UNIVERSITY THOUGHT
Publication in Natural Sciences, Vol.8, No.1, 2018, pp. 40-45.

doi: 10.5937/univtho8-17290
Original Scientific Paper

HOMOTOPY PERTURBATIONS METHOD: THEORETICAL ASPECTS & APPLICATIONS

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ABSTRACT

The application of the homotopy perturbation method (HPM) in two different research's area has been proposed in this paper. First, the HPM has been used for approximate solving of the well-known implicit equation for electrostatic surface potential of MOSFET transistor. The approximate analytical solution obtained in this case has relative simple mathematical form, and simultaneously high degree of accuracy. Next, HPM has been applied in determination of the invariant measures (IMs) of the non-linear dynamical systems with chaotic behavior. The convergence and efficiency of this method have been confirmed and illustrated in some characteristic examples of chaotic mappings.

Keywords: Homotopy perturbations, convergence, MOSFET modeling, invariant measures, chaotic maps.

INTRODUCTION

The homotopy perturbation method (HPM) belongs to the general and powerful techniques for solving the nonlinear equations of various kinds. As a combination of the well - known homotopy method in topology and classic perturbation techniques, the HPM was first introduced in pioneer works of He (1999, 2000, 2003, 2006, 2008). After that, the extensive development and application of this method to various fields of science researches has been started. For instance, the HPM has been successfully applied for obtaining analytic or approximate solutions of nonlinear differential and partial-differential equations (Biazar et al., 2009; El-Sayed et al., 2012; Gadallah & Elzaki, 2017), as well as Fredholm and Volterra integral equations (Hetmaniok et al., 2012, 2013; Dong et al., 2013). Furthermore, this method also has found significant application in solving many kinds of real based problems, mainly in the physical sciences (Zeb et al., 2014; Kevkić et al., 2017, 2018).

Let us emphasize that the HPM is increasingly being used also in scientific fields, such as, for example, environmental protection and epidemiology (Khan et al., 2014; Adamu et al., 2017). Thus, the wide variety of applications indicates to the flexibility and importance of this method. Finally, let us point out that HPM has been improving, developing and modifying, until to the present time. Consequently, today exist various solver techniques that are based, to a greater or lesser extent, on the basic HPM assumptions (Noor & Khan, 2012; Zhang et al., 2015; Tripathi & Mishra, 2016; Bota & Caruntu, 2017). Here is given a brief theoretical background about the HPM and some sufficient conditions to its convergence. Further, the HPM has been applied in two research's area, where still not observed its any significant application.

As a novel approach, the HPM technique firstly has been applied for solving of the implicit relation between the electrostatic surface potential of an n-channel MOSFET transistor with terminal voltages. Approximate surface potential obtained in this way shows relative simple mathematical form, and at the same times a high degree of accuracy. Indeed, these properties of solution have crucial importance from the physical as well as design point of view. Further, the application of the HPM in determination of the invariant measures (IMs) of the non-linear dynamical systems with chaotic behavior has been investigated. For this purpose, the convergence and efficiency of the HPM has been confirmed and illustrated with some characteristic examples of chaotic mappings.

METHODOLOGY OF THE HPM

For simple illustration of the basic concepts of HPM, we consider the following nonlinear equation:

$$\mathcal{N}\left[f(x)\right] = 0. \tag{1}$$

Here, $f : \mathbb{R} \to \mathbb{R}$ is unknown function and $\mathcal{N}(\cdot)$ is nonlinear operator defined on some functional domain Ω. In solving Eq. (1) by using the HPM technique is assumed the introduction of *homotopy* $\mathcal{H} : \Omega \times [0, 1] \to \mathbb{R}$ such that, for an arbitrary $u \in \Omega$:

$$\mathcal{H}[u, 0] = \mathcal{L}[u], \quad \mathcal{H}[u, 1] = \mathcal{N}[u], \tag{2}$$

where $\mathcal{L}[u]$ is a linear operator, defined on the same domain Ω. More precisely, if denote with $p \in [0, 1]$ the so-called *embedding parameter*, then homotopy $\mathcal{H}$ can be defined as a function

$$\mathcal{H}[u, p] = (1 - p)\,\mathcal{L}[u] + p\,\mathcal{N}[u], \tag{3}$$

which obviously satisfies the both of Eqs. (2). According to this, for the unknown function $f(x)$, we can form the so-called *homotopy equation*:

$$\mathcal{H}\left[f(x), p\right] = 0. \tag{4}$$

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When $p = 0$, Eqs. (2)-(4) give the following linear equation:

$$\mathcal{L}[f(x)] = 0, \tag{5}$$

for which is assumed that has a unique solution $f_0(x)$, usually called *initial solution* or *initial approximation*. On the other hand, when $p = 1$ the same Eqs. (2)-(4) obviously give the nonlinear Eq. (1).

The basic assumption of HPM is that general solution $F(x, p)$ of the homotopy Eq. (4) can be expressed as the power series in p:

$$F(x, p) = \sum_{k=0}^{+\infty} p^k f_k(x). \tag{6}$$

From here, we get immediately $F(x, 0) = f_0(x)$ as the initial solution of linear Eq. (5), i.e. as the solution of homotopy Eq. (4) when $p = 0$. On the other hand, the solution of the "main" Eq. (1), or equivalently Eq. (4) when $p = 1$, will be:

$$f(x) = \lim_{p\to 1^-} F(x, p) = \sum_{k=0}^{\infty} f_k(x). \tag{7}$$

on the condition of convergence the series in Eq. (7). In this case, according to Abel theorem, immediately follows:

Theorem 1. *If series $\sum_{k=0}^{\infty} f_k(x)$ converges, then function $F(x, p)$ is continuous from the left at $p = 1$, i.e. Eq. (7) holds.*

Notice that, if the conditions of the previous theorem are fulfilled, then series $\sum_{k=0}^{\infty} f_k(x)$ represents the solution of the homotopy Eq. (4) when $p = 1$, i.e., it is solution of the nonlinear Eq. (1). Moreover, the solution $f(x)$ of Eq. (1) can be estimated by the so-called *HPM-approximations*:

$$\begin{aligned} \widehat{f_0}(x) &= f_0(x), \\ \widehat{f_k}(x) &= \widehat{f_{k-1}}(x) + f_k(x) = \sum_{j=1}^{k} f_j(x), \qquad k = 1, 2, \ldots \end{aligned} \tag{8}$$

Obviously, series $\sum_{k=0}^{\infty} f_k(x)$ is a solution of Eq. (1) if and only if the series $\{\widehat{f_k}(x)\}$ converges to the unknown function $f(x)$. Sufficient conditions for this convergence can be also given by the following statement, which is a special case of Banach fixed point theorem:

Theorem 2. *Let Ω is Banach space with sup-norm $\|\cdot\|$ and the series $\{f_k(x)\}$ defined on Ω. If for some $\alpha \in (0, 1)$ and $k \geq 1$ the inequality $\|f_k\| \leq \alpha \|f_{k-1}\|$ holds, then series $\{\widehat{f_k}(x)\}$, defined by Eqs. (8), uniformly converges to the unique solution $f(x)$ of Eq. (1).*

Proof. According to assumptions of the theorem and Eqs. (8), for an arbitrary $k, m > 0$ we have that:

$$\begin{aligned} \|\widehat{f_{k+m}} - \widehat{f_k}\| &\leq \|\widehat{f_{k+m}} - \widehat{f_{k+m-1}}\| + \cdots \|\widehat{f_{k+1}} - \widehat{f_k}\| \\ &\leq \|f_{k+m}\| + \cdots + \|f_{k+1}\| \\ &\leq (\alpha^m + \cdots + \alpha)\, \|f_k\| \\ &\leq \alpha^{k+1} \frac{1-\alpha^m}{1-\alpha} \|f_0\|. \end{aligned}$$

Thus, $\lim_{k,m\to\infty} \|\widehat{f_{k+m}} - \widehat{f_k}\| = 0$, i.e. $\{\widehat{f_k}(x)\}$ is a Cauchy sequence in Banach space Ω. It implies that $\{\widehat{f_k}(x)\}$ is uniformly convergent, and its limit is uniquely determined by Eq. (7). □

In following will be describe some practical applications of the aforementioned HPM methodology.

MODELING SURFACE POTENTIAL IN MOSFET TRANSISTORS

The most of MOSFET transistor models are based on charge sheet approximation and the incrementally linear relationship between the inversion charge density and the surface potential (van Langevelde & Klaassen, 2000; Chen & Gildenblat, 2001). Their combination gives following implicit relation between the surface potential ψ_s and gate voltage V_G:

$$V_G - V_{FB} - \psi_s = \gamma \sqrt{\psi_s + u_T \exp\left(\frac{\psi_s - 2\phi_F - V_{ch}}{u_T}\right)}. \tag{9}$$

where V_{FB} is the flat band voltage, u_T is the thermal voltage, ϕ_F is Fermi potential, V_{ch} is the channel potential and γ is the body factor defined by $\sqrt{2\, q\, \varepsilon_{Si} N_A}/C_{ox}$. Here N_A is acceptor concentration in homogeneously doped channel and $C_{ox} = \varepsilon_{ox}/t_{ox}$ is the oxide capacitance per unit area, t_{ox} is the gate oxide capacitance per unit area, as ε_{ox} is the oxide permittivity.

Eq. (9) makes the base of all so-called surface potential based (SPBM) MOSFET models which are the most accurate physically based MOSFET models. However, it is obvious that Eq. (9) can be solved with respect to ψ_s only numerically, what represent the main drawback of the SPBM from the design and physical point of view. Note that, after some elementary transformations, the Eq. (9) can be rearranged in the following, dimensionless form

$$A \exp\left(\frac{y - C}{u_T}\right) - y^2 + (2x + \gamma^2)\, y - x^2 = 0, \tag{10}$$

where $A := \gamma^2 u_T$, $x := V_G - V_{FB}$, $C := 2\phi_F + V_{ch}$ and $y := \psi_s(x)$ is an unkown function. To find the (approximative) solution of Eq. (10), we apply the HPM technique. Firstly, we construct the homotopy equation

$$(1 - p)\, \mathcal{L}[Y(x; p)] + p\, \mathcal{N}[Y(x; p)] = 0, \tag{11}$$

where $p \in (0, 1)$ is the embedding parameter, as

$$\mathcal{L}[Y(x; p)] = Y(x; p) - f_0(x)$$

is a *linear part*, and

$$\mathcal{N}[Y(x; p)] = A \exp\left(\frac{Y(x; p) - C}{u_T}\right) - Y^2(x; p) + (2x + \gamma^2)\, Y(x; p) - x^2$$

is a *non-linear ("true") part* of Eq. (10).

When $p = 0$ the homotopy Eq. (11) obviously becomes $\mathcal{L}[Y(x; 0)] = 0$, with the unique initial solution $Y(x; 0) = f_0(x)$. Similarly, for $p = 1$, Eq. (11) becomes $\mathcal{N}[Y(x; 1)] = 0$, and it is equivalent to Eq. (10), with the "main" solution $Y(x; 1) \equiv \psi_s(x)$.

Now, according to aforementioned facts, solution of the homotopy Eq. (11) can be expressed as the power series in p:

$$Y(x;p) = \sum_{k=0}^{\infty} p^k f_k(x). \tag{12}$$

From here, we get $f_0(x) := Y(x;0)$ as the initial solution of the Eq. (11), obtained for $p = 0$. On the other hand, the "main" solution of the Eq. (11), obtained for $p = 1$, will be

$$\psi_s(x) \equiv Y(x;1) = \lim_{p\to 1^-} Y(x;p) = \sum_{k=0}^{\infty} f_k(x). \tag{13}$$

Thus, it represents the solution of the Eq. (10), on the condition of convergence the series in (13).

Now, substituting Eq. (12) in the homotopy Eq. (11), and by using Taylor's expansion of the exponential term, we obtain

$$\begin{aligned}(1-p)\sum_{k=1}^{+\infty} p^k f_k(x) = p\Bigg\{A\sum_{j=0}^{\infty}\frac{1}{j!\,u_T^j}\left[\sum_{k=0}^{\infty}p^k f_k(x)\right]^j \\ -\left[\sum_{k=0}^{\infty}p^k f_k(x)\right]^2 + B(x)\sum_{k=0}^{\infty}p^k f_k(x) - C(x)\Bigg\},\end{aligned} \tag{14}$$

where we denoted $B(x) := 2x + \gamma^2 + 1$ and $C(x) := x^2 + f_0(x)$. By equating expressions with the identical powers $p^k,\ k = 1, 2, \ldots$ we obtain the explicit expression of the functions $\{f_k(x)\}$. They can be expressed, recursively, with the following recurrence relations:

$$\begin{aligned}
f_1(x) &= A\exp\left(\frac{f_0(x)}{u_T}\right) - f_0(x)^2 + B(x)f_0(x) - C(x),\\
f_2(x) &= A\exp\left(\frac{f_0(x)}{u_T}\right)\cdot\frac{f_1(x)}{u_T} - 2f_0(x)f_1(x) + B(x)f_1(x),\\
f_3(x) &= A\exp\left(\frac{f_0(x)}{u_T}\right)\cdot\left[\frac{f_2(x)}{u_T} + \frac{f_1(x)^2}{2u_T^2}\right] - \big[2f_0(x)f_2(x)\\
&\quad + f_1(x)^2\big] + B(x)f_2(x),\\
&\ \vdots
\end{aligned}$$

In the general case, by using the induction method, it can be easily shown that, for $k \geq 2$ and $m! \leq k < (m+1)!$, hold the following equalities:

$$\begin{aligned}
f_k(x) &= A\exp\left(\frac{f_0(x)}{u_T}\right)\cdot\sum_{j=1}^{m}\frac{1}{j!\,u_T^j}\sum_{i_1+\cdots+i_j=k-1} f_{i_1}(x)\cdots f_{i_j}(x)\\
&\quad -\sum_{j=0}^{k-1} f_j(x)f_{k-j-1}(x) + B(x)f_{k-1}(x).
\end{aligned} \tag{15}$$

At last, the equalities above give the appropriate estimates of surface potential $y = \psi_s(x)$:

$$\begin{aligned}
\widehat{\psi}_s^{(0)}(x) &= f_0(x),\\
\widehat{\psi}_s^{(k)}(x) &= \widehat{\psi}_s^{(k-1)}(x) + f_k(x) = \sum_{j=1}^{k} f_j(x), \qquad k = 1, 2, \ldots
\end{aligned}$$

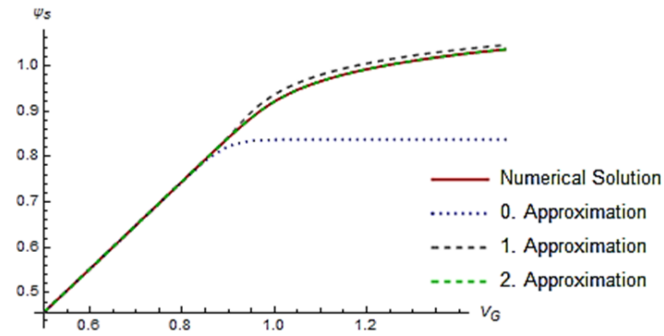

Figure 1. The HPM-approximations of the surface potential $\psi_s(V_G)$.

Fig.1 shows the surface potential versus gate voltage V_G. The convergences of the HPM-approximations are illustrated, along with the numerically obtained solution of Eq. (9). For initial solution was used the *the interpolation function*:

$$f_0(x) = \psi_{wi}(x) - u_T\log\left[1 + \exp\left(\frac{\psi_{wi}(x) - 2\phi_F - V_{ch}}{x}\right)\right], \tag{16}$$

where

$$\psi_{s_{wi}}(x) = \left(-\frac{\gamma}{2} + \sqrt{x + \frac{\gamma^2}{4}}\right)^2 \tag{17}$$

is an approximation of the surface potential ψ_s in the so-called *weak inversion region* (i.e. when $\psi_s < 2\phi_F + V_{ch}$). As it easily can be seen, already for $k \geq 2$, the HPM-approximations $\{\widehat{\psi}_s^{(k)}(x)\}$ give precise approximations of the surface potential $\psi_s(x)$.

DETERMINATION OF INVARIANT MEASURES

In the researching of nonlinear chaotic dynamical models special attention is paid to determining their potential stochastic characteristics. Then, the precise analyses of the behavior of these models commonly needs the using of *the invariant (probabilistic) measures* (*IMs*). One-dimensional nonlinear chaotic model is an usually defined by the operator $T : A \to A$, where $A \subseteq \mathbb{R}$. If chaotic map $T(x)$ has a finite set of inverse branches $T^{-1}(x) = \{g_1(x), \ldots, g_\ell(x)\}$, the determination of its IM is based on solving the well-known *Frobenius-Perron equation*:

$$f(x) = \sum_{y\in T^{-1}(x)}\frac{f(y)}{|T'(y)|}. \tag{18}$$

Here, $f : \mathbb{R} \to \mathbb{R}$ is unknown probability density function which corresponds to appropriate IM, i.e. such that, for any Borel set $B \subseteq \mathbb{R}$, satisfies the following T*-invariant condition:*

$$\int_B f(x)\,\mathrm{d}x = \int_B f\circ T(x)\,\mathrm{d}x. \tag{19}$$

Since, as it is known, there no exists a general procedure to solve Frobenius-Perron Eq. (18) here is proposed one of the possible way of its solving, based on the HPM. For this purpose, we construct the following *homotopy equation*:

$$(1-p)\big[F(x;p) - f_0(x)\big] + p\left[F(x;p) - \sum_{y\in T^{-1}(x)}\frac{F(y;p)}{|T'(y)|}\right] = 0. \tag{20}$$

Here, $p \in [0,1]$ is embedding parameter and $F(x,p)$ is solution of the homotopy Eq. (20) expressed as the power series given by Eq. (6). Thus, the solution of Frobenius-Perron Eq. (18) can be obtain in the same way as in Eq. (7) , i.e. by substituting the power series from Eq. (6) in Eq. (20). After some computations, the following equations get ones:

$$\begin{aligned} f_1(x) &= \sum_{y\in T^{-1}(x)} \frac{f_0(y)}{|T'(y)|} - f_0(x), \\ f_k(x) &= \sum_{y\in T^{-1}(x)} \frac{f_{k-1}(y)}{|T'(y)|}, \quad k \geq 2. \end{aligned} \tag{21}$$

Using Eqs. (21), functions $\{f_k(x)\}$ can be obtained recursively for an arbitrary $k = 1, 2, \ldots$ The appropriate *HPM-approximations* of the unknown function $f(x)$ are

$$\widehat{f_k}(x) := \sum_{j=0}^{k} f_j(x), \quad k = 0, 1, 2, \ldots \tag{22}$$

and their convergence, under some sufficiently conditions, was proven in Stojanović et al. (2018). In the following, some examples of application of the HPM in determining IMs will be described.

Example 3 (Λ***-map***).* On the closed unit interval $[0,1]$ consider the so-called *Lambda (*Λ*) map* (Fig. 2, left panel):

$$T_a(x) = \begin{cases} \dfrac{x}{a}, & 0 \leq x \leq a, \\ \dfrac{1-x}{1-a}, & a < x \leq 1, \end{cases}$$

where $a \in (0,1)$ is a predefined parameter. In this case, we have:

$$\left|T_a'(x)\right| = \begin{cases} \dfrac{1}{a}, & 0 \leq x \leq a, \\ \dfrac{1}{1-a}, & a < x \leq 1, \end{cases}$$

and $T_a^{-1}(x) = \{ax, 1-(1-a)x\}$ is the set of inverse branches. Thus, the Frobenius-Perron equation is

$$f(x) = af(ax) + (1-a)f(1-(1-a)x).$$

If we take, as an initial approximation $f_0(x) \equiv 1$, then the first of Eqs. (21) gives:

$$f_1(x) = af_0(ax) + (1-a)f_0(1-(1-a)x) - f_0(x) \equiv 0.$$

After that, using the second of Eqs. (21), immediately follows $f_k(x) \equiv 0$, for any $k \geq 2$. In that way, the initial approximation $f_0(x) \equiv 1$ is exact solution of the Frobenius-Perron Eq. (18), i.e. the invariant probability measure for this map is standard Lebesgue measure.

*Example 4 (***Truncated** Λ**-map).** Consider again the closed unit interval $[0,1]$ and the fallowing map (Fig. 2, right panel):

$$T_a(x) = \begin{cases} \dfrac{x}{a}, & 0 \leq x < a, \\ -x+a+1, & a \leq x \leq 1, \end{cases}$$

where, in the same way as in the previous case, $a \in (0,1)$ is a predefined parameter. Here, we have:

$$\left|T_a'(x)\right| = \begin{cases} \dfrac{1}{a}, & 0 \leq x < a, \\ 1, & a \leq x \leq 1, \end{cases}$$

and the set of inverse branches is $T_a^{-1}(x) = \{g_1(x), g_2(x)\}$, where:

$$g_1(x) = ax, \quad 0 \leq x \leq 1,$$

$$g_2(x) = \begin{cases} 0, & 0 \leq x < a, \\ -x+a+1, & a \leq x \leq 1, \end{cases}$$

According to this, here the Frobenius-Perron Eq. (18) takes the following form:

$$f(x) = \begin{cases} a\,f(g_1(x)), & 0 \leq x < a, \\ a\,f(g_1(x)) + f(g_2(x)), & a \leq x \leq 1. \end{cases} \tag{23}$$

Now, if we take, as in previous example $f_0(x) \equiv 1$ and apply Eqs. (21), we obtain:

$$\begin{aligned} f_1(x) &= \begin{cases} a\,f_0(g_1(x)) - f_0(x), & 0 \leq x < a, \\ a\,f_0(g_1(x)) + f_0(g_2(x)) - f_0(x), & a \leq x \leq 1, \end{cases} \\ &= \begin{cases} a-1, & 0 \leq x < a, \\ a, & a \leq x \leq 1, \end{cases} \end{aligned}$$

$$\begin{aligned} f_2(x) &= \begin{cases} a\,f_1(g_1(x)), & 0 \leq x < a, \\ a\,f_1(g_1(x)) + f_1(g_2(x)), & a \leq x \leq 1, \end{cases} \\ &= \begin{cases} a\,(a-1), & 0 \leq x < a, \\ a^2, & a \leq x \leq 1, \end{cases} \end{aligned}$$

$$\begin{aligned} f_3(x) &= \begin{cases} a\,f_2(g_1(x)), & 0 \leq x < a, \\ a\,f_2(g_1(x)) + f_2(g_2(x)), & a \leq x \leq 1, \end{cases} \\ &= \begin{cases} a^2\,(a-1), & 0 \leq x < a, \\ a^3, & a \leq x \leq 1, \quad \text{etc.} \end{cases} \end{aligned}$$

In general, using the induction method, it can be easily proven that equalities:

$$f_k(x) = \begin{cases} a^{k-1}\,(a-1), & 0 \leq x < a, \\ a^k, & a \leq x \leq 1, \end{cases}$$

hold for an arbitrary $k \geq 1$. According to thus obtained series $\{f_k(x)\}$, the solution of Frobenius-Perron equation in this case is as following:

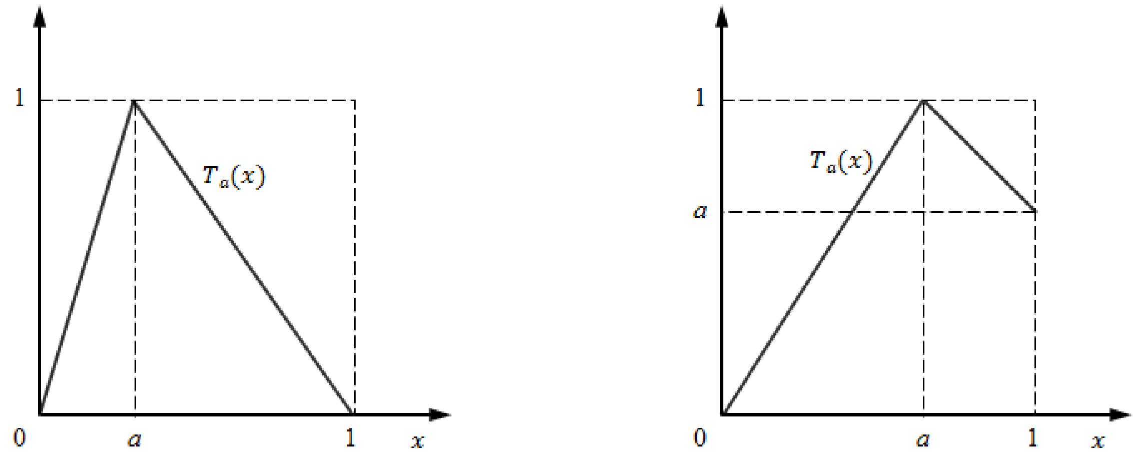

Figure 2. Graphs of the Λ-map (panel left) and the truncated Λ-map (panel right).

$$f(x) := \sum_{k=0}^{\infty} f_k(x) = \begin{cases} 1 + (a-1)\sum_{k=1}^{\infty} a^{k-1}, & 0 \le x < a, \\ \sum_{k=0}^{\infty} a^k, & a \le x \le 1, \end{cases}$$

$$= \begin{cases} 0, & 0 \le x < a, \\ \dfrac{1}{1-a}, & a \le x \le 1, \end{cases} \tag{24}$$

i.e. it is concentrated on the interval $[a, 1]$. It can be easily proved that the last expression in Eq. (24) represents the invariant probability density of truncated Λ-map, i.e. the exact solution of the Frobenius-Perron Eq. (23). Also, notice that, according to the well-known facts about geometric series, the above HPM procedure converges for arbitrary $a \in (0, 1)$.

CONCLUSION

This paper describes the Homotopy perturbation method (HPM) representing the powerful technique for solving of nonlinear equations of various kinds. The HPM has been introduced in approximate solving of the well-known implicit relation between the electrostatic surface potential and terminal voltages in MOSFET transistor. As the second application of the HPM has been chosen the determination of the invariant measures (IMs) of the non-linear dynamical systems with chaotic behavior.

In the first case, the simple mathematical form and high degree of accuracy of obtained HPM approximate analytical solution lead to the improved surface potential based MOSFET model preferring for various circuit simulation programs. On the other hand, due to the second application of HPM procedure is confirmed that invariant measures of the chaotic mappings can be easily analytically determined. Two considered examples have been presented to elucidate the efficiency and implementation of the HPM in solving of various kind of research problems.

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