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ABSTRACT MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The present paper is an addendum to our recent work on abstract time-fractional equations of the following form:

$$\begin{aligned}&\mathbf{D}_t^{\alpha_n}u(t)+\sum_{i=1}^{n-1}A_i\mathbf{D}_t^{\alpha_i}u(t)=A\mathbf{D}_t^{\alpha}u(t)+f(t),\quad t>0,\\&u^{(k)}(0)=u_k,\quad k=0,\cdots,\lceil\alpha_n\rceil-1,\end{aligned}\tag{0.1}$$

where $n\in\mathbb{N}\setminus\{1\}$, A and $A_1,\cdots,A_{n-1}$ are closed linear operators on a sequentially complete locally convex space X, $0\leq\alpha_1<\cdots<\alpha_n$, $0\leq\alpha<\alpha_n$, $f(t)$ is an X-valued function, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α [2]. We analyze the existence and uniqueness of solutions of the equation (0.1), and prove a Ljubich type uniqueness theorem in this context.

1. Introduction and Preliminaries

During the past three decades or so, there has been an explosion of interest in fractional differential equations and fractional dynamics, primarily from their invaluable importance in modeling of various phenomena appearing in physics, chemistry, mathematical biology and engineering. After significant research of E. Bazhlekova [2], it became clear that the abstract Volterra integro-differential equations provide a general framework for the analysis of a large class of abstract time-fractional equations with Caputo derivatives. For more details about fractional differential equations and Volterra integro-differential equations, the reader may consult the monographs by K. Diethelm [8], V. Kiryakova [13] and K. S. Miller-B. Ross, I. Podlubny, J. Prüss [28]-[30].

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In this paper, we continue the analysis of the abstract multi-term fractional differential equation (0.1); cf. also [2]-[4] and [14]-[27]. The most important subcase of (0.1) is, without any doubt, the abstract Cauchy problem (ACP_n):

$$(ACP_n):\begin{cases} u^{(n)}(t)+A_{n-1}u^{(n-1)}(t)+\cdots+A_1u'(t)+A_0u(t)=0,\ t\geq 0,\\ u^{(k)}(0)=u_k,\ k=0,\cdot\cdot\cdot,n-1,\end{cases}$$

whose qualitative properties have been studied in a series of papers by T.-J. Xiao and J. Liang (cf. the monograph [31] for a comprehensive exposition of results). As some other very important subcases of (0.1), we quote the abstract Basset-Boussinesq-Oseen equation

$$u'(t)-A\mathbf{D}_t^{\alpha}u(t)+u(t)=f(t),\quad t\geq 0\quad (\alpha\in(0,1)), \tag{1.1}$$

which describes the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity, and the equation

$$u''(t)+A\mathbf{D}_t^{\alpha}u(t)+u(t)=f(t),\ t\geq 0\quad (\alpha\in(1,2)), \tag{1.2}$$

which models an oscillation process with fractional damping term. The study of equations (1.1)-(1.2) has been initiated by C. Lizama in his joint papers with H. Prado [26] and A. Karczewska [27] (cf. [19] and [22] for more details on the C-wellposedness of equations (1.1)-(1.2) in some classes of Banach or Fréchet spaces). Of importance is also to stress that H. Jiang, F. Liu, I. Turner and K. Burrage have recently analyzed in [11] the following scalar multi-term time-space Caputo-Riesz fractional advection diffusion equation

$$\mathbf{D}_t^{\alpha_n}u(t,x)+\sum_{j=1}^{n-1}c_j\mathbf{D}_t^{\alpha_j}u(t,x)=k_\beta\frac{\partial^\beta u(t,x)}{\partial|x|^\beta}+k_\gamma\frac{\partial^\gamma u(t,x)}{\partial|x|^\gamma}, \tag{1.3}$$

for $t\geq 0,\ 0\leq x\leq L$, where $n\in\mathbb{N}\setminus\{1\}$, $c_1,\cdot\cdot\cdot,c_{n-1}\in\mathbb{C}$, $0\leq\alpha_1<\cdots<\alpha_n\leq 2$, $0<\beta<1$, $1<\gamma\leq 2$, $k_\beta,\ k_\gamma>0$, $L>0$ and $\frac{\partial^\beta u(t,x)}{\partial|x|^\beta}$ denotes the Riesz fractional operator of order β (cf. [5] for various evolution models of equation (1.3) and its backwards analogue); concerning abstract (multi-term) fractional differential equations involving Riemann-Liouville or Liouville-Grünwald fractional derivatives, the reader may consult [3], [9]-[10], [12] and [25]. Finally, there is no need to say that it would be really difficult to include all the relevant subcases of (0.1) not mentioned so far; thus we will refer the reader to [15] and the references cited there for further information on the subject.

The organization of paper is as follows. In the first part of Section 2, we shall recall some known definitions and assertions necessary for our further work [20]. In Theorem 2.1-Theorem 2.2 and Remark 2.1, we investigate mutual relations between k-regularized (C_1,C_2)-existence and uniqueness propagation families (k-regularized C-resolvent propagation families) and k-regularized (C_1,C_2)-existence and uniqueness families (k-regularized C-resolvent families). The analysis of these relations is very complicated in the general case, so that we must focus our attention on the case in which $A_j = c_j I$ for all $i \in \mathbb{N}_{n-1}$. If so, then we will be able to express any single operator family $(R_i(t))_{t\in[0,\tau)}$ of a locally equicontinuous k-regularized C-resolvent propagation family $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ for (0.1) in terms of $(R_0(t))_{t\in[0,\tau)}$ (cf. Remark 2.1(ii), and [31, pp. 116-119] for a slightly different approach in the case of the abstract Cauchy problem (ACP_n)). Further on, we would like to note that perturbations of existence and uniqueness families for the abstract Cauchy problem (0.1) have not been well-studied in the corresponding literature [6-7, 20, 23-24, 33–34], even in the case of equations of integer order. In Theorem 2.3, we transfer results of R. deLaubenfels [7, Section VI] to existence and uniqueness families for the equation (0.1); as an application, we consider in Example 2.1 bounded perturbations of a damped Klein-Gordon equation in L^p-spaces [33, 20]. Notice also that T.-J. Xiao and J. Liang [32] have considered Desch-Schappacher multiplicative and additive perturbation type theorems for C_1-existence families for the abstract Cauchy problem $u^{(n)}(t) = Au(t)$ $(t \geq 0)$; $u^{(k)}(0) = u_k$ $(0 \leq k \leq n)$, in the setting of Banach spaces. We shall examine the possibility of extension of their results to the abstract multi-term fractional differential equations in a separate paper. The existence and uniqueness of solutions of the equation (0.1) are analyzed in the third section of paper.

Unless stated otherwise, X denotes a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short, and the abbreviation $\circledast_X$, or simply $\circledast$, stands for the fundamental system of seminorms which defines the topology of X. By $L(X)$ we denote the space of all continuous linear mappings from X into X. Let $\mathcal{B}$ be the family of bounded subsets of E and let $p_B(T) := \sup_{x\in B} p(Tx)$, $p \in \circledast$, $B \in \mathcal{B}$, $T \in L(X)$. Then $p_B(\cdot)$ is a seminorm on $L(X)$ and the system $(p_B)_{(p,B)\in\circledast\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(X)$. Henceforth A denotes a closed linear operator acting on X, $C \in L(X)$ is an injective operator, and the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\,ds$. The domain, resolvent set, range, point spectrum and adjoint operator of A are denoted by $D(A)$, $\rho(A)$, $R(A)$, $\sigma_p(A)$ and A^*, respectively. Since no confusion seems likely, we will identify A with its graph. Recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by

$$\rho_C(A) := \Big\{\lambda \in \mathbb{C} \ ; \ \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(X)\Big\}.$$

Suppose now that Y is also a sequentially complete locally convex space over the field of complex numbers. Then we denote by $L(Y, X)$ the space which consists of all bounded linear operators from Y into X. By $\circledast_Y$ and I we denote the fundamental system of seminorms which defines the topology on Y, and the identity operator on X, respectively. If $0 < \tau \leq \infty$, then a strongly continuous operator family $(W(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ is said to be locally equicontinuous iff, for every $T \in (0, \tau)$ and for every $p \in \circledast_X$, there exist $q_p \in \circledast_Y$ and $c_p > 0$ such that $p(W(t)y) \leq c_p q_p(y)$, $y \in Y$, $t \in [0, T]$.

Given $s \in \mathbb{R}$ in advance, set $\lfloor s\rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$ and $\lceil s\rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $\mathbb{N}_l := \{1, \cdot \cdot \cdot, l\}$, $\mathbb{N}_l^0 := \{0, 1, \cdot \cdot \cdot, l\}$, $0^\zeta := 0$, $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ ($\zeta > 0, t > 0$) and $g_0 :=$ the Dirac δ-distribution; the symbol δ_{kl} denotes the Kronecker delta. If $\omega > 0$, then we say that a function $f : (\omega, \infty) \to X$ belongs to the class $LT{-}X$, if there exists a function $h(\cdot) \in C([0, \infty) : X)$ such that, for every $p \in \circledast$, there exists $M_p > 0$ satisfying $p(h(t)) \leq M_p e^{\omega t}$, $t \geq 0$, and $f(t) = \int_0^\infty e^{-\lambda t} h(t)\, dt$, $\lambda > \omega$. We assume henceforth that A and $A_1, \cdot \cdot \cdot, A_{n-1}$ are closed linear operators on X as well as that $0 \leq \alpha_1 < \cdot \cdot \cdot < \alpha_n$ and $0 \leq \alpha < \alpha_n$. Set $m_j := \lceil \alpha_j \rceil$, $1 \leq j \leq n$, $m := m_0 := \lceil \alpha \rceil$, $A_0 := A$ and $\alpha_0 := \alpha$.

Let $\alpha > 0$, let $\beta \in \mathbb{R}$, and let the Mittag-Leffler function $E_{\alpha,\beta}(z)$ be defined by $E_{\alpha,\beta}(z) := \sum_{n=0}^\infty z^n/\Gamma(\alpha n + \beta)$, $z \in \mathbb{C}$; here we assume that $1/\Gamma(\alpha n + \beta) = 0$ if $\alpha n + \beta \in -\mathbb{N}_0$. Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$. Then we define the Wright function $\Phi_\gamma(t)$ by $\Phi_\gamma(t) := \mathcal{L}^{-1}(E_\gamma(-\lambda))(t)$, $t \geq 0$, where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform (cf. [2, Section 1.3] for more details about the Mittag-Leffler and Wright functions).

Suppose $0 < \tau \leq \infty$, $k \in C([0, \tau))$, $k \neq 0$, $a \in L^1_{loc}([0, \tau))$ and $a \neq 0$. We refer the reader to [17]-[20] and [23] for the notions of various types of (local) (a, k)-regularized (C_1, C_2)-existence and uniqueness families, (a, k)-regularized C-resolvent families and their subgenerators.

2. Further Results on the C-Wellposedness of (0.1)

The notions of strong solutions of the abstract Cauchy problem (0.1), and its C-wellposedness, are understood in the sense of [20, Definition 2.1]. If $u(t) \equiv u(t; u_0, \cdot \cdot \cdot, u_{m_n-1})$, $t \geq 0$ is a strong solution of (0.1), with $f(t) \equiv 0$ and initial values $u_0, \cdot \cdot \cdot, u_{m_n-1} \in R(C)$, then we can integrate both sides of (0.1) α_n-times, and make use of the equality [2, (1.21)]. In such a way, we get that the function $u(t)$, $t \geq 0$ satisfies the following integral equation:

$$u(\cdot) - \sum_{k=0}^{m_n-1} u_k g_{k+1}(\cdot) + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j \left[u(\cdot) - \sum_{k=0}^{m_j-1} u_k g_{k+1}(\cdot) \right]$$
$$= g_{\alpha_n-\alpha} * A \left[u(\cdot) - \sum_{k=0}^{m-1} u_k g_{k+1}(\cdot) \right]. \tag{2.1}$$

The following definition can be compared with Definition 2.2 of [21].

Definition 2.1. Suppose $0 < \tau \leq \infty$, $k \in C([0, \tau))$, $C_1 \in L(Y, X)$, and $C_2 \in L(X)$ is injective.

(i) A strongly continuous operator family $(E(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ is said to be a (local, if $\tau < \infty$) (k, C_1)-existence family for (0.1) iff, for every $y \in Y$ and $t \in [0, \tau)$, the following holds: $A_j(g_{\alpha_n-\alpha_j} * E)(\cdot)y \in C([0, \tau) : X)$ for $0 \leq j \leq n - 1$, and

$$E(t)y+\sum_{j=1}^{n-1}A_j\big(g_{\alpha_n-\alpha_j}*E\big)(t)y-A\big(g_{\alpha_n-\alpha}*E\big)(t)y=k(t)C_1y. \tag{2.2}$$

(ii) A strongly continuous operator family $(U(t))_{t\in[0,\tau)}\subseteq L(X)$ is said to be a (local, if $\tau<\infty$) (k,C_2)-uniqueness family for (0.1) iff, for every $t\in[0,\tau)$ and $x\in\bigcap_{0\leq j\leq n-1}D(A_j)$, the following holds:

$$U(t)x+\sum_{j=1}^{n-1}\big(g_{\alpha_n-\alpha_j}*U(\cdot)A_jx\big)(t)-\big(g_{\alpha_n-\alpha}*U(\cdot)Ax\big)(t)x=k(t)C_2x. \tag{2.3}$$

(iii) Suppose $X=Y$. Then a strongly continuous family $((E(t))_{t\in[0,\tau)},(U(t))_{t\in[0,\tau)})\subseteq L(X)\times L(X)$ is said to be a (local, if $\tau<\infty$) (k,C_1,C_2)-existence and uniqueness family for (0.1) iff $(E(t))_{t\in[0,\tau)}$ is a (k,C_1)-existence family for (0.1), and $(U(t))_{t\in[0,\tau)}$ is a (k,C_2)-uniqueness family for (0.1).

(iv) Suppose $C=C_1=C_2$. Then a strongly continuous operator family $(R(t))_{t\in[0,\tau)}\subseteq L(X)$ is said to be a (local, if $\tau<\infty$) (k,C)-resolvent family for (0.1) iff $(R(t))_{t\in[0,\tau)}$ is a (k,C)-uniqueness family for (0.1), $R(t)A_j\subseteq A_jR(t)$, for $0\leq j\leq n-1$ and $t\in[0,\tau)$, as well as $R(t)C=CR(t)$, $t\in[0,\tau)$, and $CA_j\subseteq A_jC$, for $0\leq j\leq n-1$.

The notion of (exponential) analyticity of various types of (C_1,C_2)-existence and uniqueness families for (0.1) is taken in the sense of [20, Definition 1.2(ii)]. We refer the reader to [20, Definition 3.1] for the notions of k-regularized (C_1,C_2)-existence and uniqueness families and k-regularized C-resolvent families for (0.1); if $k(t)=1$, $t\in[0,\tau)$, then any such operator family is simply called a (C_1,C_2)-existence and uniqueness family for (0.1) (C-resolvent family for (0.1)). Notice that the following assertions hold:

(i) Suppose $(E(t))_{t\in[0,\tau)}\subseteq L(Y,X)$ is a strongly continuous operator family. Then $(E(t))_{t\in[0,\tau)}$ is a (local) (k,C_1)-existence family for (0.1) iff $((g_{m_n-1}*E)(t))_{t\in[0,\tau)}$ is a k-regularized C_1-existence family for (0.1).

(ii) Suppose $(U(t))_{t\in[0,\tau)}\subseteq L(X)$ is a strongly continuous operator family. Then $(U(t))_{t\in[0,\tau)}$ is a (local) $(k*g_{m_n-1},C_2)$-uniqueness family for (0.1) iff $(U(t))_{t\in[0,\tau)}$ is a (local) k-regularized C_2-uniqueness family for (0.1).

The reader may consult [20, Definition 2.7] for the notions of strong (mild) solutions of the following inhomogeneous Volterra equation:

$$u(t)+\sum_{j=1}^{n-1}\big(g_{\alpha_n-\alpha_j}*A_ju\big)(t)=f(t)+\big(g_{\alpha_n-\alpha}*Au\big)(t),\quad t\in[0,T]. \tag{2.4}$$

One can similarly define the notion of a strong (mild) solution of the problem (2.1).

Given $i\in\mathbb{N}^0_{m_n-1}$ in advance, set $D_i:=\{j\in\mathbb{N}_{n-1}:m_j-1\geq i\}$.

Definition 2.2. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $C,\ C_1,\ C_2 \in L(X)$, C and C_2 are injective. A sequence $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ of strongly continuous operator families in $L(X)$ is called a (local, if $\tau < \infty$):

(i) k-regularized C_1-existence propagation family for (0.1) iff $R_i(0) = (k * g_i)(0)C_1$ and the following holds:

$$\Big[R_i(\cdot)x - \big(k * g_i\big)(\cdot)C_1x\Big] + \sum_{j\in D_i} A_j\Big[g_{\alpha_n-\alpha_j} * \Big(R_i(\cdot)x - \big(k * g_i\big)(\cdot)C_1x\Big)\Big]$$

$$+ \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} A_j\big(g_{\alpha_n-\alpha_j} * R_i\big)(\cdot)x$$

$$= \begin{cases} A\big(g_{\alpha_n-\alpha} * R_i\big)(\cdot)x, & m-1 < i,\ x \in X, \\ A\Big[g_{\alpha_n-\alpha} * \big(R_i(\cdot)x - \big(k * g_i\big)(\cdot)C_1x\big)\Big](\cdot), & m-1 \geq i,\ x \in X, \end{cases} \tag{2.5}$$

for any $i = 0, \cdots, m_n - 1$.

(ii) k-regularized C_2-uniqueness propagation family for (0.1) iff $R_i(0) = (k * g_i)(0)C_2$ and

$$\Big[R_i(\cdot)x - \big(k * g_i\big)(\cdot)C_2x\Big] + \sum_{j\in D_i} g_{\alpha_n-\alpha_j} * \Big[R_i(\cdot)A_jx - \big(k * g_i\big)(\cdot)C_2A_jx\Big]$$

$$+ \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} \big(g_{\alpha_n-\alpha_j} * R_i(\cdot)A_jx\big)(\cdot)$$

$$= \begin{cases} \big(g_{\alpha_n-\alpha} * R_i(\cdot)Ax\big)(\cdot), & m-1 < i, \\ g_{\alpha_n-\alpha} * \Big[R_i(\cdot)Ax - \big(k * g_i\big)(\cdot)C_2Ax\Big](\cdot), & m-1 \geq i, \end{cases} \tag{2.6}$$

for any $x \in \bigcap_{0\leq j\leq n-1} D(A_j)$ and $i \in \mathbb{N}^0_{m_n-1}$.

(iii) k-regularized C-resolvent propagation family for (0.1), in short k-regularized C-propagation family for (0.1), if $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a k-regularized C-uniqueness propagation family for (0.1), and if for every $t \in [0,\tau)$, $i \in \mathbb{N}^0_{m_n-1}$ and $j \in \mathbb{N}^0_{n-1}$, one has $R_i(t)A_j \subseteq A_jR_i(t)$, $R_i(t)C = CR_i(t)$ and $CA_j \subseteq A_jC$.

As mentioned in the introductory part, the case in which the operator A_j is a scalar multiple of the identity operator for all $j \in \mathbb{N}_{n-1}$ (cf. [21] for more details) is very specific. In the subsequent theorems, we shall prove certain relations between (k, C_1, C_2)-existence and uniqueness families ((k, C)-resolvent families) and k-regularized (C_1, C_2)-existence and uniqueness propagation families (k-regularized C-resolvent propagation families).

Theorem 2.1. (i) *Let $C,\ C_1 \in L(X)$, and let C be injective. Suppose that $A_j \in L(X)$ and $A_jA_l = A_lA_j$ for $1 \leq j,\ l \leq n-1$ and $A_jA \subseteq AA_j$ for $1 \leq j \leq n-1$. Suppose $(E(t))_{t\in[0,\tau)}$ is a (k, C_1)-existence family for* (0.1), *and $(R(t))_{t\in[0,\tau)}$ is a (k, C)-resolvent family for* (0.1). *Put, for every $x \in X$, $t \in [0,\tau)$, and $i = m, \cdots, m_n - 1$,*

$$E_i(t)x := \big(g_i * E\big)(t)x + \sum_{j\in D_i} A_j\big(g_{\alpha_n-\alpha_j+i} * E\big)(t)x,$$

and, for every $x \in X,\ t \in [0,\tau),$ *and* $i = 0,\cdots,m-1,$

$$E_i(t)x := \big(k * g_i\big)(t)C_1x - \sum_{j\in \mathbb{N}_{n-1}\setminus D_i} A_j\big(g_{\alpha_n-\alpha_j+i} * E\big)(t)x.$$

Define also $R_i(t)x$ *by replacing respectively* $E(t)$ *and* C_1 *with* $R(t)$ *and* C *in the above formulae. Then* $((E_0(t))_{t\in[0,\tau)},\cdots,(E_{m_n-1}(t))_{t\in[0,\tau)})$ *is a* k*-regularized* C_1*-existence propagation family for* (0.1) *and* $((R_0(t))_{t\in[0,\tau)},\cdot\cdot\cdot,(R_{m_n-1}(t))_{t\in[0,\tau)})$ *is a* k*-regularized* C*-resolvent propagation family for* (0.1)*. Furthermore,* (2.5) *holds for* $((R_0(t))_{t\in[0,\tau)},\cdot\cdot\cdot,(R_{m_n-1}(t))_{t\in[0,\tau)})$*, provided that* (2.2) *holds for* $(R(t))_{t\in[0,\tau)}$*.*

(ii) *Let the following condition hold:*

$$c_j \in \mathbb{C} \text{ and } A_j = c_jI \text{ for all } j \in \mathbb{N}_{n-1}. \tag{2.7}$$

Suppose $((R_0(t))_{t\in[0,\tau)},\cdots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ *is a* k*-regularized* C_1*-existence propagation family for* (0.1)*, resp.* k*-regularized* C_2*-uniqueness propagation family for* (0.1) *(*k*-regularized* C*-resolvent propagation family for* (0.1)*), and* $m = 0$*. Define, for every* $t > 0$*,*

$$b(t) := \mathcal{L}^{-1}\Bigg(\Bigg(1+\sum_{j\in D_0} c_j\lambda^{\alpha_j-\alpha_n}\Bigg)^{-1} - 1\Bigg)(t),$$

and $R(t)x := R_0(t)x + \big(b * R_0\big)(t)x, \quad t \in [0,\tau),\ x \in X$*. Then* $(R(t))_{t\in[0,\tau)}$ *is a* (k,C_1)*-existence family for* (0.1)*, resp.* (k,C_2)*-uniqueness family for* (0.1) *(*(k,C)*-resolvent family for* (0.1)*). Furthermore, if* (2.5) *holds for* $((R_0(t))_{t\in[0,\tau)},\cdot\cdot\cdot,(R_{m_n-1}(t))_{t\in[0,\tau)})$*, then* (2.2) *holds with* $(R(t))_{t\in[0,\tau)}$ *in place of* $(E(t))_{t\in[0,\tau)}$*.*

Proof. We wil only prove the assertion (i) in the case of existence families. Suppose first $i \in \mathbb{N}^0_{m_n-1}$ and $m-1 < i$. Then it is clear that $(E_{1,i}(t))_{t\in[0,\tau)}$ is a strongly continuous operator family. Using the definition of $(E_{1,i}(t))_{t\in[0,\tau)}$ and $(E(t))_{t\in[0,\tau)}$, as well as the equalities $A_jA_l = A_lA_j$ for $1 \leq j,\ l \leq n-1$ and $A_jA \subseteq AA_j$ for $1 \leq j \leq n-1$, we get that, for every $t \in [0,\tau)$ and $x \in X$,

$$A\big(g_{\alpha_n-\alpha} * E_{1,i}\big)(t)x = g_i * \Bigg[E_1(\cdot)x - k(\cdot)C_1x + \sum_{j=1}^{n-1} A_j\big(g_{\alpha_n-\alpha_j} * E_1\big)(\cdot)x\Bigg](t)$$
$$+ A\Bigg(\sum_{j\in D_i} A_j\big(g_{\alpha_n-\alpha_j+i+\alpha_n-\alpha} * E_1\big)(t)x\Bigg)$$

$$
\begin{aligned}
&= g_i * \Big[E_1(\cdot)x - k(\cdot)C_1x + \sum_{j=1}^{n-1} A_j\big(g_{\alpha_n-\alpha_j} * E_1\big)(\cdot)x \Big](t) \\
&\quad + \sum_{j\in D_i} A_j\Bigg(g_{\alpha_n-\alpha_j+i} * \Big[R_1(\cdot) - k(\cdot)C_1 + \sum_{l=1}^{n-1} A_l\big(g_{\alpha_n-\alpha_l} * R_1\big)(\cdot)\Big]\Bigg)(t)x \\
&= \Big[R_{1,i}(\cdot)x - \big(k*g_i\big)(\cdot)C_1x\Big](t) + \sum_{j\in\mathbb{N}\setminus D_i} A_j\Big(g_{\alpha_n-\alpha_j} * \big[g_i * R_1\big]\Big)(t)x \\
&\quad + \sum_{j\in D_i} A_j\Bigg(g_{\alpha_n-\alpha_j+i} * \Big[R_1(\cdot) - k(\cdot)C_1 + \sum_{l=1}^{n-1} A_l\big(g_{\alpha_n-\alpha_l} * R_1\big)(\cdot)\Big]\Bigg)(t)x \\
&= \Big[R_{1,i}(\cdot)x - \big(k*g_i\big)(\cdot)C_1x\Big](t) \\
&\quad + \sum_{j\in\mathbb{N}\setminus D_i} A_j\Bigg(g_{\alpha_n-\alpha_j} * \Big[R_{1,i}(\cdot) - \sum_{l\in D_i} A_l\big(g_{\alpha_n-\alpha_j+i} * R_1\big)(\cdot)\Big]\Bigg)(t)x \\
&\quad + \sum_{j\in D_i} \Bigg(g_{\alpha_n-\alpha_j} * \Big[R_{1,i}(\cdot)x - \sum_{l\in D_i} A_l\big(g_{\alpha_n-\alpha_j+i} * R_1\big)(\cdot)x - \big(k*g_i\big)(\cdot)C_1x \Big] \\
&\quad + g_{\alpha_n-\alpha_j+i} * \sum_{l=1}^{n-1} A_l\big(g_{\alpha_n-\alpha_l} * R_1\big)(\cdot)x \Bigg)(t).
\end{aligned}
$$

Since, for every $t \in [0,\tau)$ and $x \in X$,

$$
\begin{aligned}
&- \sum_{j\in\mathbb{N}\setminus D_i} \sum_{l\in D_i} A_jA_l\big(g_{\alpha_n-\alpha_l+i+\alpha_n-\alpha_j} * R_1\big)(t)x - \sum_{j\in D_i}\sum_{l\in D_i} A_jA_l\big(g_{\alpha_n-\alpha_l+i+\alpha_n-\alpha_j} * R_1\big)(t)x \\
&= \sum_{l=1}^{n-1}\sum_{j\in D_i} A_jA_l\big(g_{\alpha_n-\alpha_l+i+\alpha_n-\alpha_j} * R_1\big)(t)x,
\end{aligned}
$$

(apply the substitution $(j,l) \hookrightarrow (l,j)$), the above implies (2.5), with $(R_i(t))_{t\in[0,\tau)}$ replaced by $(E_{1,i}(t))_{t\in[0,\tau)}$. The proof in case $m-1 \geq i$ is similar and therefore omitted. $\square$

Let (2.7) hold. Then the relations between k-regularized C-resolvent propagation families and (k,C)-resolvent families are not clear in the case $m > 0$. Here we recognize the following subcases:

(a) There exists $i \in \mathbb{N}^0_{m_n-1}$ such that $i > m-1$. Then the consideration is trivial provided that, for such an index i, one has $D_i = \emptyset$ (cf. Theorem 2.1); because of that, we shall assume in the further analysis of this subcase that $D_i \neq \emptyset$.

(b) $m = m_n$ and, for every $i \in \mathbb{N}^0_{m_n-1}$, one has $\mathbb{N}_{n-1} \setminus D_i = \emptyset$. That is the worst case possible and here we have hat the function $u(t) = \sum_{k=0}^{m_n-1} u_k g_{k+1}(t)$, $t \geq 0$ is a strong solution of (0.1) since $\mathbf{D}_t^{\beta} g_{\gamma}(t) \equiv 0$, provided $\beta > \gamma - 1 > 0$.

(c) $m = m_n$, and there exists $i \in \mathbb{N}^0_{m_n-1}$ such that $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$.

The proof of following theorem, which considers the cases (a) and (c) in more detail, is omitted for the sake of brevity.

Theorem 2.2. (i) *Let $m > 0$, and let $i \in \mathbb{N}^0_{m_n-1}$ satisfy $i > m - 1$ and $D_i \neq \emptyset$. Suppose $l \in \mathbb{N}$ and there exists $j \in \mathbb{N}_{n-1} \setminus D_i$ such that $\alpha_j - \alpha_n + l > i$. Let $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ be a k-regularized C_1-existence propagation family for (0.1), resp. k-regularized C_2-uniqueness propagation family for (0.1) (k-regularized C-resolvent propagation family for (0.1)). Suppose, additionally, that for every $x \in X$ and $p \in \circledast$, the mapping $t \mapsto R_i^{(l)}(t)x$, $t \in [0, \tau)$ is continuous with $R_i^{(j)}(0)x = 0$ for all $j \in \mathbb{N}^0_{l-1}$, and that, for every $p \in \circledast$ and $t \in [0, \tau)$, there exist $c_{p,t} > 0$ and $q_{p,t} \in \circledast$ such that $p\big(R_i^{(l)}(t)x\big) \leq c_{p,t} q_{p,t}(x)$, $x \in X$. Put, for every $t \in [0, \tau)$ and $x \in X$,*

$$R(t)x := \left[\mathcal{L}^{-1}\left(\frac{1}{\lambda^{l-i} + \sum_{j\in D_i} c_j \lambda^{\alpha_j-\alpha_n-i+l}}\right) * R_i^{(l)}(\cdot)\right](t)x.$$

Then $(R(t))_{t\in[0,\tau)}$ is a (k, C_1)-existence family for (0.1), resp. (k, C_2)-uniqueness family for (0.1) ((k, C)-resolvent propagation family for (0.1)). Furthermore, if (2.5) holds for $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$, then (2.2) holds with $(R(t))_{t\in[0,\tau)}$ in place of $(E(t))_{t\in[0,\tau)}$.

(ii) *Let $m = m_n$, and let $i \in \mathbb{N}^0_{m_n-1}$ be such that $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$. Suppose $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a k-regularized C_1-existence propagation family for (0.1), resp. k-regularized C_2-uniqueness propagation family for (0.1) (k-regularized C-resolvent propagation family for (0.1)). Let $l \in \mathbb{N}$, let $c(t) \in C([0, \tau))$ satisfy $\tilde{c}(\lambda) = \tilde{k}(\lambda) \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} c_j \lambda^{\alpha_j-\alpha_n}$, for λ sufficiently large, and let there exist $j \in \mathbb{N}_{n-1} \setminus D_i$ such that $\alpha_j - \alpha_n + l > i$. Suppose, additionally, that for every $x \in X$ and $p \in \circledast$, the mapping $t \mapsto R_i^{(l)}(t)x$, $t \in [0, \tau)$ is continuous with $R_i^{(j)}(0)x = 0$ for all $j \in \mathbb{N}^0_{l-1}$ and that, for every $p \in \circledast$ and $t \in [0, \tau)$, there exist $c_{p,t} > 0$ and $q_{p,t} \in \circledast$ such that $p\big(R_i^{(l)}(t)x\big) \leq c_{p,t} q_{p,t}(x)$, $x \in X$. Put, for every $t \in [0, \tau)$ and $x \in X$,*

$$R(t)x := c(t)Cx - \left[\mathcal{L}^{-1}\left(\frac{1}{\sum_{j\in\mathbb{N}_{n-1}\setminus D_i} c_j \lambda^{\alpha_j-\alpha_n-i+l}}\right) * R_i^{(l)}(\cdot)\right](t)x. \tag{2.8}$$

Then $(R(t))_{t\in[0,\tau)}$ is a (k, C_1)-existence family for (0.1), resp. (k, C_2)-uniqueness family for (0.1) ((k, C)-resolvent family for (0.1)). Furthermore, if (2.5) holds for $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$, then (2.2) holds with $(R(t))_{t\in[0,\tau)}$ in place of $(E(t))_{t\in[0,\tau)}$.

Remark 2.1. Let (2.7) hold.

(i) Let $m = 0$. Then it is checked at once that there exist $M \geq 1$ and $\omega \geq 0$ such that $\int_0^t |b(s)|\,ds \leq Me^{\omega t}$, $t \geq 0$, which implies that the properties of (q)-exponential equicontinuity and local equicontinuity are stable under passing from k-regularized C_1-existence propagation families (C_2-uniqueness propagation families, C-resolvent propagation families) to (k, C_1)-existence families ((k, C_2)-uniqueness families, (k, C)-resolvent families). Suppose now that $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ is an exponentially equicontinuous, analytic k-regularized C_1-existence family (C_2-uniqueness family, C-resolvent propagation family) of angle $\delta \in (0, \pi/2]$. Owing to [18, Theorem 3.4(i)], we have that $(R(t))_{t\geq 0}$ is an exponentially equicontinuous, analytic (k, C_1)-existence family ((k, C_2)-uniqueness family, (k, C)-resolvent family) of angle δ.

(ii) Let $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ be a locally equicontinuous k-regularized C-resolvent propagation family for (0.1), and let $m = 0$. Then it is not difficult to see, with the help of Theorem 2.1-Theorem 2.2, that the following equality holds, for every $x \in D(A)$, $i \in \mathbb{N}^0_{m_n-1}$ and $t \in [0, \tau)$,

$$\begin{aligned} R_i(t)x =& g_i * \Big[R_0(\cdot)x + \big(b * R_0\big)(\cdot)x\Big](t) \\ &+ \sum_{j\in D_i} c_j \Big[g_{\alpha_n-\alpha_j+i} * \big(R_0(\cdot)x + \big(b * R_0(\cdot)x\big)\big)\Big](t)x; \end{aligned} \tag{2.9}$$

furthermore, (2.9) holds for every $x \in X$, $i \in \mathbb{N}^0_{m_n-1}$ and $t \in [0, \tau)$, provided (2.5).

(iii) Consider the situation of [20, Example 5.1(b)]. Then there exists an exponentially equicontinuous, analytic k_1-regularized I-resolvent propagation family $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ for the corresponding problem (0.1), with $k_1(t)$ being defined by $k_1(t) = \mathcal{L}^{-1}(\exp(-a_1\lambda^{b_1}))(t)$, $t \geq 0$ for certain positive real numbers $a_1 > 0$ and $b_1 \in (0, 1)$. Under some additional assumptions (very natural in the theory of convoluted operator families), we may apply Theorem 2.2 in the construction of an exponentially equicontinuous, analytic (k_1, I)-resolvent family for (0.1).

The following proposition is very similar to [20, Proposition 3.2] and [23, Proposition 2.7]. Because of that, we shall omit the proof.

Proposition 2.1. *Suppose* (2.7) *holds,* $((R_{1,0}(t))_{t\in[0,\tau)}, \cdots, (R_{1,m_n-1}(t))_{t\in[0,\tau)})$ *is a k-regularized C_1-existence propagation family for* (0.1), $\mathbb{N}_{n-1} \backslash D_i \neq \emptyset$ *provided* $m-1 \geq i$, *and* $((R_{2,0}(t))_{t\in[0,\tau)}, \cdots, (R_{2,m_n-1}(t))_{t\in[0,\tau)})$ *is a locally equicontinuous k-regularized C_2-uniqueness propagation family for* (0.1). *Then, for every* $i \in \mathbb{N}^0_{m_n-1}$, *one has* $C_2R_{1,i}(t) = R_{2,i}(t)C_1$, $t \in [0, \tau)$.

Before discussing some perturbation properties of (0.1), we would like to observe that Proposition 2.1 can be proved under slightly weakened assumptions (see e.g. the formulation of [20, Proposition 2.3]). Consider now the abstract Cauchy problem

$$\begin{aligned}(2.10)\qquad & \mathbf{D}_t^{\alpha_n}u(t)+\sum_{i=1}^{n-1}\bigl(A_i+B_i\bigr)\mathbf{D}_t^{\alpha_i}u(t)=(A+B)\mathbf{D}_t^{\alpha}u(t)+f(t),\quad t>0,\\ & u^{(k)}(0)=u_k,\quad k=0,\cdot\cdot\cdot,\lceil\alpha_n\rceil-1,\end{aligned}$$

where B and $B_1,\cdot\cdot\cdot,B_{n-1}$ are closed linear operators on E. Put $B_0:=B$.

The following theorem is an important extension of [7, Theorem 6.2-Theorem 6.3].

Theorem 2.3. (i) *Suppose $Y=X$, $(E(t))_{t\in[0,\tau)}\subseteq L(X)$ is a (local) C_1-existence family for* (0.1)*, $D_j\in L(E)$ and $B_j=C_1D_j$ ($j\in\mathbb{N}_{n-1}^0$). Suppose that the following conditions hold:*

(a) *For every $p\in\circledast_X$ and for every $T\in(0,\tau)$, there exists $c_{p,T}>0$ such that $p\bigl(E^{(m_n-1)}(t)x\bigr)\le c_{p,T}p(x),\quad x\in E,\ t\in[0,T]$.*

(b) *For every $p\in\circledast_X$, there exists $c_p>0$ such that $p\bigl(D_jx\bigr)\le c_pp(x)$, $j\in\mathbb{N}_{n-1}^0$, $x\in E$.*

(c) *$\alpha_n-\alpha_{n-1}\ge 1$ and $\alpha_n-\alpha\ge 1$.*

Then there exists a (local) C_1-existence propagation family $(R(t))_{t\in[0,\tau)}$ for (2.10)*. If $\tau=\infty$ and if, for every $p\in\circledast_X$, there exist $M\ge 1$ and $\omega\ge 0$ such that*

$$(2.11)\qquad p\bigl(E^{(m_n-1)}(t)x\bigr)\le Me^{\omega t}p(x),\quad t\ge 0,\ x\in E,$$

then $(R(t))_{t\ge 0}$ is exponentially equicontinuous, and moreover, $(R(t))_{t\ge 0}$ also satisfies the condition (2.11)*, with possibly different numbers $M\ge 1$ and $\omega>0$.*

(ii) *Suppose $Y=X$, $(U(t))_{t\in[0,\tau)}\subseteq L(X)$ is a (local) $(1,C_2)$-uniqueness family for* (0.1)*, $D_j\in L(E)$ and $B_j=D_jC_2$ ($j\in\mathbb{N}_{n-1}^0$) Suppose that* (b)-(c) *hold, and that* (a) *holds with $(E^{(m_n-1)}(t))_{t\in[0,\tau)}$ replaced by $(U(t))_{t\in[0,\tau)}$ therein. Then there exists a (local) $(1,C_2)$-uniqueness family $(W(t))_{t\in[0,\tau)}$ for* (2.10)*. If $\tau=\infty$ and if, for every $p\in\circledast_X$, there exist $M\ge 1$ and $\omega\ge 0$ such that* (2.11) *holds, then $(W(t))_{t\ge 0}$ is exponentially equicontinuous, and moreover, $(W(t))_{t\ge 0}$ also satisfies the condition* (2.11)*, with possibly different numbers $M\ge 1$ and $\omega>0$.*

Proof. We will only outline the main details of proof. Put

$$K_0(t)x:=-\Biggl\{E^{(m_n-1)}*\Biggl[\sum_{j=1}^{n-1}g_{\alpha_n-\alpha_j}D_j+g_{\alpha_n-\alpha}D\Biggr]\Biggr\}(t)x,\quad t\in[0,\tau),\ x\in X.$$

Then the assumption (c) implies that, for every fixed $x\in X$, one has $K_0(\cdot)x\in C^1([0,\tau):X)$ and

$$K_0'(t)x:=-\Biggl\{E^{(m_n-1)}*\Biggl[\sum_{j=1}^{n-1}g_{\alpha_n-\alpha_j-1}D_j+g_{\alpha_n-\alpha-1}D\Biggr]\Biggr\}(t)x,\quad t\in[0,\tau).$$

Denote by $f^{*,k}(t)$ the kth convolution power of a function $f(t)$. By assumptions (a)-(b), we have that, for every $t \in [0,\tau)$ and $x \in X$, the series

$$L(t)x := -\Big[K_0(t)x + \big(K_0 * K_0'\big)(t)x + \cdots + \big(K_0 * K_0'^{,*k}\big)(t)x + \cdots\Big]$$

converges, uniformly on compacts of $[0,\tau)$. Furthermore, the operator family $(L(t))_{t\in[0,\tau)} \subseteq L(X)$ is strongly continuous and, for every $x \in X$, the unique solution of the following integral equation:

$$R_{m_n-1}(t)x = E^{(m_n-1)}(t)x + \int_0^t K_0'(t-s)R_{m_n-1}(s)x\,ds, \quad t \in [0,\tau), \tag{2.12}$$

is given by

$$R_{m_n-1}(t)x = E^{(m_n-1)}(t)x + \int_0^t L'(t-s)E^{(m_n-1)}(s)x\,ds, \quad t \in [0,\tau); \tag{2.13}$$

cf. also [30, Theorem 0.5, Corollary 0.3]. It is not difficult to prove that $(R_{m_n-1}(t))_{t\in[0,\tau)}$ is a strongly continuous operator family in $L(X)$. Define now $R(t)x := (g_{m_n-1} * R_{m_n-1})(t)x$, $t \in [0,\tau)$, $x \in X$. Applying the functional equation for $(E(t))_{t\in[0,\tau)}$ twice, it is checked at once that $(R(t))_{t\in[0,\tau)}$ satisfies

$$\begin{aligned} &R^{(m_n-1)}(t)x + \sum_{j=1}^{n-1}\big(A_j + B_j\big)\Big(g_{\alpha_n-\alpha_j} * R^{(m_n-1)}\Big)(t)x \\ &\quad - \big(A + B\big)\Big(g_{\alpha_n-\alpha} * R^{(m_n-1)}\Big)(t)x = C_1x, \quad t \in [0,\tau),\ x \in X, \end{aligned}$$

if we prove that, for every $t \in [0,\tau)$ and $x \in X$,

$$\begin{aligned} &\Big(dK_0 * R_{m_n-1}\Big)(t)x + \left(E^{(m_n-1)} * \sum_{j=0}^{n-1} g_{\alpha_n-\alpha_j-1}(\cdot)D_j * R_{m_n-1}\right)(t)x \\ &\quad + \left(C_1 * \sum_{j=0}^{n-1} g_{\alpha_n-\alpha_j-1}(\cdot)D_j * R_{m_n-1}\right)(t)x \\ &\quad + \sum_{j=1}^{n-1} B_j\Big(g_{\alpha_n-\alpha_j} * R_{m_n-1}\Big)(t)x - B\Big(g_{\alpha_n-\alpha} * R_{m_n-1}\Big)(t)x = 0. \end{aligned}$$

But, the last equality is an immediate consequence of (2.12). In the case that $(E^{(m_n-1)}(t))_{t\geq 0}$ satisfies (2.11), then it is not difficult to see, with the help of (2.12)-(2.13), that $(R(t))_{t\in[0,\tau)}$ also satisfies the same condition, with possibly different numbers $M \geq 1$ and $\omega > 0$. The proof of (ii) is quite similar. Define, for every $t \in [0,\tau)$ and $x \in X$,

$$Q_0(t)x := -\left\{\left[\sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j}D_j + g_{\alpha_n-\alpha}D\right] * U\right\}(t)x,$$

and $Z(t)x := -\Big[Q_0(t)x + \big(Q_0 * Q_0'\big)(t)x + \cdots + \big(Q_0 * Q_0^{\prime,*k}\big)(t)x + \cdots\Big]$.

The resulting $(1, C_2)$-uniqueness family $(W(t))_{t\in[0,\tau)}$ is the unique solution of the following integral equation:

$$W(t)x = U(t)x + \int_0^t W(t-s)Q'(s)x\,ds, \quad t \in [0,\tau),\ x \in X,$$

which is given by $W(t)x = U(t)x + \int_0^t U(t-s)Z'(s)x\,ds$, $t \in [0,\tau)$, $x \in X$. The proof is completed through a routine argument. ☐

The analysis of persistence of differential and analytical properties under perturbations described in Theorem 2.3 will appear elsewhere.

Remark 2.2.
(i) It is worth noting that the proof of the preceding theorem is a slight modification of the corresponding proofs of [30, Theorem 6.1] and [24, Theorem 2.12], established for abstract Volterra equations of non-scalar type. Using the method given in the proofs of aforementioned theorems, one can similarly clarify some results on time-dependent perturbations of (0.1).
(ii) It is not clear how one can prove an analogue of Theorem 2.3(ii) in the case of a (local) C_2-uniqueness family for (0.1).

Example 2.1. Suppose $1 \le p \le \infty$, $X := L^p(\mathbb{R})$, $a \in \mathbb{R}$, $r > 0$, $\vartheta(\cdot) \in W^{1,\infty}(\mathbb{R})$, $1/2 < \gamma \le 1$, $T > 0$, $\chi \in C([0,T] : X)$, and $\frac{d}{dt}(g_{2\gamma-1} * \frac{d}{dx}\chi(t,\cdot)) \in C([0,T] : X)$. Put $A_1 := ad/dx$ and $Au := r\Delta u - \vartheta(\cdot)u$ with maximal distributional domain. In [20, Example 5.3], we have recently considered the following fractional analogue of the damped Klein-Gordon equation:

$$\mathbf{D}_t^{2\gamma}u(t,x) + a\frac{\partial}{\partial x}\mathbf{D}_t^{\gamma}u(t,x) - r\Delta_x u(t,x) + \vartheta(x)u(t,x) = \chi(t,x), \tag{2.14}$$

where $t > 0$, $x \in \mathbb{R}$, and $u(0,x) = \phi(x)$, $u_t(0,x) = \psi(x)$, $x \in \mathbb{R}$, continuing the analysis of T.-J. Xiao and J. Liang [33, Example 4.1], in which it has been assumed that $\gamma = 1$. Since $\alpha_2 - \alpha_1 = \gamma < 1$ for $1/2 < \gamma < 1$, it is clear that Theorem 2.3 cannot be applied in this case, unfortunately. If $\gamma = 1$, then Theorem 2.3 is applicable and we shall consider, as an illustration, the following perturbed problem of (2.14):

$$\begin{aligned}
&\frac{\partial^2 u(t,x)}{\partial t^2} + a\frac{\partial^2 u(t,x)}{\partial x\partial t} + \int_{-\infty}^{+\infty} f(x-y)\frac{\partial u(s,y)}{\partial s}\,dy \\
&- r\frac{\partial^2 u(t,x)}{\partial x^2} + \vartheta(x)u(t,x) + \int_{-\infty}^{+\infty} g(x-y)u(t,y)\,dy = 0, \quad t > 0,\ x \in \mathbb{R},
\end{aligned} \tag{2.15}$$

where $u(0,x) = \phi(x)$, $u_t(0,x) = \psi(x)$, $x \in \mathbb{R}$, with $f,\ g \in W^{1,1}(\mathbb{R})$. Making use of Theorem 2.3 and the corresponding result with $f = g = 0$, we get that for each

$\mu_0 \in \rho(A_1)$ there exists a global exponentially bounded $(\mu_0 - A_1)^{-1}$-existence family for (2.15). By [20, Theorem 3.7], it readily follows that, for every $\phi \in W^{3,p}(\mathbb{R})$ and $\psi \in W^{3,p}(\mathbb{R})$, there exists a strong solution $u(t,x)$ of the problem (2.15) as well as that there exist $M \geq 1$ and $\omega \geq 0$ such that the following estimate holds for each $t \geq 0$: $\|u(t,x)\|_{L^p(\mathbb{R})} \leq Me^{\omega t}\Big[\|\phi\|_{W^{1,p}(\mathbb{R})} + \|\psi\|_{W^{1,p}(\mathbb{R})}\Big]$. To prove the uniqueness of solutions to (2.15), notice that there exist a number $\omega \geq 0$ and a strongly continuous operator family $(U(t))_{t\geq 0} \subseteq L(L^p(\mathbb{R}))$ such that, for every $h \in W^{2,p}(\mathbb{R})$, $\int_0^\infty e^{-\lambda t}U(t)h\,dt = \Big(\lambda^2 + \lambda A_1 + A_0\Big)^{-1}h$, $\Re\lambda > \omega$; cf. [33, Example 4.1]. Let $B_0 \in L(L^p(\mathbb{R}))$ and $B_1 \in L(L^p(\mathbb{R}))$ be defined by $B_0h = f * h$ and $B_0h = g * h$ for any $h \in L^p(\mathbb{R})$. Suppose, for the time being, that $||f||_{L^1(\mathbb{R})}$ is a sufficiently small positive real number. Making use of the complex characterization theorem for the Laplace transform and [31, Theorem 1.1.11], in this case we obtain that there exist a sufficiently large number $\omega' > \omega$ and a strongly continuous operator family $(U_1(t))_{t\geq 0} \subseteq L(L^p(\mathbb{R}))$ such that the operator $(\lambda^2 + \lambda(A_1 + B_1) + (A_0 + B_0))^{-1}$ exists in $L(L^p(\mathbb{R}))$ for any $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega'$, and that

$$\begin{aligned}
&\lambda^{-2}\Big(\lambda^2 + \lambda\big(A_1 + B_1\big) + \big(A_0 + B_0\big)\Big)^{-1}h \\
&= \lambda^{-2}\left(I + \Big(\lambda^2 + \lambda A_1 + A_0\Big)^{-1}\big(\lambda B_1 + B_0\big)\right)^{-1}\Big(\lambda^2 + \lambda A_1 + A_0\Big)^{-1}h \\
&= \int_0^\infty e^{-\lambda t}U_1(t)h\,dt, \quad \Re\lambda > \omega',\ h \in L^p(\mathbb{R}).
\end{aligned}$$

Therefore, if $||f||_{L^1(\mathbb{R})}$ is sufficiently small, then we may apply [20, Theorem 3.5(iii)] in order to see that there exists a global exponentially bounded g_3-regularized I-uniqueness family for the perturbed problem (2.10). The final conclusion now follows from [20, Theorem 3.4(ii)] and the fact that, for every (sufficiently small) $c > 0$, the function $v(t,x) := u(ct,x)$, $t \geq 0$, $x \in \mathbb{R}$ satisfies the equation

$$\begin{aligned}
&\frac{\partial^2 v(t,x)}{\partial t^2} + ac\frac{\partial^2 v(t,x)}{\partial x \partial t} + \int_{-\infty}^{+\infty} cf(x-y)\frac{\partial v(s,y)}{\partial s}\,dy \\
&- rc^2\frac{\partial^2 u(t,x)}{\partial x^2} + c^2\vartheta(x)v(t,x) + \int_{-\infty}^{+\infty} c^2 g(x-y)v(t,y)\,dy = 0, \quad t > 0,\ x \in \mathbb{R},
\end{aligned}$$

where $v(0,x) = 0$, $v_t(0,x) = 0$, $x \in \mathbb{R}$, provided that $u(t,x)$ satisfies (2.15) with $\phi = \psi = 0$. Finally, we would like to mention that the slightly better results on the C-wellposedness of equation (2.15) can be obtained in the case that the function

$\vartheta(\cdot)$ is a positive constant (cf. [15]), and that Theorem 2.3 can be also applied to differential operators considered in [23, Example 2.11-Example 2.12].

3. Existence and Uniqueness of Solutions to (0.1)

We start this section by stating the following existence type theorem.

Theorem 3.1. *Suppose A, $A_1, \cdot\cdot\cdot, A_{n-1}$ are closed linear operators on X, $\omega > 0$, $L(X) \ni C$ is injective and $u_0, \cdot\cdot\cdot, u_{m_n-1} \in X$. Set $P_\lambda := \lambda^{\alpha_n-\alpha} + \sum_{j=1}^{n-1} \lambda^{\alpha_j-\alpha} A_j - A$, $\lambda \in \mathbb{C} \setminus \{0\}$. Let the following conditions hold:*

(i) *The operator P_λ is injective for $\lambda > \omega$ and $D(P_\lambda^{-1}C) = X$, $\lambda > \omega$.*

(ii) *If $0 \leq j \leq n-1$, $0 \leq k \leq m_n - 1$, $m-1 < k$, $1 \leq l \leq n-1$, $m_l - 1 \geq k$ and $\lambda > \omega$, then $Cu_k \in D(P_\lambda^{-1}A_l)$,*

$$A_j\left\{\lambda^{\alpha_j}\left[\lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}Cu_k + \sum_{l\in D_k}\lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k\right]\right.$$
$$\left. - \sum_{l=0}^{m_j-1}\delta_{kl}\lambda^{\alpha_j-1-l}Cu_k\right\} \in LT - X \tag{3.1}$$

and

$$\lambda^{\alpha_n}\left[\lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}Cu_k + \sum_{l\in D_k}\lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k\right] - \lambda^{\alpha_n-1-k}Cu_k \in LT - X.$$

(iii) *If $0 \leq j \leq n-1$, $0 \leq k \leq m_n - 1$, $m-1 \geq k$, $\mathbb{N}_{n-1} \setminus D_k \neq \emptyset$, $s \in \mathbb{N}_{n-1} \setminus D_k$ and $\lambda > \omega$, then $Cu_k \in D(A_s)$, $\sum_{l\in\mathbb{N}_{n-1}\setminus D_k} \lambda^{\alpha_l-\alpha-k-1}A_lCu_k \in D(P_\lambda^{-1})$,*

$$A_j\left\{\lambda^{\alpha_j}\left[\lambda^{-k-1}Cu_k - P_\lambda^{-1}\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\lambda^{\alpha_l-\alpha-k-1}A_lCu_k\right]\right.$$
$$\left. - \sum_{l=0}^{m_j-1}\delta_{kl}\lambda^{\alpha_j-1-l}Cu_k\right\} \in LT - X \tag{3.2}$$

and

$$\lambda^{\alpha_n}\left[\lambda^{-k-1}Cu_k - P_\lambda^{-1}\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\lambda^{\alpha_l-\alpha-k-1}A_lCu_k\right] - \lambda^{\alpha_n-1-k}Cu_k \in LT - X. \tag{3.3}$$

Then the abstract Cauchy problem (0.1) *has a strong solution, with u_k replaced by Cu_k ($0 \leq k \leq m_n - 1$).*

Proof. Suppose, for the time being, $0 \leq k \leq m_n - 1$ and $m - 1 < k$. Denote by $F_{k,n} : (\omega, \infty) \to X$ the function satisfying that, for every $p \in \circledast$, there exists $M_p > 0$ such that $p(F_{k,n}(t)) \leq M_p e^{\omega t}$, $t \geq 0$, and

$$\int_0^\infty e^{-\lambda t} F_{k,n}(t)\,dt = \lambda^{\alpha_n}\Biggl[\lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}Cu_k + \sum_{l\in D_k}\lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k\Biggr] - \lambda^{\alpha_n-1-k}Cu_k. \tag{3.4}$$

Then it is straightforward to see that there exists a function $u_k : (\omega,\infty) \to X$ such that, for every $p \in \circledast$, there exists $M_p' > 0$ satisfying $p(u_k(t)) \le M_p' e^{\omega t}$, $t \ge 0$, and

$$\int_0^\infty e^{-\lambda t} u_k(t)\,dt = \lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}Cu_k + \sum_{l\in D_k}\lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k, \quad \lambda > \omega.$$

The Laplace transform can be used to prove that:

$$\bigl(g_{m_n} * F_{k,n}\bigr)(t) = \Bigl[g_{m_n-\alpha_n} * \bigl(u_k(\cdot) - g_{k+1}(\cdot)Cu_k\bigr)\Bigr](t), \quad t \ge 0. \tag{3.5}$$

This implies $\bigl(g_{\alpha_n} * F_{k,n}\bigr)(t) = u_k(t) - g_{k+1}(t)Cu_k$, $t \ge 0$, $u \in C^{m_n-1}([0,\infty) : X)$ and $u_k^{(j)}(0) = \delta_{kj}Cu_k$ for $0 \le j \le m_n - 1$. Due to (3.5), we have $\mathbf{D}_t^{\alpha_n}u_k(t) = F_{k,n}(t)$, $t \ge 0$. It is not difficult to show that, for every $j \in \mathbb{N}_{n-1}^0$, $\mathbf{D}_t^{\alpha_j}u_k$ is defined as well as that

$$\int_0^\infty e^{-\lambda t}\mathbf{D}_t^{\alpha_j}u_k(t)\,dt = \lambda^{\alpha_j}\int_0^\infty e^{-\lambda t}u_k(t)\,dt - \sum_{l=0}^{m_j-1}u_k^{(l)}(0)\lambda^{\alpha_j-1-l}, \quad \lambda > \omega.$$

Notice now that the condition (3.1) in combination with [31, Theorem 1.1.10] implies that the mapping $t \mapsto A_j\mathbf{D}_t^{\alpha_j}u_k(t)$, $t \ge 0$ is well defined, continuous as well as that

$$\int_0^\infty e^{-\lambda t}A_j\mathbf{D}_t^{\alpha_j}u_k(t)\,dt = A_j\Biggl[\lambda^{\alpha_j}\int_0^\infty e^{-\lambda t}u_k(t)\,dt - \sum_{l=0}^{m_j-1}u_k^{(l)}(0)\lambda^{\alpha_j-1-l}\Biggr], \ \lambda > \omega. \tag{3.6}$$

Having in mind (3.4), (3.6) and the definition of P_λ, a simple calculation yields that:

$$\int_0^\infty e^{-\lambda t}\Bigl[\mathbf{D}_t^{\alpha_n}u_k(t) + A_{n-1}\mathbf{D}_t^{\alpha_{n-1}}u_k(t) + \cdots + A_1\mathbf{D}_t^{\alpha_1}u_k(t) - A\mathbf{D}_t^{\alpha}u_k(t)\Bigr]\,dt = 0,$$

which implies that $u_k(\cdot)$ is a strong solution of the problem (0.1) with $u_k^{(j)}(0) = \delta_{kj}Cu_k$. Suppose now $0 \le k \le m_n - 1$ and $m - 1 \ge k$. Then one can similarly prove, with the help of conditions (3.2)-(3.3), that the function $t \mapsto u_k(t) := \mathcal{L}^{-1}(\lambda^{-k-1}Cu_k - P_\lambda^{-1}\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\lambda^{\alpha_l-\alpha-l-1}A_lCu_k)(t)$, $t \ge 0$, is a strong solution of the problem (0.1) with $u_k^{(j)}(0) = \delta_{kj}Cu_k$. Define $u(t) := \sum_{k=0}^{m_n-1}u_k(t)$, $t \ge 0$. Then it is clear that $u(\cdot)$ is a strong solution of the abstract Cauchy problem (0.1). $\square$

Remark 3.1. (i) Let $0 \leq k \leq m_n - 1$ and $m-1 < k$. Then Theorem 3.1 continues to hold if we replace the term $\lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}Cu_k + \sum_{l\in D_k} \lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k$ i.e., the Laplace transform of $u_k(t)$, in (3.1) by

$$\lambda^{-k-1}Cu_k - \sum_{l\in\mathbb{N}_{n-1}\setminus D_k} \lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k + \lambda^{-k-1}P_\lambda^{-1}ACu_k;$$

in this case, one has to assume that $Cu_k \in D(P_\lambda^{-1}A_l)$, provided $0 \leq l \leq n-1$, $k > m_l - 1$ and $\lambda > \omega$. Notice also that a similar modification can be made in the case $0 \leq k \leq m_n - 1$ and $m-1 \geq k$. As a matter of fact, one can replace the term $\lambda^{-k-1}Cu_k - P_\lambda^{-1} \sum_{l\in\mathbb{N}_{n-1}\setminus D_k} \lambda^{\alpha_l-\alpha-k-1}A_lCu_k$ i.e., the Laplace transform of $u_k(t)$, in (3.2)-(3.3) by

$$\lambda^{\alpha_n-\alpha-k-1}P_\lambda^{-1}Cu_k + \sum_{l\in D_k} \lambda^{\alpha_l-\alpha-k-1}P_\lambda^{-1}A_lCu_k - \lambda^{-k-1}P_\lambda^{-1}ACu_k;$$

in this case, one has to assume that $Cu_k \in D(P_\lambda^{-1}A_l)$, provided $0 \leq l \leq n-1$, $m_l - 1 \geq k$ and $\lambda > \omega$.

(ii) Consider now the situation of the abstract Cauchy problem (ACP$_n$), i.e., suppose that $\alpha_j = j$, $j \in \mathbb{N}_n^0$. Keeping in mind the proof of [31, Lemma 2.2.1, pp. 54-55], it readily follows that the condition:

$$\lambda^j A_j P_\lambda^{-1} Cu_{n-1}, \quad \lambda^j A_j \sum_{i=0}^{k} \lambda^{i-k-1}P_\lambda^{-1}A_iCu_k \in LT - X,$$

for any $k \in \mathbb{N}_{n-2}^0$ and $j \in \mathbb{N}_{n-1}^0$, implies (3.1). Therefore, Theorem 3.1 can be viewed as a generalization of the above-mentioned result.

Now we shall state and prove the Ljubich uniqueness theorem for abstract time-fractional equations of the form (0.1).

Theorem 3.2. *Let $\lambda > 0$, let $L(X) \ni C$ be injective, and let $D(P_{n\lambda}^{-1}C) = X$, $n \in \mathbb{N}$. Suppose, additionally, that for every positive real number $\sigma > 0$ and for every null sequence $(x_n)_{n\in\mathbb{N}}$ in X, one has:*

$$\lim_{n\to\infty} e^{-n\lambda\sigma}P_{n\lambda}^{-1}Cx_n = 0. \tag{3.7}$$

Then, for every $u_0, \cdots, u_{m_n-1} \in X$, the abstract Cauchy problem (0.1) has at most one strong (integral) solution.

Proof. Clearly, it suffices to show the uniqueness of integral solutions of the abstract Cauchy problem (0.1) with $u^{(k)}(0) = u_k = 0$, $k \in \mathbb{N}_{m_n-1}^0$. Let $u(t)$ be such a solution. Then, for every $n \in \mathbb{N}$ and $t \geq 0$, one has:

$$
\begin{aligned}
P_{n\lambda} &\int_0^t e^{n\lambda(t-s)} Cu(s)\,ds \\
&= \big(n\lambda\big)^{\alpha_n-\alpha} \int_0^t e^{n\lambda(t-s)} \Bigg[\big(g_{\alpha_n-\alpha} * CAu\big)(s) - \sum_{j=1}^{n-1} \big(g_{\alpha_n-\alpha_j} * CA_j u\big)(s) \Bigg]\,ds \\
&\quad + \sum_{j=1}^{n-1} \big(n\lambda\big)^{\alpha_j-\alpha} CA_j \int_0^t e^{n\lambda(t-s)} u(s)\,ds - CA \int_0^t e^{n\lambda(t-s)} u(s)\,ds \\
&= \Bigg[\big(n\lambda\big)^{\alpha_n-\alpha} \int_0^t e^{n\lambda(t-s)} \big(g_{\alpha_n-\alpha} * CAu\big)(s)\,ds - CA \int_0^t e^{n\lambda(t-s)} u(s)\,ds \Bigg] \\
&\quad + \sum_{j=1}^{n-1} \big(n\lambda\big)^{\alpha_j-\alpha} CA_j \int_0^t e^{n\lambda(t-s)} u(s)\,ds \\
&\quad - \big(n\lambda\big)^{\alpha_n-\alpha} \int_0^t e^{n\lambda(t-s)} \big(g_{\alpha_n-\alpha_j} * CA_j u\big)(s)\,ds.
\end{aligned}
\tag{3.8}
$$

Keeping in mind [31, Lemma 1.5.5, p. 23] and its proof, we obtain that there exist numbers $M_0, \cdots, M_{n-1} \geq 1$ and $k_0, \cdots, k_{n-1} \in \mathbb{N}$ such that, for every $p \in \circledast$, $t \geq 0$, $n \in \mathbb{N}$ and $j \in \mathbb{N}_{n-1}^0$,

$$
p\Bigg(\big(n\lambda\big)^{\alpha_n-\alpha} \int_0^t e^{n\lambda(t-s)} \big(g_{\alpha_n-\alpha} * Au\big)(s)\,ds - A \int_0^t e^{n\lambda(t-s)} u(s)\,ds \Bigg)
$$

$$
= p\Bigg(\big(n\lambda\big)^{\alpha_n-\alpha} \int_0^t \Bigg(\int_0^\infty \frac{\big(sn\lambda + \varsigma\big)^{\alpha_n-\alpha-1}}{\Gamma(\alpha_n-\alpha)} e^{-\varsigma}\,d\varsigma \Bigg) Au(t-s)\,ds \Bigg) \tag{3.9}
$$

$$
\leq M_0 \big(1 + n + |\lambda|\big)^{k_0} \int_0^t p(Au(s))\,ds, \tag{3.10}
$$

and

$$
p\Bigg(\big(n\lambda\big)^{\alpha_j-\alpha} A_j \int_0^t e^{n\lambda(t-s)} u(s)\,ds - \big(n\lambda\big)^{\alpha_n-\alpha} \int_0^t e^{n\lambda(t-s)} \big(g_{\alpha_n-\alpha_j} * A_j u\big)(s)\,ds \Bigg)
$$

$$(3.11)\quad = p\Biggl(\bigl(n\lambda\bigr)^{\alpha_n-\alpha}\int\limits_0^t\Biggl[\int_0^s e^{n\lambda(s-r)}g_{\alpha_n-\alpha_j}(r)\,dr - e^{n\lambda s}\bigl(n\lambda\bigr)^{\alpha_j-\alpha_n}\Biggr]A_ju(t-s)\,ds\Biggr)$$

$$(3.12)\quad \leq M_j\bigl(1+n+|\lambda|\bigr)^{k_j}\int\limits_0^t p\bigl(A_ju(s)\bigr)\,ds.$$

Using (3.8)-(3.12), it is not difficult to see that, for every $\sigma > 0$ and $t \geq 0$, we have $\lim_{n\to\infty} e^{-n\lambda\sigma}\int_0^t e^{n\lambda(t-s)}Cu(s)\,ds = 0$. Since $\lim_{n\to\infty}\int_{t-\sigma}^t e^{n\lambda(t-s-\sigma)}Cu(s)\,ds = 0$ for $0 \leq \sigma \leq t$, we obtain that $\lim_{n\to\infty}\int_0^{t-\sigma} e^{n\lambda(t-s-\sigma)}Cu(s)\,ds = 0$. By [16, Lemma 3.5(iii)], one gets $Cu(t) = 0$, $t \geq 0$, which completes the proof by the injectivity of C. □

If $\alpha_n - \alpha_j \in \mathbb{N}$, $j \in \mathbb{N}_{n-1}^0$, then the formulae (3.9) and (3.11) imply that it suffices to suppose (instead of a slightly stronger condition (3.7)) that, for every $\sigma > 0$ and $x \in X$, one has $\lim_{n\to\infty} e^{-n\lambda\sigma}P_{n\lambda}^{-1}Cx = 0$; keeping this observation in mind, it readily follows that Theorem 3.2 provides a generalization of [14, Theorem 2.3.23] and [31, Lemma 2.3.1, pp. 67-68]. Notice also that the analysis given in [18, Remark 2.2] enables one to see that Theorem 3.2 provides a proper generalization of [16, Theorem 3.6] in barreled spaces.

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