2 K^2 - \alpha_1 \alpha_2 K + \alpha_2^2 I_r)$$

and

$$(\alpha_1 K^2 + \alpha_2 I_r)^{-1} = \frac{1}{\alpha_1 + \alpha_2} (\alpha_1^2 K - \alpha_1 \alpha_2 K^2 + \alpha_2^2 I_r).$$

**Theorem 2.1** Let  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ ,  $\alpha_1 \neq 0$ ,  $\alpha_1^3 + \alpha_2^3 \neq 0$ ,  $\alpha_1^3 + \alpha_3^3 \neq 0$  and  $s, k \in \mathbb{N}$ . If  $A, B \in C^{n \times n}$  be commuting generalized projectors such that  $A+B \in C_n^{GP}$  or  $A, B \in C^{n \times n}$  be commuting hypergeneralized projectors such that  $A \perp B$ ,  $A \perp B$  then  $\alpha_1 I_n + \alpha_2 A^s + \alpha_3 B^k$  is nonsingular and

$$(\alpha_1 I_n + \alpha_2 A^s + \alpha_3 B^k)^{-1} = \frac{1}{\alpha_1^3 + \alpha_2^3} [\alpha_1^2 (A^s)^3 + \alpha_2^2 (A^s)^2 - \alpha_1 \alpha_2 A^s] + (I - AA^\dagger) [\alpha_1 I_n + \alpha_3 B^k]^{-1}. \quad (3)$$

**Proof.** (1) Let  $A, B \in C^{n \times n}$  be commuting generalized projectors such that  $A+B \in C_n^{GP}$ . By Theorem 5 in (Groß & Trenkler, 1997), we have that for generalized projectors  $A, B$   $A+B \in C_n^{GP} \Leftrightarrow AB = 0 = BA$ .

If  $A$  is given by (1) and  $r(A) = r$ , then  $B$  has the form

$$B = U \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} U^*, \quad (4)$$

where  $G \in C^{(n-r) \times (n-r)}$  is a generalized projector. Then

$$\alpha_1 I_n + \alpha_2 A^s + \alpha_3 B^k = U \begin{bmatrix} \alpha_1 I_r + \alpha_2 K^s & 0 \\ 0 & \alpha_1 I_{n-r} + \alpha_3 G^k \end{bmatrix} U^*,$$

where

$$K^s = \begin{cases} I_r, & s \equiv_3 0 \\ K, & s \equiv_3 1 \\ K^2, & s \equiv_3 2. \end{cases} \quad (5)$$

and

$$G^k = \begin{cases} P_{R(G)}, & k \equiv_3 0 \\ G, & k \equiv_3 1 \\ G^2, & k \equiv_3 2. \end{cases} \quad (6)$$

Obviously,  $\alpha_1 I_n + \alpha_2 A^s + \alpha_3 B^k$  is nonsingular if and only if  $\alpha_1 I_r + \alpha_2 K^s$  and  $\alpha_1 I_{n-r} + \alpha_3 G^k$  are nonsingular. By Lemma 2.1 it follows that  $\alpha_1 I_r + \alpha_2 K^s$  is nonsingular for every  $s \in \mathbb{N}$  and

$$(\alpha_1 I_r + \alpha_2 K^s)^{-1} = \begin{cases} (\alpha_1 + \alpha_2)^{-1} I_r, & s \equiv_3 0 \\ \frac{1}{\alpha_1^3 + \alpha_2^3} (\alpha_2^2 K^* - \alpha_1 \alpha_2 K + \alpha_1^2 I_r), & s \equiv_3 1 \\ \frac{1}{\alpha_1^3 + \alpha_2^3} (\alpha_2^2 K - \alpha_1 \alpha_2 K^* + \alpha_1^2 I_r), & s \equiv_3 2 \end{cases}. \quad (7)$$

Since  $G^3$  is an orthogonal projector and  $(\alpha_1 I_{n-r})^3 + (\alpha_3 G^k)^3 = \alpha_1^3 I_{n-r} + \alpha_3^3 G^3$ , we get that  $(\alpha_1 I_{n-r})^3 + (\alpha_3 G^k)^3$  is nonsingular for all constants  $\alpha_1, \alpha_3 \in \mathbb{C} \setminus \{0\}$  such that  $\alpha_1^3 + \alpha_3^3 \neq 0$ .

From the invertibility of  $(\alpha_1 I_{n-r})^3 + (\alpha_3 G^k)^3$ , it follows that  $\alpha_1 I_{n-r} + \alpha_3 G^k$  is nonsingular. Now,

$$(\alpha_1 I_n + \alpha_2 A^s)^{-1} = U \begin{bmatrix} (\alpha_1 I_r + \alpha_2 K^s)^{-1} & 0 \\ 0 & (\alpha_1 I_{n-r} + \alpha_3 G^k)^{-1} \end{bmatrix} U^*, \quad (8)$$

where  $(\alpha_1 I_r + \alpha_2 K^s)^{-1}$  is given by (7). Obviously, the form (8) is equivalent to the form (3).

(2) Let  $A, B \in C^{n \times n}$  be commuting hypergeneralized projectors such that  $A \perp B$ . Since  $A, B \in C_n^{EP}$  it follows that  $A \perp B \Leftrightarrow AB = 0 \Leftrightarrow BA = 0$ .

The proof is similar to item (1).  $\square$

In subsequent consideration, the first part of Theorem 2.5 in (Tošić & Cvetković-Ilić, 2013) plays a crucial role.

**Theorem 2.2** (Tošić & Cvetković-Ilić, 2013). Let  $A \in C_r^{n \times n}$  and  $B \in C^{n \times n}$  be generalized projectors and let  $k, l \in N$ ,  $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \{0\}$ . If  $A + B \in C_n^{GP}$ , then the following conditions are equivalent:

- (i)  $R(A) \oplus R(B) = C^{n \times 1}$
- (ii)  $N(A) \oplus N(B) = C^{n \times 1}$
- (iii)  $R(A) \cap R(B) = \{0\}$  and  $N(A) \cap N(B) = \{0\}$
- (iv)  $\alpha_1 A^k + \alpha_2 B^l$  is nonsingular.

As a corollary we get the following result:

**Corollary 2.4** Let  $A \in C_r^{n \times n}$  and  $B \in C^{n \times n}$  be generalized projectors and let  $k, l \in N$ ,  $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \{0\}$ . If  $A + B \in C_n^{GP}$ , then the following conditions are equivalent:

- (i)  $\alpha_1 A^k + \alpha_2 B^l$  is nonsingular,
- (ii)  $A + B$  is nonsingular.

Also, we need the following lemma:

**Lemma 2.2** (Mišić et al., 2016). Let  $P_1 \in C_r^{n \times n}$  and  $P_2 \in C^{n \times n}$  be orthogonal projectors,  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ ,  $\alpha_1 \neq 0$ ,  $\alpha_1 - \alpha_2 \neq 0$  and  $\alpha_1 - \alpha_3 \neq 0$ . If  $P_1 P_2 = 0 = P_2 P_1$ , then  $\alpha_1 I_n - \alpha_2 P_1 - \alpha_3 P_2$  is nonsingular.

Theorem 2.10 in (Tošić et al., 2011) presents necessary and sufficient conditions for the invertibility of  $\alpha_1 A + \alpha_2 B + \alpha_3 C$ . The following theorem presents also necessary and sufficient conditions for the invertibility of  $\alpha_1 A^s + \alpha_2 B^k + \alpha_3 C^l$ .

**Theorem 2.3** Let  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C} \setminus \{0\}$ ,  $\alpha_1^3 + \alpha_2^3 \neq 0$ ,  $\alpha_1^3 + \alpha_3^3 \neq 0$  and  $s, k, l \in N$ . If  $A, B, C$  are generalized

projectors such that  $B + C \in C_n^{GP}$  or  $A, B, C$  are hypergeneralized projectors such that  $B \perp C$ , then  $\alpha_1 A^s + \alpha_2 B^k + \alpha_3 C^l$  is nonsingular if and only if  $(I_n - AA^\dagger)(B + C) + AA^\dagger$  is nonsingular.

**Proof. (1)** Let  $A, B, C$  are generalized projectors. By Theorem 5 in (Groß & Trenkler, 1997), we have that for generalized projectors  $B, C$ ,  $B + C \in C_n^{GP} \Leftrightarrow BC = 0 = CB$ .

Suppose that  $A$  has the form (1) and  $r(A) = r$ . Then  $B$  has the form

$$B = U \begin{bmatrix} D & 0 \\ 0 & G \end{bmatrix} U^*, \quad (9)$$

where  $D \in C^{r \times r}$  and  $G \in C^{(n-r) \times (n-r)}$  are generalized projectors and  $KD = DK$ .

Also,  $C$  has the form

$$C = U \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} U^*, \quad (10)$$

where  $M \in C^{r \times r}$  and  $N \in C^{(n-r) \times (n-r)}$  are generalized projectors and  $KM = MK$ . From  $BC = 0 = CB$  it follows that  $DM = 0 = MD$  and  $GN = 0 = NG$ , i.e.  $D + M \in C_r^{GP}$  and  $G + N \in C_{n-r}^{GP}$ , respectively.

Now,

$$\alpha_1 A^s + \alpha_2 B^k + \alpha_3 C^l = U \begin{bmatrix} \alpha_1 K^s + \alpha_2 D^k + \alpha_3 M^l & 0 \\ 0 & \alpha_2 G^k + \alpha_3 N^l \end{bmatrix} U^*,$$

where,  $K^s$  is given by (5),  $D^k$ ,  $M^l$ ,  $G^k$  and  $N^l$  are given by (6).

Remark that:

$$(\alpha_1 K^s)^3 + (\alpha_2 D^k + \alpha_3 M^l)^3 = \alpha_1^3 K^3 + \alpha_2^3 D^3 + \alpha_3^3 M^3.$$

Since  $D^3$  and  $M^l$  are orthogonal projectors, then  $\alpha_1^3 I_r + \alpha_2^3 D^3 + \alpha_3^3 M^3$ , i.e.  $(\alpha_1 K^s)^3 + (\alpha_2 D^k + \alpha_3 M^l)^3$  is nonsingular for every constants  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$  such that  $\alpha_1 \neq 0$ ,  $\alpha_1^3 + \alpha_2^3 \neq 0$  and  $\alpha_1^3 + \alpha_3^3 \neq 0$  (by Lemma 2.2).

From the invertibility of  $(\alpha_1 K^s)^3 + (\alpha_2 D^k + \alpha_3 M^l)^3$ , it follows that  $\alpha_1 K^s + \alpha_2 D^k + \alpha_3 M^l$  is nonsingular.

Also,

$$(I_n - AA^\dagger)(B + C) + AA^\dagger = U \begin{bmatrix} I_r & 0 \\ 0 & G + N \end{bmatrix} U^*,$$

Remark that the invertibility of  $\alpha_2 G^k + \alpha_3 N^l$ , is equivalent to the invertibility of  $G + N$  for every constants  $\alpha_2, \alpha_3 \in \mathbb{C} \setminus \{0\}$  (by Corollary 2.4). Hence,  $\alpha_1 A^s + \alpha_2 B^k + \alpha_3 C^l$  is nonsingular if and only if  $(I_n - AA^\dagger)(B + C) + AA^\dagger$  is nonsingular.

(2) Let  $A, B, C$  are hypergeneralized projectors. Since  $B, C \in C_n^{EP}$  it follows that:

$$B \perp C \Leftrightarrow BC = 0 \Leftrightarrow CB = 0,$$

so the proof is similar as the item (1).  $\square$

### THE MOORE-PENROSE INVERSES OF GENERALIZED PROJECTORS OR HYPERGENERALIZED PROJECTIONS

In this section, we first present the form of the Moore-Penrose inverse of  $\alpha_1 A^s + \alpha_2 B^k$ , where  $s, k \in \mathbb{N}$  and  $A, B$  are commuting generalized projectors or commuting hypergeneralized projectors. Remark that Theorem 2.1 in (Tošić et al., 2011) presents the form of the Moore-Penrose inverse of  $\alpha_1 A + \alpha_2 B$ , where  $A$  and  $B$  are commuting generalized projectors or commuting hypergeneralized projectors.

**Theorem 3.1** Let  $A \in C_r^{n \times n}$  and  $B \in C^{n \times n}$  be commuting generalized projectors or commuting hypergeneralized projectors, and let  $s, k \in \mathbb{N}$ ,  $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \{0\}$  and  $\alpha_1^3 + \alpha_2^3 \neq 0$ . Then

$$(\alpha_1 A^s + \alpha_2 B^k)^\dagger = (\alpha_1 A^s + \alpha_2 AA^\dagger B^k)^\dagger + \alpha_2^{-1} (I_n - AA^\dagger) (B^k)^2. \quad (11)$$

**Proof.** We only prove that the result holds when  $A \in C_r^{n \times n}$  and  $B \in C^{n \times n}$  are commuting generalized projectors. As for the case that  $A \in C_r^{n \times n}$  and  $B \in C^{n \times n}$  are commuting hypergeneralized projectors, the proof is similar so we will omit them.

Let  $A \in C_n^{GP}$  be of the form (1) and  $r(A) = r$ . We get that the condition  $AB = BA$  is equivalent to the fact that  $B$  has the form (9). Now,

$$\alpha_1 A^s + \alpha_2 B^k = U \begin{bmatrix} \alpha_1 K^s + \alpha_2 D^k & 0 \\ 0 & \alpha_2 G^k \end{bmatrix} U^*,$$

where  $U \in C^{n \times n}$  is unitary,  $K, D \in C^{r \times r}$  are such that  $K^3 = I_r$ ,  $K^* = K^{-1}$ ,  $D^2 = D^*$ ,  $KD = DK$  and  $G \in C^{(n-r) \times (n-r)}$  is a generalized projector such that:

$$G^k = \begin{cases} P_{R(G)}, & k \equiv_3 0 \\ G, & k \equiv_3 1 \\ G^*, & k \equiv_3 2. \end{cases}$$

Similarly as in Theorem 2.1 we conclude that  $\alpha_1 K^s + \alpha_2 D^k$  is nonsingular.

Let

$$W = U \begin{bmatrix} (\alpha_1 K^s + \alpha_2 D^k)^{-1} & 0 \\ 0 & \alpha_2^{-1} (G^k)^\dagger \end{bmatrix} U^*,$$

where

$$(G^k)^\dagger = \begin{cases} P_{R(G)}, & k \equiv_3 0 \\ G^*, & k \equiv_3 1 \\ G, & k \equiv_3 2, \end{cases} \quad (12)$$

i.e. the right hand side of (11). Obviously,  $W$  is the Moore-Penrose inverse of  $\alpha_1 A^s + \alpha_2 B^k$ .  $\square$

With the additional requirements of Theorem 3.1 it is possible to give a more precise form of Moore-Penrose inverse. The following theorem is a generalization of Corollary 2.4 in (Tošić et al., 2011).

**Theorem 3.2** Let  $s, k \in \mathbb{N}$ ,  $\alpha_1, \alpha_2 \in \mathbb{C} \setminus \{0\}$ . If  $A, B \in C_n^{GP}$  such that  $A + B \in C_n^{GP}$  or  $A, B \in C_n^{HGP}$  such that  $A \perp B$ , then:

$$(\alpha_1 A^s + \alpha_2 B^k)^\dagger = \alpha_1^{-1} (A^s)^2 + \alpha_2^{-1} (B^k)^2. \quad (13)$$

**Proof.** (1)  $A, B \in C_n^{GP}$  be such that  $A + B \in C_n^{GP}$ . Similarly as in Theorem 2.1, we can suppose that  $A$  and  $B$  have the form given by (1) and (4), respectively.

Since  $(\alpha_1 A^s + \alpha_2 B^k) = U \begin{bmatrix} \alpha_1 K^s & 0 \\ 0 & \alpha_2 G^k \end{bmatrix} U^*$ , where  $G^k$  is defined by (6), we get that:

$$(\alpha_1 A^s + \alpha_2 B^k)^\dagger = U \begin{bmatrix} \alpha_1^{-1} K^{-s} & 0 \\ 0 & \alpha_2^{-1} (G^k)^\dagger \end{bmatrix} U^*,$$

where

$$K^{-s} = \begin{cases} I_r, & s \equiv_3 0 \\ K^*, & s \equiv_3 1 \\ K, & s \equiv_3 2 \end{cases} \quad (14)$$

and  $G^\dagger$  is defined by (12), i.e.  $(\alpha_1 A^s + \alpha_2 B^k)^\dagger$  is defined by (13).

(2) Let  $A, B \in C_n^{HGP}$  be such that  $A \perp B$ . Since  $A, B \in C_n^{EP}$ , it follows that  $A \perp B \Leftrightarrow AB = 0 \Leftrightarrow BA = 0$ , so the proof is similar to the item (1).  $\square$

In the next theorem, we present the form of Moore-Penrose inverse of  $\alpha_1 A^s + \alpha_2 A^k$ , where  $s, k \in \mathbb{N}$  and  $A$  is a generalized or hypergeneralized projector. Remark that it is a corollary of Theorem 3.1 and that it is also Corollary 2.5 in (Tošić et al., 2011).

**Theorem 3.3** Let  $A \in C_r^{n \times n}$  be a generalized or hypergeneralized projector and let  $\alpha_1, \alpha_2 \in \mathbb{C}$ ,  $\alpha_1^3 + \alpha_2^3 \neq 0$  and  $s, k \in \mathbb{N}$ . Then:

$$(\alpha_1 A^s + \alpha_2 A^k)^\dagger = \frac{1}{\alpha_1 + \alpha_2} [\alpha_1^2 (A^s)^2 + \alpha_2^2 (A^k)^2 - \alpha_1 \alpha_2 A^s A^k]. \quad (15)$$

Proof. Suppose that generalized projector  $A$  has the form

(1). Then  $\alpha_1 A^s + \alpha_2 A^k$  has the form

$$\alpha_1 A^s + \alpha_2 A^k = U \begin{bmatrix} \alpha_1 K^s + \alpha_2 K^k & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

where

$$\alpha_1 K^s + \alpha_2 K^k = \begin{cases} (\alpha_1 + \alpha_2) I_r, & s \equiv_3 0, k \equiv_3 0 \\ \alpha_1 I_r + \alpha_2 K, & s \equiv_3 0, k \equiv_3 1 \\ \alpha_1 I_r + \alpha_2 K^*, & s \equiv_3 0, k \equiv_3 2 \\ \alpha_1 K + \alpha_2 I_r, & s \equiv_3 1, k \equiv_3 0 \\ (\alpha_1 + \alpha_2) K, & s \equiv_3 1, k \equiv_3 1 \\ \alpha_1 K + \alpha_2 K^*, & s \equiv_3 1, k \equiv_3 2 \\ \alpha_1 K^* + \alpha_2 I_r, & s \equiv_3 2, k \equiv_3 0 \\ \alpha_1 K^* + \alpha_2 K, & s \equiv_3 2, k \equiv_3 1 \\ (\alpha_1 + \alpha_2) K^*, & s \equiv_3 2, k \equiv_3 2 \end{cases}$$

By Lemma 2.1 and Lemma 2 in (Baksalary et al., 2008) it follows that  $\alpha_1 K^s + \alpha_2 K^k$  is nonsingular for every  $s, k \in \mathbb{N}$  and

$$(\alpha_1 K^s + \alpha_2 K^k)^{-1} = \begin{cases} (\alpha_1 + \alpha_2)^{-1} I_r, & s \equiv_3 0, k \equiv_3 0 \\ \frac{1}{\alpha_1 + \alpha_2} (\alpha_2^2 K^* - \alpha_1 \alpha_2 K + \alpha_1^2 I_r), & s \equiv_3 0, k \equiv_3 1 \\ \frac{1}{\alpha_1 + \alpha_2} (\alpha_2^2 K - \alpha_1 \alpha_2 K^* + \alpha_1^2 I_r), & s \equiv_3 0, k \equiv_3 2 \\ \frac{1}{\alpha_1 + \alpha_2} (\alpha_1^2 K^* - \alpha_1 \alpha_2 K + \alpha_2^2 I_r), & s \equiv_3 1, k \equiv_3 0 \\ (\alpha_1 + \alpha_2)^{-1} K^*, & s \equiv_3 1, k \equiv_3 1 \\ \frac{1}{\alpha_1 + \alpha_2} (\alpha_2^2 K + \alpha_1^2 K^* - \alpha_1 \alpha_2 I_r), & s \equiv_3 1, k \equiv_3 2 \\ \frac{1}{\alpha_1 + \alpha_2} (\alpha_1^2 K - \alpha_1 \alpha_2 K^* + \alpha_2^2 I_r), & s \equiv_3 2, k \equiv_3 0 \\ \frac{1}{\alpha_1 + \alpha_2} (\alpha_1^2 K + \alpha_2^2 K^* - \alpha_1 \alpha_2 I_r), & s \equiv_3 2, k \equiv_3 1 \\ (\alpha_1 + \alpha_2)^{-1} K, & s \equiv_3 2, k \equiv_3 2 \end{cases} \quad (17)$$

Let

$$W = U \begin{bmatrix} (\alpha_1 K^s + \alpha_2 K^k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

i.e. the right hand side of (15). Obviously,  $W$  is Moore-Penrose inverse of  $\alpha_1 A^s + \alpha_2 A^k$ . As for the case that  $A$  is a hypergeneralized projector, the proof is similar so we omit it.  $\square$

Also, we will consider the star partial ordering, introduced by Drazin (1978). Let us recall that for the matrices  $A, B \in C^{n \times n}$ , a matrix  $A$  is less or equal than  $B$  with respect to the star partial ordering, denoted by  $A \leq^* B$ , if  $A^* A = A^* B$  and  $AA^* = BA^*$ . If  $A \in C_n^{EP}$ , then for any  $B \in C^{n \times n}$ ,  $A \leq^* B \Leftrightarrow AB = A^2 = BA$ .

Theorem 2.7 in (Tošić et al., 2011) presents the form of Moore-Penrose inverse of  $\alpha_1 A^s + \alpha_2 B^k$ . In the next theorem, we give the alternate form of Moore-Penrose inverse of  $\alpha_1 A^s + \alpha_2 B^k$ .

**Theorem 3.4** Let  $\alpha_1, \alpha_2 \in \mathbb{C}$ ,  $\alpha_2 \neq 0$ ,  $\alpha_1^3 + \alpha_2^3 \neq 0$  and  $s, k \in \mathbb{N}$ . If  $A \in C_r^{n \times n}$  and  $B \in C^{n \times n}$  be generalized projectors such that  $B - A \in C_n^{GP}$  or  $A \in C_n^{EP}$ ,  $B \in C_n^{HGP}$  such that  $A \leq^* B$ , then:

$$(\alpha_1 A^s + \alpha_2 B^k)^\dagger = \frac{1}{\alpha_1 + \alpha_2} [\alpha_1^2 (A^s)^2 + \alpha_2^2 (A^k)^2 - \alpha_1 \alpha_2 A^s A^k] + \alpha_2^{-1} (I - AA^\dagger) (B^k)^2.$$

**Proof.** (1) By Theorem 6 [5], it follows that  $B - A \in C_n^{GP}$  if and only if  $AB = A^2 = BA$ . Suppose that  $A$  has the form (1) and  $B$  has the form given by (9). From  $AB = A^2 = BA$ , we get that

$$B = U \begin{bmatrix} K & 0 \\ 0 & G \end{bmatrix} U^*,$$

where  $G \in C^{(n-r) \times (n-r)}$  is a generalized projector. Now  $\alpha_1 A^s + \alpha_2 B^k$  has the form

$$\alpha_1 A^s + \alpha_2 B^k = U \begin{bmatrix} \alpha_1 K^s + \alpha_2 K^k & 0 \\ 0 & \alpha_2 G^k \end{bmatrix} U^*,$$

where  $\alpha_1 K^s + \alpha_2 K^k$  is given by (16) and  $G^k$  is given by (6). By Lemma 2.1 and Lemma 2 in (Baksalary et al., 2008) it follows that  $\alpha_1 K^s + \alpha_2 K^k$  is nonsingular for every  $s, k \in \mathbb{N}$  and  $(\alpha_1 K^s + \alpha_2 K^k)^{-1}$  is given by (17). Obviously

$$(\alpha_1 A^s + \alpha_2 B^k)^\dagger = U \begin{bmatrix} (\alpha_1 K^s + \alpha_2 K^k)^{-1} & 0 \\ 0 & \alpha_2^{-1} (G^k)^\dagger \end{bmatrix} U^*.$$

(2) Under the assumptions  $A \in C_n^{EP}$ ,  $B \in C_n^{HGP}$  and  $A \leq B$ , we get  $A \in C_n^{HGP}$  by Theorem 3 in (Groß & Trenkler, 1997). Then  $B$  has the form  $B = U \begin{bmatrix} \Sigma K & 0 \\ 0 & G \end{bmatrix} U^*$ . Since  $B \in C_n^{HGP}$ , then  $G \in C_{n-r}^{HGP}$ . Now the proof is similar to the item (1).  $\square$

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