BAZILEVIČ p -VALENT FUNCTIONS ASSOCIATED WITH GENERALIZED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The aim of this paper is to introduce and study a new class of Bazilevič p -valent function of order β by using the subordination concept between this function and a generalized derivative operator. Some interesting properties are also obtained.

1. INTRODUCTION

Let $\mathcal{A}_p$ the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (z \in \mathbb{U}, p \in \mathbb{N}),$$

which are analytic and p -valent in the unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. For $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that f is subordinate to g if there exists an analytic function ω in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. We denote this subordination by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Now, we define new generalized differential operator $D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)$ of analytic p -valent functions as follows.

Definition 1.1. Let f be in the class $\mathcal{A}_p$, then we have

$$D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z) = z^p + \sum_{n=1}^{\infty} \left[\frac{p + (\lambda_1 + \lambda_2)n + b}{p + \lambda_2 n + b} \right]^m \frac{(a_1)_n \cdots (a_r)_n a_{p+n} z^{p+n}}{(b_1)_n \cdots (b_s)_n n!},$$

where $p \in \mathbb{N}$, $m, b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ ($i = 1, \dots, r$, $q = 1, \dots, s$) and $r \leq s + 1$; $r, s \in \mathbb{N}_0$, and $(x)_n$ is the Pochhammer

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symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1, & n = 0, \\ x(x+1) \cdots (x+n-1), & n = \{1, 2, 3, \dots\}. \end{cases}$$

It follows from the above definition that

$$(1.1) \quad (p + \lambda_2 n + b) D_{\lambda_1, \lambda_2, p}^{m+1, b}(a_i, b_q) f(z) = (p + \lambda_2 n - p\lambda_1 + b) D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f(z) + p\lambda_1 z \left(D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f(z) \right)'.$$

Remark 1.1. It should be remarked that the linear operator $\mathcal{D}_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f(z)$ is a generalization of many operators considered earlier. Let us see some of the examples:

- For $\lambda_2 = b = 0$, the operator $\mathcal{D}_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f$ reduces to the operator was given by Selvaraj and Karthikeyan [18].
- For $m = 0$, the operator $\mathcal{D}_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f$ reduces to the operator was given by El-Ashwah [9].
- For $m = 0$ and $p = 1$, the operator $\mathcal{D}_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f$ reduces to the well-known operator introduced by Dziok and Srivastava [8].
- For $m = 0, r = 2, s = 1$ and $p = 1$, we obtain the operator which was given by Hohlov [12].
- For $r = 1, s = 0, a_1 = 1, \lambda_1 = 1, \lambda_2 = b = 0$ and $p = 1$, we get the operator introduced by Sălăgean [17].
- For $r = 1, s = 0, a_1 = 1, \lambda_2 = b = 0$ and $p = 1$, we get the generalized Sălăgean derivative operator introduced by Al-Oboudi [1].
- For $m = 0, r = 1, s = 0, a_1 = \delta + 1$ and $p = 1$, we obtain the operator introduced by Ruscheweyh [16].
- For $r = 1, s = 0, a_1 = \delta + 1$ and $p = 1$, we obtain the operator studied by El-Yagubi and Darus [10], [11].
- For $m = 0, r = 2$ and $s = 1, a_2 = 1$ and $p = 1$, we obtain the operator studied by Carlson and Shaffer [4].
- For $r = 1, s = 0, a_1 = 1, \lambda_2 = 0$ and $p = 1$, we get the operator introduced by Cătăs [5].

By making use of the differential operator $D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)$ and the principle of subordination between Bazilevič p -valent functions, we introduce and investigate the following subclass of $\mathcal{A}_p$.

Definition 1.2. Let $f \in \mathcal{A}_p$ is said to be in the class $S_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q, \gamma, \beta, A, B)$ if it satisfies the following subordination condition

$$(1 - \gamma) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f(z)}{z^p} \right)^\beta + \gamma \left(\frac{D_{\lambda_1, \lambda_2, p}^{m+1, b}(a_i, b_q) f(z)}{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f(z)} \right) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f(z)}{z^p} \right)^\beta \prec \frac{1 + Az}{1 + Bz} \quad (p \in \mathbb{N}, z \in \mathbb{U}),$$

where $\gamma \in \mathbb{C}$, $\Re(\beta) > 0$, $m, b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ ($i = 1, \dots, r$, $q = 1, \dots, s$), $r \leq s + 1$; $r, s \in \mathbb{N}_0$, $-1 \leq B \leq 1$ and $A \neq B \in \mathbb{N}_0$.

Clearly, if we put $p = 1, m = b = 0, \lambda_1 = \lambda_2 = 1, r = 1, s = 0$ and $a_1 = 1$ in the class $S_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q, \gamma, \beta, A, B)$, then we obtain the class of Bazilevič functions studied by Liu and Noor [14].

To prove our main result, the following lemmas are required.

Lemma 1.1. [1] *Let $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $h(0) = 1$. Assume also the function $\wp(z)$ given by*

$$(1.2) \quad \wp(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \dots$$

be analytic in $\mathbb{U}$. If

$$\wp(z) + \frac{z\wp'(z)}{\delta} \prec h(z) \quad \{\Re(\delta) \geq 0; \delta \neq 0, z \in \mathbb{U}\},$$

then

$$(1.3) \quad \wp(z) \prec \psi(z) = \frac{\delta}{n} z^{-(\frac{\delta}{n})} \int_0^z t^{(\frac{\delta}{n})-1} h(t) dt \prec h(z) \quad (z \in \mathbb{U}),$$

and ψ is the best dominant.

Lemma 1.2. [19] *Let $q(z)$ be a convex univalent function in $\mathbb{U}$ and let $\sigma, \eta \in \mathbb{C}$ with*

$$\Re \left(1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left(\frac{\sigma}{\eta} \right) \right\}.$$

If the function p is analytic in $\mathbb{U}$ and

$$\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 1.3. [15] *Let q be convex univalent in $\mathbb{U}$ and $k \in \mathbb{C}$. Further assume that $\Re(k) > 0$. If $p(z) \in H[q(0), 1] \cap Q$ and $p(z) + k z p'(z)$ is univalent in $\mathbb{U}$, then*

$$q(z) + k z q'(z) \prec p(z) + k z p'(z)$$

implies $q(z) \prec p(z)$ and $q(z)$ is the best subdominant.

2. MAIN RESULTS

In what follows we aim to study some interesting properties of the class $S_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q, \gamma, \beta, A, B)$.

Theorem 2.1. Let $f(z) \in S_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q, \gamma, \beta, A, B)$, then

$$\begin{aligned} \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta &\prec \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} - 1} du \\ &\prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \end{aligned}$$

where $\gamma \in \mathbb{C}$, $\Re(\beta) > 0$, $p \in \mathbb{N}$, $m, b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ ($i = 1, \dots, r, q = 1, \dots, s$), $r \leq s + 1$; $r, s \in \mathbb{N}_0$, $-1 \leq B \leq 1$ and $A \neq B \in \mathbb{N}_0$.

Proof. Let

$$(2.1) \quad p(z) = \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \quad (z \in \mathbb{U}).$$

Then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. By taking the derivative in the both sides in equality (2.1) and using (1.1), we get

$$\begin{aligned} (1 - \gamma) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta + \gamma \left(\frac{D_{\lambda_1, \lambda_2, p}^{m+1, b}(a_i, b_q)f(z)}{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)} \right) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \\ (2.2) \quad = p(z) + \frac{p\lambda_1\gamma}{\beta(p + \lambda_2 n + b)} zp'(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}). \end{aligned}$$

By applying Lemma 1.1 in the last equation, we get

$$\begin{aligned} \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta &\prec \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} z^{-\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma}} \int_0^z t^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} - 1} \frac{1 + At}{1 + Bt} dt \\ (2.3) \quad &= \frac{\zeta}{n} \int_0^1 u^{\frac{\zeta}{n} - 1} \frac{1 + Azu}{1 + Bzu} du \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \end{aligned}$$

where $\zeta = \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 \gamma}$. □

Theorem 2.2. Let $q(z)$ be univalent in $\mathbb{U}$. Suppose also that $q(z)$ satisfies

$$(2.4) \quad \Re \left(1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left(\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 \gamma} \right) \right\}.$$

If $f(z) \in \mathcal{A}_p$ is satisfying the following subordination

$$\begin{aligned} (1 - \gamma) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta + \gamma \left(\frac{D_{\lambda_1, \lambda_2, p}^{m+1, b}(a_i, b_q)f(z)}{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)} \right) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \\ (2.5) \quad \prec q(z) + \frac{p\gamma\lambda_1}{(p + \lambda_2 n + b)\beta} zq'(z), \end{aligned}$$

then $\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p}\right)^\beta \prec q(z)$, and $q(z)$ is the best dominant.

Proof. Let $p(z)$ be defined by (2.1). We know that (2.2) is true. Combining (2.2) and (2.5), we see that

$$(2.6) \quad p(z) + \frac{p\lambda_1\gamma}{\beta(p + \lambda_2n + b)}zp'(z) \prec q(z) + \frac{p\lambda_1\gamma}{\beta(p + \lambda_2n + b)}zq'(z).$$

By using Lemma 1.2 and (2.6), we get the assertion of Theorem 2.2. $\square$

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 2.2, we get the following result.

Corollary 2.1. *Let $\gamma \in \mathbb{C}$ and $-1 \leq B < A \leq 1$. Suppose also that $\frac{1+Az}{1+Bz}$ satisfies the condition (2.4). If $f(z) \in \mathcal{A}_p$ satisfies the following subordination*

$$(1 - \gamma) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p}\right)^\beta + \gamma \left(\frac{D_{\lambda_1, \lambda_2, p}^{m+1, b}(a_i, b_q)f(z)}{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}\right) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}f(z)}{z^p}\right)^\beta \\ \prec \frac{1 + Az}{1 + Bz} + \frac{p\lambda_1\gamma(A - B)z}{(p + \lambda_2n + b)\beta(1 + Bz)^2},$$

then $\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p}\right)^\beta \prec \frac{1+Az}{1+Bz}$ and $\frac{1+Az}{1+Bz}$ is the best dominant.

Theorem 2.3. *Let $q(z)$ be convex univalent in $\mathbb{U}$, $\gamma \in \mathbb{C}$, with $\Re(\gamma) > 0$. Also let $\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p}\right)^\beta \in H[q(0), 1] \cap Q$ and*

$$(1 - \gamma) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p}\right)^\beta + \gamma \left(\frac{D_{\lambda_1, \lambda_2, p}^{m+1, b}(a_i, b_q)f(z)}{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}\right) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p}\right)^\beta$$

be univalent in $\mathbb{U}$. If

$$q(z) + \frac{p\lambda_1\gamma}{\beta(p + \lambda_2n + b)}zq'(z) \prec (1 - \gamma) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p}\right)^\beta \\ + \gamma \left(\frac{D_{\lambda_1, \lambda_2, p}^{m+1, b}(a_i, b_q)f(z)}{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}\right) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p}\right)^\beta,$$

then $q(z) \prec \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p}\right)^\beta$ and $q(z)$ is the best subdominant.

Proof. Let $p(z)$ be defined by (2.1). Then

$$\begin{aligned} q(z) + \frac{p\lambda_1\gamma}{\beta(p + \lambda_2n + b)}zq'(z) &\prec (1 - \gamma) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \\ &\quad + \gamma \left(\frac{D_{\lambda_1, \lambda_2, p}^{m+1, b}(a_i, b_q)f(z)}{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)} \right) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \\ &= p(z) + \frac{p\lambda_1\gamma}{\beta(p + \lambda_2n + b)}zq'(z). \end{aligned}$$

An application of Lemma 1.3 yields the assertion of Theorem 2.3. $\square$

Corollary 2.2. *Let $q(z)$ be convex univalent in $\mathbb{U}$ and $-1 \leq B < A \leq 1, \gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$. Also let $\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \in H[q(0), 1] \cap Q$ and*

$$(1 - \gamma) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta + \gamma \left(\frac{D_{\lambda_1, \lambda_2, p}^{m+1, b}(a_i, b_q)f(z)}{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)} \right) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta$$

be univalent in $\mathbb{U}$. If

$$\begin{aligned} \frac{1 + Az}{1 + Bz} + \frac{p\lambda_1\gamma(A - B)z}{(p + \lambda_2n + b)\beta(1 + Bz)^2} &\prec (1 - \gamma) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \\ &\quad + \gamma \left(\frac{D_{\lambda_1, \lambda_2, p}^{m+1, b}(a_i, b_q)f(z)}{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)} \right) \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta, \end{aligned}$$

then $\frac{1+Az}{1+Bz} \prec \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta$ and $\frac{1+Az}{1+Bz}$ is the best subdominant.

Theorem 2.4. *Let $f(z) \in S_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q, \gamma, \beta, A, B)$, then*

$$\begin{aligned} \inf_{z \in \mathbb{U}} \Re \left\{ \frac{(p + \lambda_2n + b)\beta}{p\lambda_1n\gamma} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{(p + \lambda_2n + b)\beta}{p\lambda_1n\gamma} - 1} du \right\} \\ < \Re \left\{ \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \right\} \\ < \sup_{z \in \mathbb{U}} \Re \left\{ \frac{(p + \lambda_2n + b)\beta}{p\lambda_1n\gamma} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{(p + \lambda_2n + b)\beta}{p\lambda_1n\gamma} - 1} du \right\}. \end{aligned}$$

where $\gamma \in \mathbb{C}, \Re(\beta) > 0, p \in \mathbb{N}, m, b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda_2 \geq \lambda_1 \geq 0, a_i \in \mathbb{C}, b_q \in \mathbb{C} \setminus \{0, -1, -2, \dots\} (i = 1, \dots, r, q = 1, \dots, s), r \leq s + 1; r, s \in \mathbb{N}_0, -1 \leq B \leq 1$ and $A \neq B \in \mathbb{N}_0$.

Proof. Suppose that $f(z) \in S_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q, \gamma, \beta, A, B)$, then from Theorem 2.1 we know that

$$\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \prec \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} - 1} du.$$

Therefore, from the definition of the subordination, we have

$$\Re \left\{ \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \right\} > \inf_{z \in \mathbb{U}} \Re \left\{ \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} - 1} du \right\}$$

and

$$\begin{aligned} & \Re \left\{ \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \right\} \\ & < \sup_{z \in \mathbb{U}} \Re \left\{ \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} - 1} du \right\}. \quad \square \end{aligned}$$

Corollary 2.3. Let $\gamma \in \mathbb{C}$, $\Re(\beta) > 0$, $p \in \mathbb{N}$, $m, b \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ ($i = 1, \dots, r, q = 1, \dots, s$), $r \leq s + 1$; $r, s \in \mathbb{N}_0$ and $-1 \leq B < A \leq 1$. If $f(z) \in S_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q, \gamma, \beta, A, B)$. then

$$\begin{aligned} (2.7) \quad & \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} \int_0^1 \frac{1 - Au}{1 - Bu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} - 1} du < \Re \left\{ \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \right\} \\ & < \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} - 1} du \quad (z \in \mathbb{U}). \end{aligned}$$

Proof. Suppose that $f(z) \in S_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q, \gamma, \beta, A, B)$, then from Theorem 2.1 we know that

$$\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \prec \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} - 1} du.$$

Therefore, from the definition of the subordination and $A > B$, we have

$$\begin{aligned} \Re \left\{ \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q)f(z)}{z^p} \right)^\beta \right\} & > \inf_{z \in \mathbb{U}} \Re \left\{ \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} - 1} du \right\} \\ & \geq \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} \int_0^1 \inf_{z \in \mathbb{U}} \left\{ \frac{1 + Azu}{1 + Bzu} \right\} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} - 1} du \\ & > \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} \int_0^1 \frac{1 - Au}{1 - Bu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n\gamma} - 1} du, \end{aligned}$$

and

$$\begin{aligned} \Re \left\{ \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f(z)}{z^p} \right)^\beta \right\} &< \sup_{z \in \mathbb{U}} \Re \left\{ \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} - 1} du \right\} \\ &\leq \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} \int_0^1 \sup_{z \in \mathbb{U}} \left\{ \frac{1 + Azu}{1 + Bzu} \right\} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} - 1} du \\ &< \frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} - 1} du, \end{aligned}$$

which proves the result. $\square$

Corollary 2.4. Let $\gamma \in \mathbb{C}$, $\Re(\beta) > 0$, $p \in \mathbb{N}$, $m, b \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ ($i = 1, \dots, r, q = 1, \dots, s$), $r \leq s + 1; r, s \in \mathbb{N}_0$ and $-1 \leq B < A \leq 1$. If $f(z) \in S_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q, \gamma, \beta, A, B)$, then

$$\begin{aligned} \left(\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} \int_0^1 \frac{1 - Au}{1 - Bu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} - 1} du \right)^{\frac{1}{2}} &< \Re \left\{ \left(\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f(z)}{z^p} \right)^\beta \right)^{\frac{1}{2}} \right\} \\ (2.8) \quad &< \left(\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} - 1} du \right)^{\frac{1}{2}} \quad (z \in \mathbb{U}). \end{aligned}$$

Proof. Suppose that $f(z) \in S_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q, \gamma, \beta, A, B)$, then from Theorem 2.1, we have

$$\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f(z)}{z^p} \right)^\beta \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

Since $-1 \leq B < A \leq 1$, we have

$$0 \leq \frac{1 - A}{1 - B} < \Re \left\{ \left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f(z)}{z^p} \right)^\beta \right\} < \frac{1 + A}{1 + B}.$$

Thus, from the inequality (2.7), we can get the inequality (2.8). $\square$

Corollary 2.5. Let $\gamma \in \mathbb{C}$, $\Re(\beta) > 0$, $p \in \mathbb{N}$, $m, b \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $a_i \in \mathbb{C}$, $b_q \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ ($i = 1, \dots, r, q = 1, \dots, s$), $r \leq s + 1; r, s \in \mathbb{N}_0$ and $-1 \leq A < B \leq 1$. If $f(z) \in S_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q, \gamma, \beta, A, B)$, then

$$\begin{aligned} \left(\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} \int_0^1 \frac{1 + Au}{1 + Bu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} - 1} du \right)^{\frac{1}{2}} &< \Re \left\{ \left(\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, b}(a_i, b_q) f(z)}{z^p} \right)^\beta \right)^{\frac{1}{2}} \right\} \\ &< \left(\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} \int_0^1 \frac{1 - Au}{1 - Bu} u^{\frac{(p + \lambda_2 n + b)\beta}{p\lambda_1 n \gamma} - 1} du \right)^{\frac{1}{2}} \quad (z \in \mathbb{U}). \end{aligned}$$

Proof. By applying similar method as in Corollary 2.4, we get the required result. $\square$

Note that other work related to classes of Bazilevič functions can be found in [2], [3], [6], [7], [13].

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