NEW NORM INEQUALITIES OF ČEBYŠEV TYPE FOR POWER SERIES IN BANACH ALGEBRAS

S. S. DRAGOMIR, M. V. BOLDEA, AND M. MEGAN

Abstract. Let \( f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n \) be a function defined by power series with complex coefficients and convergent on the open disk \( D(0, R) \subset \mathbb{C}, R > 0 \) and \( x, y \in B \), a Banach algebra, with \( xy = yx \). In this paper we establish some new upper bounds for the norm of the Čebyšev type difference

\[
 f(\lambda) f(\lambda xy) - f(\lambda x) f(\lambda y),
\]

providing that the complex number \( \lambda \) and the vectors \( x, y \in B \) are such that the series in the above expression are convergent. These results complement the earlier results obtained by the authors. Applications for some fundamental functions such as the exponential function and the resolvent function are provided as well.

1. Introduction

Let \( B \) be an algebra. An algebra norm on \( B \) is a map \( \|\cdot\| : B \to [0, \infty) \) such that \( (B, \|\cdot\|) \) is a normed space, and further

\[
\|ab\| \leq \|a\| \|b\|,
\]

for any \( a, b \in B \). The normed algebra \( (B, \|\cdot\|) \) is a Banach algebra if \( \|\cdot\| \) is a complete norm. We assume that the Banach algebra is unital, this means that \( B \) has an identity \( 1 \) and that \( \|1\| = 1 \).

Let \( B \) be a unital algebra. An element \( a \in B \) is invertible if there exists an element \( b \in B \) with \( ab = ba = 1 \). The element \( b \) is unique; it is called the inverse of \( a \) and written \( a^{-1} \) or \( 1/a \). The set of invertible elements of \( B \) is denoted by \( \text{Inv} B \). If \( a, b \in \text{Inv} B \) then \( ab \in \text{Inv} B \) and \( (ab)^{-1} = b^{-1}a^{-1} \).

For a unital Banach algebra we also have

(i) If \( a \in B \) and \( \lim_{n \to \infty} \|a^n\|^{1/n} < 1 \), then \( 1 - a \in \text{Inv} B \);
(ii) \( \{a \in B : \|1 - b\| < 1\} \subset \text{Inv} B \);
(iii) \( \text{Inv} B \) is an open subset of \( B \);

Key words and phrases. Banach algebras, power series, exponential function, resolvent function, norm inequalities.

Received: February 25, 2015
Accepted: April 4, 2015.
41
(iv) The map \( \text{Inv} \mathcal{B} \ni a \longmapsto a^{-1} \in \text{Inv} \mathcal{B} \) is continuous.

For simplicity, we denote \( \lambda 1 \), where \( \lambda \in \mathbb{C} \) and 1 is the identity of \( \mathcal{B} \), by \( \lambda \). The resolvent set of \( a \in \mathcal{B} \) is defined by

\[
\rho(a) := \{ \lambda \in \mathbb{C} : \lambda - a \in \text{Inv} \mathcal{B} \};
\]

the spectrum of \( a \) is \( \sigma(a) \), the complement of \( \rho(a) \) in \( \mathbb{C} \), and the resolvent function of \( a \) is \( R_a : \rho(a) \to \text{Inv} \mathcal{B} \), \( R_a(\lambda) := (\lambda - a)^{-1} \). For each \( \lambda, \gamma \in \rho(a) \) we have the identity

\[
R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma).
\]

We also have that \( \sigma(a) \subset \{ \lambda \in \mathbb{C} : |\lambda| \leq \|a\| \} \). The spectral radius of \( a \) is defined as

\[
\nu(a) = \sup \{|\lambda| : \lambda \in \sigma(a)\}.
\]

If \( a, b \) are commuting elements in \( \mathcal{B} \), i.e., \( ab = ba \), then

\[
\nu(ab) \leq \nu(a) \nu(b) \quad \text{and} \quad \nu(a + b) \leq \nu(a) + \nu(b).
\]

Let \( f \) be an analytic function on the open disk \( \{ |\lambda| < R \} \) given by the power series

\[
f(\lambda) := \sum_{j=0}^{\infty} a_j \lambda^j \quad (|\lambda| < R). \]

If \( \nu(a) < R \), then the series \( \sum_{j=0}^{\infty} a_j \lambda^j \) converges in the Banach algebra \( \mathcal{B} \) because \( \sum_{j=0}^{\infty} |a_j| \|a^j\| < \infty \) and we can define \( f(a) \) to be its sum. Clearly \( f(a) \) is well defined and there are many examples of important functions on a Banach algebra \( \mathcal{B} \) that can be constructed in this way. For instance, the exponential map on \( \mathcal{B} \) denoted \( \exp \) and defined as

\[
\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \quad \text{for each} \quad a \in \mathcal{B}.
\]

If \( \mathcal{B} \) is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for \( a \) and \( b \) from \( \mathcal{B} \)

\[
\exp (a + b) = \exp (a) \exp (b).
\]

In a general Banach algebra \( \mathcal{B} \) it is difficult to determine the elements in the range of the exponential map \( \exp(\mathcal{B}) \), i.e., the element which have a “logarithm”. However, it is easy to see that if \( a \) is an element in \( \mathcal{B} \) such that \( \|1 - a\| < 1 \), then \( a \) is in \( \exp(\mathcal{B}) \). That follows from the fact that if we set

\[
b = -\sum_{n=1}^{\infty} \frac{1}{n} (1 - a)^n,
\]

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for \( \exp(b) \) yields \( \exp(b) = a \).

It is known that if \( x \) and \( y \) are commuting, i.e., \( xy = yx \), then the exponential function satisfies the property \( \exp(x) \exp(y) = \exp(y) \exp(x) = \exp(x + y) \). Also, if \( x \) is invertible and \( a, b \in \mathbb{R} \) with \( a < b \) then

\[
\int_a^b \exp(tx) \, dt = x^{-1} [\exp(bx) - \exp( ax)].
\]
Moreover, if $x$ and $y$ are commuting and $y - x$ is invertible, then
\[
\int_0^1 \exp \left((1 - s) x + sy\right) ds = \int_0^1 \exp \left(s (y - x)\right) \exp(x) ds
\]
\[= \left( \int_0^1 \exp \left(s (y - x)\right) ds \right) \exp(x)
\]
\[= (y - x)^{-1} [\exp (y - x) - I] \exp(x)
\]
\[= (y - x)^{-1} [\exp (y) - \exp(x)].
\]

Inequalities for functions of operators in Hilbert spaces may be found in the papers [3], [2] and in the recent monographs [4], [5], [7] and the references therein.

In order to state some earlier results [6] that motivate our current work we need some preparation as follows.

Let $\alpha_n$ be nonzero complex numbers and let
\[
R := \limsup |\alpha_n|^{1/n}.
\]
Clearly $0 \leq R \leq \infty$, but we consider only the case $0 < R \leq \infty$.

Denote by
\[
D(0, R) = \begin{cases}
\{ \lambda \in \mathbb{C}, |\lambda| < R\}, & \text{if } R < \infty, \\
\mathbb{C}, & \text{if } R = \infty,
\end{cases}
\]
consider the functions
\[
\lambda \mapsto f(\lambda) : D(0, R) \rightarrow \mathbb{C}, \quad f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n
\]
and
\[
\lambda \mapsto f_A(\lambda) : D(0, R) \rightarrow \mathbb{C}, \quad f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.
\]

Let $B$ be a unital Banach algebra and 1 its unity. Denote by
\[
B(0, R) = \begin{cases}
\{ x \in B : \|x\| < R\}, & \text{if } R < \infty, \\
B, & \text{if } R = \infty,
\end{cases}
\]
We associate to $f$ the map
\[
x \mapsto \tilde{f}(x) : B(0, R) \rightarrow B, \quad \tilde{f}(x) := \sum_{n=0}^{\infty} \alpha_n x^n.
\]
Obviously, $\tilde{f}$ is correctly defined because the series $\sum_{n=0}^{\infty} \alpha_n x^n$ is absolutely convergent, since $\sum_{n=0}^{\infty} \|x^n\| \leq \sum_{n=0}^{\infty} |\alpha_n| \|x\|^n$.

In addition, we assume that $s_2 := \sum_{n=0}^{\infty} n^2 |\alpha_n| < \infty$. Let $s_0 := \sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $s_1 := \sum_{n=0}^{\infty} n |\alpha_n| < \infty$.

With the above assumptions we have the following [6].
**Theorem 1.1.** Let $\lambda \in \mathbb{C}$ such that $\max\{|\lambda|, |\lambda|^2\} < R < \infty$ and let $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \leq 1$ and $xy = yx$. Then

(i) We have

\[ \| \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) - \tilde{f}(\lambda x) \tilde{f}(\lambda y) \| \leq \sqrt{2} \psi \min \{\|x - 1\|, \|y - 1\|\} f_A (|\lambda|^2), \]

where

\[ \psi^2 := s_0 s_2 - s_1^2. \]

(ii) We also have

\[ \| \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) - \tilde{f}(\lambda x) \tilde{f}(\lambda y) \| \leq \sqrt{2} \psi \min \{\|x - 1\|, \|y - 1\|\} f_A (|\lambda|) \]

\[ \times \left\{ f_A (|\lambda|) [ |\lambda| f_A (|\lambda|) + |\lambda|^2 f_A'' (|\lambda|) ] - |\lambda| f_A (|\lambda|) \right\}^{1/2}. \]

For other similar results, see [6].

In this paper we establish some new upper bounds for the norm of the Čebyshev type difference

\[ \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) - \tilde{f}(\lambda x) \tilde{f}(\lambda y) \]

provide that the complex number $\lambda$ and the vectors $x, y \in \mathcal{B}$ are such that the series in (1.4) are convergent. Applications for some fundamental functions such as the exponential function and the resolvent function are provided as well.

2. THE RESULTS

We start with the following result that is of interest in itself.

**Lemma 2.1.** Let $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$ and $x, y \in \mathcal{B}$ with $xy = yx$. If $\|y\| < 1$, $\lambda \in \mathbb{C}$ and $x \in \mathcal{B}$ with $|\lambda| \|x\| < R$, then we have the inequality

\[ \| f(\lambda) y - \tilde{f}(\lambda xy) \| \leq \frac{\|y - 1\|}{1 - \|y\|} \left[ f_A (|\lambda| \|x\|) - |\alpha_k| |\lambda|^k \|x\|^k \right], \]

for any $k \in \mathbb{N}$, $k \geq 0$.

**Proof.** We have for $m \geq 2$ and $1 \leq k \leq m - 1$ that

\[ \left( \sum_{j=0}^{m} \alpha_j \lambda^j x^j \right) y^k - \sum_{j=0}^{m} \alpha_j \lambda^j (xy)^j = \left( \sum_{j=0}^{m} \alpha_j \lambda^j x^j \right) y^k - \sum_{j=0}^{m} \alpha_j \lambda^j x^j y^j \]

\[ = \sum_{j=0}^{m} \alpha_j \lambda^j x^j (y^k - y^j) \]

\[ = \sum_{j=0, j \neq k}^{m} \alpha_j \lambda^j x^j (y^k - y^j) = A. \]
Since $y^k - y^j = \sum_{l=j}^{k-1} (y^{l-1} - y^l) = \sum_{l=j}^{k-1} y^l (y - 1)$, then by taking the norm in (2.2) we get

(2.3) $\|A\| \leq \sum_{j=0, j \neq k}^m |\alpha_j| |\lambda|^j \|x\|^j \left(\sum_{l=j}^{k-1} y^l (y - 1)\right)$

$$\leq \sum_{j=0, j \neq k}^m |\alpha_j| |\lambda|^j \|x\|^j \sum_{l=j}^{k-1} \|y\|^l \|y - 1\|$$

$$= \|y - 1\| \sum_{j=0, j \neq k}^m |\alpha_j| |\lambda|^j \|x\|^j \sum_{l=j}^{k-1} \|y\|^l : B.$$ 

By noticing that

$$\sum_{l=j}^{k-1} \|y\|^l \leq \sum_{l=0}^{m-1} \|y\|^l$$

we have

(2.4) $B \leq \|y - 1\| \sum_{l=0}^{m-1} \|y\|^l \sum_{j=0}^m |\alpha_j| |\lambda|^j \|x\|^j \left(\sum_{l=j}^{k-1} \|y\|^l - |\alpha_k| |\lambda|^k \|x\|^k\right)$

Utilising the inequalities (2.2), (2.4) we conclude that

(2.5) $\left\|\left(\sum_{j=0}^m \alpha_j \lambda^j x^j\right) y^k - \sum_{j=0}^m \alpha_j \lambda^j (xy)^j\right\|$

$$\leq \|y - 1\| \sum_{l=0}^{m-1} \|y\|^l \left(\sum_{j=0}^m |\alpha_j| |\lambda|^j \|x\|^j \left(\sum_{l=j}^{k-1} \|y\|^l - |\alpha_k| |\lambda|^k \|x\|^k\right)\right)$$

for any $m \geq 2$ and $1 \leq k \leq m - 1$.

Since the series $\sum_{j=0}^m \alpha_j \lambda^j x^j$ and $\sum_{j=0}^m \alpha_j (xy)^j$ are convergent in $B$ and, because $\|y\| < 1$, then $\sum_{l=0}^{\infty} \|y\|^l = \frac{1}{1-\|y\|}$, then by letting $m \to \infty$ in (2.5), we get the desired result (2.1).

If $k = 0$, then $\sum_{j=0}^m \alpha_j \lambda^j x^j - \sum_{j=0}^m \alpha_j \lambda^j (xy)^j = \sum_{j=1}^m \alpha_j \lambda^j x^j (1 - y^j) =: C$. Since $1 - y^j = (1 - y) (1 + y + \ldots + y^{j-1})$, $j \geq 1$, then

$$\|1 - y^j\| \leq \|y - 1\| \sum_{l=0}^{j-1} \|y\|^l \leq \|y - 1\| \sum_{l=0}^{m-1} \|y\|^l$$
and then
\begin{equation}
||C|| \leq ||y - 1|| \sum_{l=0}^{m-1} ||y||^l \sum_{j=1}^{m} |\alpha_j| |\lambda|^j ||x||^j
= ||y - 1|| \sum_{l=0}^{m-1} ||y||^l \left( \sum_{j=0}^{m} |\alpha_j| |\lambda|^j ||x||^j - |\alpha_0| \right).
\end{equation}

Letting \( m \to \infty \) in (2.6), we also obtain the inequality (2.1) for \( k = 0 \). This proves the lemma. \( \square \)

**Corollary 2.1.** Let \( f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n \) be a function defined by power series with complex coefficients and convergent on the open disk \( D(0, R) \subset \mathbb{C}, \ R > 0 \) and \( x \in \mathcal{B} \). If \( ||x|| < 1 \), \( \lambda \in \mathbb{C} \) with \( |\lambda||x|| < R \), then we have the inequality
\[
\left| \tilde{f}(\lambda x)x^k - \tilde{f}(\lambda x^2) \right| \leq \frac{||x - 1||}{1 - ||x||} \left[ f_A (|\lambda||x||) - |\alpha_k| |\lambda|^k ||x||^k \right],
\]
for any \( k \in \mathbb{N}, \ k \geq 0 \).

We can state the following result.

**Theorem 2.1.** Let \( f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n \) be a function defined by power series with complex coefficients and convergent on the open disk \( D(0, R) \subset \mathbb{C}, \ R > 0 \) and \( x, y \in \mathcal{B} \) with \( xy = yx \). If \( \lambda, \mu \in \mathbb{C} \) are such that \( |\lambda||x|| < R \) and \( ||y|| \leq 1 \) then
\begin{equation}
\left| \tilde{f}(\lambda x) \tilde{f}(\mu) - \tilde{f}(\lambda xy) \right|
\leq \frac{||x - 1||}{1 - ||y||} \left[ f_{A^2} (|\lambda||x||) f_A (|\lambda||x||) - f_{A^2} (|\lambda||\mu||x||) \right],
\end{equation}
where \( f_{A^2} (\lambda) := \sum_{n=0}^{\infty} |\lambda|^n \).

**Proof.** Utilizing Lemma 2.1 we have
\[
\left| \tilde{f}(\lambda x) \left( \sum_{k=0}^{p} \alpha_k \mu^k y^k \right) - \left( \sum_{k=0}^{p} \alpha_k \mu^k \right) \tilde{f}(\lambda xy) \right|
= \left| \sum_{k=0}^{p} \alpha_k \mu^k \left( \tilde{f}(\lambda x) y^k - \tilde{f}(\lambda xy) \right) \right|
\leq \sum_{k=0}^{p} |\alpha_k| |\mu|^k \left| \tilde{f}(\lambda x) y^k - \tilde{f}(\lambda xy) \right|
\leq \sum_{k=0}^{p} \frac{||y - 1||}{1 - ||y||} \left[ f_A (|\lambda||x||) - |\alpha_k| |\lambda|^k ||x||^k \right] |\alpha_k| |\mu|^k
= \frac{||y - 1||}{1 - ||y||} \left[ f_A (|\lambda||x||) \sum_{k=0}^{p} |\alpha_k| |\mu|^k - \sum_{k=0}^{p} |\alpha_k| |\lambda|^k |\mu|^k ||x||^k \right].
\]
for any \( p \geq 0 \).

Since all the series that are involved in the inequality from above are convergent, then by letting \( p \to \infty \) we get the desired result (2.7).

**Corollary 2.2.** Let \( f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n \) be a function defined by power series with complex coefficients and convergent on the open disk \( D(0, R) \subset \mathbb{C} \), \( R > 0 \) and \( x \in \mathcal{B} \). If \( \lambda, \mu \in \mathbb{C} \) are such that \( |\mu|, |\lambda||x| < R \) and \( \|x\| < 1 \) then

\[
\| \tilde{f}(\lambda x) \tilde{f}(\mu x) - \tilde{f}(\mu \cdot 1) \tilde{f}(\lambda x^2) \| \leq \frac{\|x - 1\|}{1 - \|x\|} \left[ f_A(|\lambda||x|) f_A(|\mu|) - f_A^2(|\lambda| |\mu||x|) \right].
\]

**Remark 2.1.** If \( \mu = \lambda \), then we get the inequality for the Čebyšev functional

\[
\| \tilde{f}(\lambda x) \tilde{f}(\lambda y) - \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) \| \leq \frac{\|y - 1\|}{1 - \|y\|} \left[ f_A(|\lambda||x|) f_A(|\lambda|) - f_A^2(|\lambda|^2 |x|) \right],
\]

provided that \( x, y \in \mathcal{B} \) with \( xy = yx \), \( \lambda \in \mathbb{C} \) are such that \( |\lambda|, |\lambda||x| < R \) and \( \|y\| < 1 \). From (2.8) we have

\[
\| \left[ \tilde{f}(\lambda x) \right]^2 - \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda x) \| \leq \frac{\|x - 1\|}{1 - \|x\|} \left[ f_A(|\lambda||x|^2) f_A(|\lambda|) - f_A^2(|\lambda| |\mu||x|) \right].
\]

We can state now the second result.

**Theorem 2.2.** Let \( f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n \) be a power series that is convergent on the open disk \( D(0, R) \subset \mathbb{C} \), \( R > 0 \). If \( x, y \in \mathcal{B} \) with \( xy = yx \) and \( \|x\|, \|y\| \leq 1 \), then we have the inequalities

\[
\| \tilde{f}(\lambda x) \tilde{f}(\lambda y) - \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) \| \leq \frac{\sqrt{2}}{2} \left( \frac{\|x - 1\|}{1 - \|x\|} \right)^{\frac{1}{2}} |\lambda| f_A(|\lambda|) \left[ f_A(|\lambda|) g_A(|\lambda|) - h_A^2(|\lambda|) \right]^{\frac{1}{2}},
\]

where

\[
f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n, \ g_A(\lambda) := \sum_{n=0}^{\infty} n^4 |\alpha_n| \lambda^n, \ h_A(\lambda) := \sum_{n=0}^{\infty} n^2 |\alpha_n| \lambda^n
\]

and \( \lambda \in D(0, R) \).

Moreover, if the series \( s_0 := \sum_{n=0}^{\infty} |\alpha_n| \), \( s_2 := \sum_{n=0}^{\infty} n^2 |\alpha_n| \) and \( s_4 := \sum_{n=0}^{\infty} n^4 |\alpha_n| \) are convergent, then we have the inequalities

\[
\| \tilde{f}(\lambda x) \tilde{f}(\lambda y) - \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) \| \leq \frac{\sqrt{2}}{2} \left( \frac{\|x - 1\|}{1 - \|x\|} \right)^{\frac{1}{2}} |\lambda| f_A(|\lambda|^2) \left[ s_0 s_4 - s_2^2 \right]^{\frac{1}{2}},
\]

for any \( \lambda \in \mathbb{C} \) with \( |\lambda|, |\lambda|^2 < R \).
Proof. We observe that

\begin{equation}
B_m := \sum_{n,j=0}^{m} \alpha_n \alpha_j \lambda^n \lambda^j (x^n - x^j) (y^n - 1) \\
= \sum_{n,j=0}^{m} \alpha_n \alpha_j \lambda^n \lambda^j (x^n y^n - x^j y^n - x^n + x^j)
\end{equation}

\begin{equation}
= \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n (xy)^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n
\end{equation}

\begin{equation}
- \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n x^n + \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n
\end{equation}

\begin{equation}
= \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n (xy)^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n.
\end{equation}

Taking the norm and using the generalized triangle inequality we have:

\begin{equation}
\|B_m\| \leq \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \|x^n - x^j\| \|y^n - 1\| := C_m.
\end{equation}

Since \(y^n - 1 = (y - 1) (y^{n-1} + ... + 1)\), we have for \(\|y\| \leq 1\) that

\[\|y^n - 1\| \leq \|y - 1\| \|y^{n-1} + ... + 1\| \leq n \|y - 1\|.
\]

If \(n > j\), then for \(\|x\| \leq 1\),

\[\|x^n - x^j\| \leq \|x^j (x^{n-j} - 1)\| \leq \|x\|^j \|x^{n-j} - 1\| \leq (n - j) \|x - 1\|.
\]

Similarly, if \(j > n\), we have \(\|x^n - x^j\| \leq (j - n) \|x - 1\|\). Therefore for any \(n, j \in \mathbb{N}\) we have

\[\|x^n - x^j\| \leq |n - j| \|x - 1\|, \|x\| \leq 1.
\]

Utilising these facts we have

\begin{equation}
C_m \leq \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j n \|x - 1\| \|y - 1\| |n - j|.
\end{equation}

Further, observe that

\[\sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j n |n - j| = \frac{1}{2} \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n - j| (n + j)
\]

\[= \frac{1}{2} \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n^2 - j^2|,
\]
therefore
\[(2.12) \quad C_m \leq \frac{1}{2} \|x - 1\| \|y - 1\| \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n^2 - j^2| := D_m.\]

Using Cauchy-Bunyakovsky-Schwarz inequality we have
\[
\sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n^2 - j^2| \\
\leq \left( \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \right)^{\frac{1}{2}} \left( \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j (n^2 - j^2)^2 \right)^{\frac{1}{2}} \\
= \left( \sum_{n=0}^{m} |\alpha_n| |\lambda|^n \right) (E_m)^{\frac{1}{2}},
\]
where
\[(2.13) \quad E_m := \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j (n^2 - j^2)^2 \\
= \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j (n^4 - 2n^2j^2 + j^4)^2 \\
= 2 \left[ \sum_{n=0}^{m} |\alpha_n| |\lambda|^n \sum_{n=0}^{m} |\alpha_n| |\lambda|^n n^4 - \left( \sum_{n=0}^{m} |\alpha_n| |\lambda|^n \right)^2 \right] .
\]

Making use of (2.9)-(2.13) we get for \(\|x\|, \|y\| \leq 1\) that
\[(2.14) \quad \left\| \sum_{i,j} \alpha_i \lambda^i \sum_{n} \alpha_n (\lambda x y)^n - \sum_{j} \alpha_j (\lambda x)^j \sum_{n} \alpha_n (\lambda y)^n \right\| \\
\leq \frac{1}{2} \|x - 1\| \|y - 1\| \sum_{n=0}^{m} |\alpha_n| |\lambda|^n \\
\times \left[ \sum_{n=0}^{m} |\alpha_n| |\lambda|^n \sum_{n=0}^{m} n^4 |\alpha_n||\lambda|^n - \left( \sum_{n=0}^{m} n^2 |\alpha_n| |\lambda|^n \right)^2 \right]^{\frac{1}{2}},
\]
for any \(m \in \mathbb{N}\).

Since all the series involved in (2.14) are convergent, then by letting \(m \to \infty\) in (2.14) we deduce the desired result
\[
\left\| \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda x y) - \tilde{f}(\lambda x) \tilde{f}(\lambda y) \right\| \\
\leq \frac{\sqrt{2}}{2} \|x - 1\| \|y - 1\| f_A(|\lambda|) \left[ f_A(|\lambda|) g_A(|\lambda|) - h_A^2(|\lambda|) \right]^{\frac{1}{2}} .
\]
Using Cauchy-Bunyakovsky-Schwarz inequality we also have
\[
\sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n^2 - j^2|
\leq \left( \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^{2n} |\lambda|^{2j} \right)^{\frac{1}{2}} \left( \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |n^2 - j^2|^2 \right)^{\frac{1}{2}}
= \left( \sum_{n=0}^{m} |\alpha_n| |\lambda|^{2n} \right) \left\{ 2 \left[ \sum_{n=0}^{m} |\alpha_n| \sum_{j=0}^{m} j^4 |\alpha_j| - \left( \sum_{n=0}^{m} n^2 |\alpha_n| \right)^2 \right] \right\} \frac{1}{2}.
\]

Making use of this inequality we then obtain in a similar way the second part of the theorem. The details are omitted. □

3. SOME EXAMPLES

Consider the function \( f : D(0,1) \rightarrow \mathbb{C} \) defined by \( f(\lambda) = (1 - \lambda)^{-1} = \sum_{k=0}^{\infty} \lambda^k \). Then
\[
f_{A^2}(\lambda) := \sum_{n=0}^{\infty} \lambda^n = (1 - \lambda)^{-1}
\]
and by (2.7), we have for \( x, y \in B \) with \( xy = yx, \|y\| < 1 \) and \( \lambda, \mu \in \mathbb{C} \) with \(|\mu|, |\lambda| \|x\| < 1\) that
(3.1) \[
\| (1 - \lambda x)^{-1} - (1 - \mu)^{-1} (1 - \lambda xy)^{-1} \| \leq \frac{\|y - 1\|}{1 - \|y\|} \left[ (1 - |\lambda| \|x\|)^{-1} (1 - |\mu|)^{-1} - (1 - |\lambda| |\mu| \|x\|)^{-1} \right].
\]
In particular, if \(|\lambda|, \|x\| < 1\), then
(3.2) \[
\| (1 - \lambda x)^{-1} - (1 - \lambda)^{-1} (1 - \lambda xy)^{-1} \| \leq \frac{\|y - 1\|}{1 - \|y\|} \left[ (1 - |\lambda| \|x\|)^{-1} (1 - |\lambda|)^{-1} - (1 - |\lambda|^2 \|x\|)^{-1} \right].
\]
We also have for \(|\lambda|, \|x\| < 1\) that
(3.3) \[
\| (1 - \lambda x)^{-2} - (1 - \lambda)^{-1} (1 - \lambda x^2)^{-1} \| \leq \frac{\|x - 1\|}{1 - \|x\|} \left[ (1 - |\lambda| \|x\|)^{-1} (1 - |\lambda|)^{-1} - (1 - |\lambda|^2 \|x\|)^{-1} \right].
\]

If we consider the function \( f(\lambda) = (1 + \lambda)^{-1} = \sum_{k=0}^{\infty} (-1)^k \lambda^k \), then the inequalities (3.1)-(3.3) also hold with “+” instead of “−” in the left hand side expressions such as \((1 - \lambda x)^{-1}\) etc.
We consider the modified Bessel function functions of the first kind

\[ I_\nu(\lambda) := \left( \frac{1}{2} \lambda \right)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4} \lambda^2)^k}{k! \Gamma(\nu + k + 1)}, \lambda \in \mathbb{C} \]

where \( \Gamma \) is the Gamma function and \( \nu \) is a real number. An integral formula is

\[ I_\nu(\lambda) = \frac{1}{\pi} \int_0^\pi e^{\lambda \cos \theta} \cos (\nu \theta) \, d\theta - \frac{\sin (\nu \pi)}{\pi} \int_0^\infty e^{-\lambda \cosh t - \nu t} \, dt, \]

which simplifies for \( \nu \) an integer \( n \) to

\[ I_n(\lambda) = \frac{1}{\pi} \int_0^\pi e^{\lambda \cos \theta} \cos (n \theta) \, d\theta. \]

For \( n = 0 \) we have

\[ I_0(\lambda) = \frac{1}{\pi} \int_0^\pi e^{\lambda \cos \theta} \, d\theta = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \lambda^k, \lambda \in \mathbb{C}. \]

Now, if we consider the exponential function \( f(\lambda) = \exp (\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k, \) then for \( \rho > 0 \) we have

\[ f_{\lambda^2}(\rho) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \rho^k = I_0(2\sqrt{\rho}). \]

Making use of the inequality (2.7), we have for \( x, y \in \mathcal{B} \) with \( xy = yx, \|y\| < 1 \) and \( \lambda, \mu \in \mathbb{C} \) that

\[ \| \exp (\lambda x + \mu \cdot 1) - \exp (\lambda (xy + \mu \cdot 1)) \| \leq \frac{1 - \|y\|}{1 - \|y\|} \left[ \exp (|\lambda| \|x\| + |\mu|) - I_0 \left( 2|\lambda| \|x\| \right) \right]. \]

In particular, we have

\[ \| \exp (\lambda (x + y)) - \exp (\lambda (xy + 1)) \| \leq \frac{1 - \|y - 1\|}{1 - \|y\|} \left[ \exp (|\lambda| (\|x\| + 1)) - I_0 \left( 2|\lambda| \sqrt{\|x\|} \right) \right]. \]

We also have for \( \|x\| < 1 \)

\[ \| \exp (2\lambda x) - \exp (\lambda (x^2 + 1)) \| \leq \frac{\|x - 1\|}{1 - \|x\|} \left[ \exp (|\lambda| (\|x\| + 1)) - I_0 \left( 2|\lambda| \sqrt{\|x\|} \right) \right] \]

for any \( \lambda \in \mathbb{C} \). If we take \( \lambda = 1 \), then we get

\[ \| \exp (2x) - \exp (x^2 + 1) \| \leq \frac{\|x - 1\|}{1 - \|x\|} \left[ \exp (\|x\| + 1) - I_0 \left( 2\sqrt{\|x\|} \right) \right] \]

for \( \|x\| < 1 \).
REFERENCES


1Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia. E-mail address: sever.dragomir@vu.edu.au URL: http://rgmia.org/dragomir

2School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

3Mathematics and Statistics, Banat University of Agricultural Sciences and Veterinary Medicine Timișoara, 119 Calea Aradului, 300645 Timișoara, România

4Department of Mathematics, West University of Timișoara, B-dul V. Pârvan 4, 1900-Timișoara, România