CLUSTERS AND VARIOUS VERSIONS OF WIENER-TYPE INVARIANTS

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ABSTRACT. The Wiener type invariant $W^{(\lambda)}(G)$ of a simple connected graph $G$ is defined as the sum of the terms $d(u, v)^{\lambda}$ over all unordered pairs $\{u, v\}$ of vertices of $G$, where $d(u, v(G))$ denotes the distance between the vertices $u$ and $v$ in $G$ and $\lambda$ is an arbitrary real number. The cluster $G_1\{G_2\}$ of a graph $G_1$ and a rooted graph $G_2$ is the graph obtained by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, and by identifying the root vertex of the $i$-th copy of $G_2$ with the $i$-th vertex of $G_1$, for $i = 1, 2, \ldots, |V(G_1)|$. In this paper, we study the behavior of three versions of Wiener type invariant under the cluster product. Results are applied to compute several distance-based topological invariants of bristled and bridge graphs by specializing components in clusters.

1. Introduction

In this paper, we consider connected finite graphs without any loops or multiple edges. Let $G$ be such a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $\deg_G(u)$ the degree of the vertex $u$ in $G$ and by $V(e)$ the set of two end vertices of the edge $e$ of $G$. The distance $d(u, v | G)$ between the vertices $u$ and $v$ of $G$ is the length of any shortest path in $G$ connecting them.

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modelled by means of molecular-graph-based structure-descriptors, which are also referred to as topological indices [11, 25]. The vertex version of the Wiener index is the first reported distance-based topological index which was introduced in 1947 by Wiener [26], who used it for modeling the shape of organic molecules and for calculating several of their physico-chemical properties. The Wiener index $W(G)$ of

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is defined as $W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v|G)$. Details on the Wiener index can be found in [6, 9, 10, 18, 20, 28].

The definition of the Wiener index can be generalized by the following definition [10, 12]

$$W^{(\lambda)}(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v|G)^{\lambda},$$

where $\lambda$ is an arbitrary real number. The index $W^{(\lambda)}(G)$ is called the vertex Wiener type invariant of $G$. It is easy to see that, $W^{(0)}(G) = \binom{|V(G)|}{2}$ and $W^{(1)}(G) = W(G)$. The so-called hyper-Wiener index $WW(G)$ of $G$ [22] was shown [19] to be equal to $\frac{1}{2}W^{(2)}(G) + \frac{1}{2}W^{(1)}(G)$, and the so-called Tratch-Stankevich-Zefirov index $TSZ(G)$ of $G$ [24] was shown [18] to be equal to $\frac{1}{6}W^{(3)}(G) + \frac{1}{2}W^{(2)}(G) + \frac{1}{3}W^{(1)}(G)$. Recall that, the hyper-Wiener and the Tratch-Stankevich-Zefirov indices were originally defined in terms completely different from the presently considered Wiener-type invariants; for details see [22, 24].

Edge versions of the Wiener index based on distance between all pairs of edges of a graph were introduced in 2009 [5, 15, 17]. Two possible distances between the edges $e = uv$ and $f = zt$ of a graph $G$ can be considered. The first distance is denoted by $d_0(e, f|G)$ and defined as

$$d_0(e, f|G) = \begin{cases} d_1(e, f|G) + 1, & e \neq f, \\ 0, & e = f, \end{cases}$$

where $d_1(e, f|G) = \min\{d(u, z|G), d(u, t|G), d(v, z|G), d(v, t|G)\}$. It is easy to see that, $d_0(e, f|G) = d(e, f|L(G))$, where $L(G)$ is the line graph of $G$.

The second distance is denoted by $d_4(e, f|G)$ and defined as

$$d_4(e, f|G) = \begin{cases} d_2(e, f|G), & e \neq f, \\ 0, & e = f, \end{cases}$$

where $d_2(e, f|G) = \max\{d(u, z|G), d(u, t|G), d(v, z|G), d(v, t|G)\}$. Related to these two distances, two edge versions of the Wiener index can be defined. The first and second edge-Wiener indices of $G$ are denoted by $W_{e_0}(G)$ and $W_{e_4}(G)$, respectively and defined as

$$W_{e_i}(G) = \sum_{\{e,f\} \subseteq E(G)} d_i(e, f|G), \quad i \in \{0, 4\}.$$  

Obviously, $W_{e_0}(G) = W(L(G))$. For more information on the edge-Wiener indices see [1, 3, 21], and especially the recent survey [14].

The definitions of the edge-Wiener indices can be generalized by the following definition [16]

$$W^{(\lambda)}_{e_i}(G) = \sum_{\{e,f\} \subseteq E(G)} d_i(e, f|G)^{\lambda}, \quad i \in \{0, 4\},$$

where $\lambda$ is an arbitrary real number. The indices $W^{(\lambda)}_{e_0}(G)$ and $W^{(\lambda)}_{e_4}(G)$ are called the first and the second edge-Wiener type invariants of $G$, respectively. It is easy
to see that, $W_{ve}^{(0)}(G) = \left(\frac{1}{2}|E(G)|\right)$ and $W_{ve}^{(1)}(G) = W_{ve}(G)$, $i \in \{0, 4\}$. The first and the second edge hyper-Wiener indices of $G$ are denoted by $WW_{ve}(G)$ and $WW_{e}(G)$, respectively and defined as [16]

$$WW_{ve}(G) = \frac{1}{2}W_{ve}^{(1)}(G) + \frac{1}{2}W_{ve}^{(2)}(G), \ i \in \{0, 4\}.$$ 

In analogy with definitions of the vertex version and edge versions of the Wiener index, vertex-edge versions of the Wiener index were introduced based on distance between vertices and edges in a graph [4, 17]. Two possible distances between a vertex $u$ and an edge $e = ab$ of a graph $G$ can be considered. The first distance is denoted by $D_1(u, e \mid G)$ and defined as

$$D_1(u, e \mid G) = \min\{d(u, a \mid G), d(u, b \mid G)\},$$

and the second one is denoted by $D_2(u, e \mid G)$ and defined as

$$D_2(u, e \mid G) = \max\{d(u, a \mid G), d(u, b \mid G)\}.$$ 

Corresponding to these two distances, two vertex-edge versions of the Wiener index can be introduced. The first and the second vertex-edge Wiener indices of $G$ are denoted by $W_{ve_1}(G)$ and $W_{ve_2}(G)$, respectively and defined as

$$W_{ve_1}(G) = \sum_{u \in V(G)} \sum_{e \in E(G)} D_1(u, e \mid G), \ i \in \{1, 2\}.$$ 

The definitions of the vertex-edge Wiener indices can be generalized as follows

$$W_{ve_1}^{(\lambda)}(G) = \sum_{u \in V(G)} \sum_{e \in E(G); u \notin V(e)} D_1(u, e \mid G)^\lambda,$$

$$W_{ve_2}^{(\lambda)}(G) = \sum_{u \in V(G)} \sum_{e \in E(G)} D_2(u, e \mid G)^\lambda,$$

where $\lambda$ is an arbitrary real number. The indices $W_{ve_1}(G)$ and $W_{ve_2}(G)$ are called the first and the second vertex-edge Wiener type invariants of $G$, respectively. Note that, if $\lambda$ is a positive number then $W_{ve_1}^{(\lambda)}(G) = \sum_{u \in V(G)} \sum_{e \in E(G)} D_1(u, e \mid G)^\lambda$. Obviously, $W_{ve_1}^{(0)}(G) = |E(G)|(|V(G)| - 2)$, $W_{ve_1}^{(1)}(G) = |V(G)||E(G)|$, and $W_{ve_1}^{(1)}(G) = W_{ve_1}(G)$, $i \in \{1, 2\}$. We denote the first and the second vertex-edge hyper-Wiener indices of $G$ by $WW_{ve_1}(G)$ and $WW_{ve_2}(G)$, respectively and define as

$$WW_{ve_1}(G) = \frac{1}{2}W_{ve_1}(G) + \frac{1}{2}W_{ve_1}^{(2)}(G), \ i \in \{1, 2\}.$$ 

It is well-known that many graphs of general, and in particular of chemical interest, arise from simpler graphs via various graph operations sometimes known as graph products. It is, hence, important to understand how certain invariants of such graph operations are related to the corresponding invariants of their components. In this paper, we compute the vertex version, edge versions, and vertex-edge versions of the Wiener type invariant for an important graph product called cluster. Results are
applied for $t$-fold bristled graphs and bridge graphs by specializing components in clusters. We encourage the reader to consult [1, 2, 3, 7, 8, 13, 23, 27, 28] for more information on computing topological indices of graph products.

2. Main Results

The cluster $G_1\{G_2\}$ of a graph $G_1$ and a rooted graph $G_2$ is the graph obtained by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, and by identifying the root vertex of the $i$-th copy of $G_2$ with the $i$-th vertex of $G_1$, for $i = 1, 2, \ldots, |V(G_1)|$. Some topological indices of clusters have been computed previously [2, 8, 28]. In this section, we determine the vertex Wiener, edge-Wiener, and vertex-edge Wiener-type invariants of the cluster $G_1\{G_2\}$. Throughout this section, we denote the root vertex of $G_2$ by $w$, the degree of $w$ in $G_2$ by $\omega$, and the copy of $G_2$ whose root is identified with the vertex $u \in V(G_1)$ by $G_2^u$. Also for $i \in \{1, 2\}$, we denote the order and size of the graph $G_i$ by $n_i$ and $m_i$, respectively.

2.1. Wiener type invariant of clusters. In this section, we determine the vertex Wiener type invariant of the cluster $G_1\{G_2\}$.

Let $\lambda$ be a real number. For $u \in V(G)$, we define

$$d^{(\lambda)}(u \mid G) = \sum_{v \in V(G) - \{u\}} d(u, v)^\lambda.$$ 

It is easy to check that, $d^{(0)}(u \mid G) = |V(G)| - 1$.

Theorem 2.1. If $\lambda$ is a positive integer, then the vertex Wiener type invariant of $G_1\{G_2\}$ is given by

$$W^{(\lambda)}(G_1\{G_2\}) = W^{(\lambda)}(G_1) + n_1W^{(\lambda)}(G_2) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \left[ 2d^{(\lambda-i)}(w \mid G_2) \right. + \left. \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w \mid G_2) d^{(\lambda-i-j)}(w \mid G_2) \right].$$

Proof. We partition the sum in the definition of $W^{(\lambda)}(G_1\{G_2\})$ into four sums as follows.

The first sum $S_1$ consists of contributions to $W^{(\lambda)}(G_1\{G_2\})$ of pairs of vertices from $G_1$. For vertices $x, y \in V(G_1)$, $d(x, y \mid G_1\{G_2\}) = d(x, y \mid G_1)$. Taking this into account, we have

$$S_1 = \sum_{\{x, y\} \subseteq V(G_1)} d(x, y \mid G_1)^\lambda = W^{(\lambda)}(G_1).$$

The second sum $S_2$ is taken over all pairs of vertices in $G_2^u$, and then over all $u \in V(G_1)$. For vertices $x, y \in V(G_2^u)$, $d(x, y \mid G_1\{G_2\}) = d(x, y \mid G_2)$. Taking this
into account, we have

$$S_2 = \sum_{u \in V(G_1)} \sum_{x,y \in V(G_2)} d(x, y | G_2)^\lambda = n_1 W^{(\lambda)}(G_2).$$

The third sum $S_3$ is taken over all pairs of vertices $x, y \in V(G_1 \{G_2\})$ such that $x \in V(G_1)$ and $y \in V(G_2) - \{w\}$, where $u \in V(G_1)$ and $u \neq x$. In this case, $d(x, y | G_1 \{G_2\}) = d(x, u | G_1) + d(y, w | G_2)$. So,

$$S_3 = \sum_{x \in V(G_1)} \sum_{u \in V(G_1) - \{x\}} \sum_{y \in V(G_2) - \{w\}} [d(x, u | G_1) + d(y, w | G_2)]^\lambda$$

$$= 2 \sum_{i=0}^\lambda \binom{\lambda}{i} W^{(i)}(G_1) d^{(\lambda-i)}(w | G_2).$$

The last sum $S_4$ is taken over all pairs of non-root vertices from different copies of $G_2$. For such a pair $x \in V(G_2) - \{w\}$ and $y \in V(G_2) - \{w\},$

$$d(x, y | G_1 \{G_2\}) = d(x, w | G_2) + d(u, v | G_1) + d(y, w | G_2).$$

So,

$$S_4 = \sum_{\{u, v\} \subseteq V(G_1) x \in V(G_2) - \{w\} y \in V(G_2) - \{w\}} \sum [d(x, w | G_2) + d(u, v | G_1) + d(w, y | G_2)]^\lambda$$

$$= \sum_{i=0}^\lambda \binom{\lambda}{i} W^{(i)}(G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w | G_2) d^{(\lambda-i-j)}(w | G_2).$$

The formula for $W^{(\lambda)}(G_1 \{G_2\})$ follows upon addition of the quantities $S_1$, $S_2$, $S_3$, and $S_4$.

By Theorem 2.1, we can reproduce the result of Theorem 5 in [28] about the Wiener index of the cluster of two graphs.

**Corollary 2.1.** The Wiener index of $G_1 \{G_2\}$ is given by

$$W(G_1 \{G_2\}) = n_2^2 W(G_1) + n_1 W(G_2) + 2n_2 \binom{n_1}{2} d^{(1)}(w | G_2).$$

Using Theorem 2.1, we can also get the formulae for the hyper-Wiener index and $TSZ$ index of $G_1 \{G_2\}$.

**Corollary 2.2.** The hyper-Wiener index of $G_1 \{G_2\}$ is given by

$$WW(G_1 \{G_2\}) = n_2^2 WW(G_1) + n_1 WW(G_2) + 2n_2 d^{(1)}(w | G_2) W(G_1)$$

$$+ \left( \binom{n_1}{2} \right) \left[ n_2 d^{(2)}(w | G_2) + d^{(1)}(w | G_2)^2 + n_2 d^{(1)}(w | G_2) \right].$$
Corollary 2.3. The Tratch-Stankevich-Zefirov index of \( G_1 \{ G_2 \} \) is given by

\[
TSZ(G_1 \{ G_2 \}) = n_2^2 TSZ(G_1) + n_1 TSZ(G_2) + 2n_2 d^{(3)}(w | G_2) WW(G_1)
+ \left[ n_2 d^{(2)}(w | G_2) + d^{(1)}(w | G_2)^2 + n_2 d^{(1)}(w | G_2) \right] W(G_1)
+ \left( \frac{1}{3} n_2 \left( \frac{n_1}{2} \right) \right) \left[ d^{(3)}(w | G_2) + 3d^{(2)}(w | G_2) + 2d^{(1)}(w | G_2) \right]
+ \left( \frac{n_1}{2} \right) d^{(1)}(w | G_2) \left[ d^{(2)}(w | G_2) + d^{(1)}(w | G_2) \right].
\]

2.2. Edge-Wiener type invariants of clusters. In this section, we determine the edge-Wiener type invariants of the cluster \( G_1 \{ G_2 \} \).

Let \( \lambda \) be a real number and let \( u \in V(G) \). We define

\[
D_1^{(\lambda)}(u | G) = \sum_{e \in E(G); u \notin V(e)} D_1(u, e | G)^\lambda,
\]

\[
D_2^{(\lambda)}(u | G) = \sum_{e \in E(G)} D_2(u, e | G)^\lambda.
\]

Note that, if \( \lambda \) is a positive number then \( D_1^{(\lambda)}(u | G) = \sum_{e \in E(G)} D_1(u, e | G)^\lambda \). In particular for \( \lambda = 0 \), \( D_1^{(0)}(u | G) = |E(G)| - \deg_G(u) \) and \( D_2^{(0)}(u | G) = |E(G)| \).

Theorem 2.2. If \( \lambda \) is a positive integer, then the edge-Wiener type invariants of \( G_1 \{ G_2 \} \) are given by

\begin{enumerate}
\item \( W_{e_0}^{(\lambda)}(G_1 \{ G_2 \}) = W_{e_0}^{(\lambda)}(G_1) + n_1 W_{e_0}^{(\lambda)}(G_2) + 2m_1 \omega \\
+ \sum_{i=0}^{\lambda} \binom{\lambda}{i} \left[ \omega^2 W^{(i)}(G_1) + \omega W_{e_1}^{(i)}(G_1) + 2m_1 D_1^{(i)}(w | G_2) \right] \\
+ \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} \left[ W_{e_1}^{(j)}(G_1) + 2\omega W^{(j)}(G_1) \right] D_1^{(i-j)}(w | G_2) \\
+ W^{(j)}(G_1) \sum_{k=0}^{i-j} \binom{i-j}{k} D_1^{(k)}(w | G_2) D_1^{(i-j-k)}(w | G_2) \right],
\]

\item \( W_{e_4}^{(\lambda)}(G_1 \{ G_2 \}) = W_{e_4}^{(\lambda)}(G_1) + n_1 W_{e_4}^{(\lambda)}(G_2) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} W_{e_2}^{(i)}(G_1) D_2^{(\lambda-i)}(w | G_2) \\
+ \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda - j}{j} D_2^{(j)}(w | G_2) D_2^{(\lambda-i-j)}(w | G_2).
\end{enumerate}

Proof. (i) We partition the sum in the definition of \( W_{e_0}^{(\lambda)}(G_1 \{ G_2 \}) \) into four sums as follows.
The first sum $S_1$ consists of contributions to $W_{e_0}^{(\lambda)}(G_1\{G_2\})$ of pairs of edges from $G_1$. For edges $e, f \in E(G_1)$, $d_0(e, f | G_1\{G_2\}) = d_0(e, f | G_1)$. Taking this into account, we have

$$S_1 = \sum_{\{e,f\} \subseteq E(G_1)} d_0(e, f | G_1)^\lambda = W_{e_0}^{(\lambda)}(G_1).$$

The second sum $S_2$ is taken over all pairs of edges in $G_2^u$, and then over all $u \in V(G_1)$. For edges $e, f \in E(G_2^u)$, $d_0(e, f | G_1\{G_2\}) = d_0(e, f | G_2)$. Taking this into account, we have

$$S_2 = \sum_{u \in V(G_1)} \sum_{\{e,f\} \subseteq E(G_2^u)} d_0(e, f | G_2)^\lambda = n_1 W_{e_0}^{(\lambda)}(G_2).$$

The third sum $S_3$ is taken over all pairs of edges $e, f \in E(G_1\{G_2\})$ such that $e \in E(G_1)$ and $f \in E(G_2^u)$, where $u \in V(G_1)$. In this case,

$$d_0(e, f | G_1\{G_2\}) = D_1(u, e | G_1) + D_1(w, f | G_2) + 1.$$

So,

$$S_3 = \sum_{u \in V(G_1)} \sum_{e \in E(G_1)} \sum_{u \notin V(e) f \in E(G_2^u)} \sum_{u \notin V(f)} [D_1(u, e | G_1) + D_1(w, f | G_2) + 1]^\lambda$$

In order to compute the sum $S_3$, we partition it into four sums $S_{31}, S_{32}, S_{33}$ and $S_{34}$ as follows. The sum $S_{31}$ is equal to

$$S_{31} = \sum_{u \in V(G_1)} \sum_{e \in E(G_1)} \sum_{u \notin V(e) f \in E(G_2^u)} [D_1(u, e | G_1) + D_1(w, f | G_2) + 1]^\lambda$$

$$= \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{\lambda} \binom{\lambda}{j} W^{(j)}_{\in V_1}(G_1) D_1^{(i-j)}(w | G_2).$$

The sum $S_{32}$ is equal to

$$S_{32} = \sum_{u \in V(G_1)} \sum_{e \in E(G_1)} \sum_{u \notin V(e) f \in E(G_2^u)} \sum_{u \in V(f)} [D_1(u, e | G_1) + 1]^\lambda = \omega \sum_{\lambda} \binom{\lambda}{i} W_{\in V_1}^{(i)}(G_1).$$

The sum $S_{33}$ is equal to

$$S_{33} = \sum_{u \in V(G_1)} \sum_{e \in E(G_1)} \sum_{u \notin V(e) f \in E(G_2^u)} \sum_{u \notin V(f)} [D_1(w, f | G_1) + 1]^\lambda$$

$$= 2m_1 \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_1^{(i)}(w | G_2).$$

The sum $S_{34}$ is equal to

$$S_{34} = \sum_{u \in V(G_1)} \sum_{e \in E(G_1)} \sum_{u \in V(e) f \in E(G_2^u)} \sum_{u \in V(f)} 1^\lambda = 2m_1 \omega.$$
By adding the quantities $S_{31}$, $S_{32}$, $S_{33}$ and $S_{34}$, we obtain

$$S_3 = 2m_1 \omega + \sum_{i=0}^{\lambda} \binom{\lambda}{i} \left[ \omega W_{v_1}^i(G_1) + 2m_1 D_1^{(i)}(w | G_2) \right]$$

$$+ \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} W_{v_1}^j(G_1) D_1^{(i-j)}(w | G_2).$$

The last sum $S_4$ is taken over all pairs of edges from different copies of $G_2$. For such a pair $e \in E(G_2^n)$ and $f \in E(G_2^n)$,

$$d_0(e, f | G_1 \{G_2\}) = D_1(w, e | G_2) + D_1(w, f | G_2) + d(u, v | G_1) + 1.$$

So,

$$S_4 = \sum_{\{u, v\} \subseteq V(G_1)} \sum_{e \in E(G_2^n)} \sum_{f \in E(G_2^n)} [D_1(w, e | G_2) + D_1(w, f | G_2) + d(u, v | G_1) + 1]^\lambda.$$

Similar to the previous case, we partition the sum $S_4$ into four sums $S_{41}$, $S_{42}$, $S_{43}$ and $S_{44}$ as follows. The sum $S_{41}$ is equal to

$$S_{41} = \sum_{\{u, v\} \subseteq V(G_1)} \sum_{e \in E(G_2^n)} \sum_{f \in E(G_2^n)} [D_1(w, e | G_2) + D_1(w, f | G_2) + d(u, v | G_1) + 1]^\lambda$$

$$= \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} W_{v_1}^j(G_1) \sum_{k=0}^{i-j} \binom{i-j}{k} D_1^{(k)}(w | G_2) D_1^{(i-j-k)}(w | G_2).$$

The sum $S_{42}$ is equal to

$$S_{42} = \sum_{\{u, v\} \subseteq V(G_1)} \sum_{e \in E(G_2^n)} \sum_{f \in E(G_2^n)} [D_1(w, e | G_2) + d(u, v | G_1) + 1]^\lambda$$

$$= \omega \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} W_{v_1}^j(G_1) D_1^{(i-j)}(w | G_2).$$

The sum $S_{43}$ is equal to

$$S_{43} = \sum_{\{u, v\} \subseteq V(G_1)} \sum_{e \in E(G_2^n)} \sum_{f \in E(G_2^n)} [D_1(w, f | G_2) + d(u, v | G_1) + 1]^\lambda.$$

It is clear that, $S_{43} = S_{42}$. So,

$$S_{43} = \omega \sum_{i=0}^{\lambda} \binom{\lambda}{i} \sum_{j=0}^{i} \binom{i}{j} W_{v_1}^j(G_1) D_1^{(i-j)}(w | G_2).$$
The sum $S_{44}$ is equal to

$$S_{44} = \sum_{\{u,v\} \subseteq V(G_1)} \sum_{e \in E(G_2)} \sum_{w \in V(e) \cap V(f)} [d(u,v \mid G_1) + 1]^\lambda = \omega^2 \sum_{i=0}^\lambda \binom{\lambda}{i} W^{(i)}(G_1).$$

By adding the quantities $S_{41}$, $S_{42}$, $S_{43}$ and $S_{44}$, we obtain

$$S_4 = \omega^2 \sum_{i=0}^\lambda \binom{\lambda}{i} W^{(i)}(G_1) + \sum_{i=0}^\lambda \binom{\lambda}{i} \sum_{j=0}^i \binom{i}{j} W^{(j)}(G_1) \left[ 2\omega D_1^{(i-j)}(w \mid G_2) + \sum_{k=0}^{i-j} \binom{i-j}{k} D_1^{(k)}(w \mid G_2) D_1^{(i-j-k)}(w \mid G_2) \right].$$

Now, the formula for $W^{(\lambda)}_{e_0}(G_1 \{G_2\})$ is obtained by adding the quantities $S_1$, $S_2$, $S_3$ and $S_4$.

(iii) Using the similar argument as in the proof of part (i), we can get the desired result. \(\square\)

Using Theorem 2.2, we can get the formulae for the edge-Wiener and edge-hyper-Wiener indices of the cluster $G_1 \{G_2\}$.

**Corollary 2.4.** The first and second edge-Wiener indices of $G_1 \{G_2\}$ are given by

(i) $W_{e_0}(G_1 \{G_2\}) = W_{e_0}(G_1) + n_1 W_{e_0}(G_2) + m_2 W_{e_1}(G_1) + m_2^2 W(G_1)$

$$+ \left[2m_2 \binom{n_1}{2} + n_1 m_1\right] D_1^{(1)}(w \mid G_2) + \binom{n_1}{2} m_2^2 + n_1 m_1 m_2,$$

(ii) $W_{e_4}(G_1 \{G_2\}) = W_{e_4}(G_1) + n_1 W_{e_4}(G_2) + m_2 W_{e_2}(G_1) + m_2^2 W(G_1)$

$$+ \left[2m_2 \binom{n_1}{2} + n_1 m_1\right] D_1^{(1)}(w \mid G_2).$$

**Corollary 2.5.** The first and second edge hyper-Wiener indices of $G_1 \{G_2\}$ are given by

(i) $WW_{e_0}(G_1 \{G_2\}) = WW_{e_0}(G_1) + n_1 WW_{e_0}(G_2) + m_2 WW_{e_1}(G_1) + n_1 m_1 m_2$

$$+ m_2^2 WW(G_1) + \left[m_2 + D_1^{(1)}(w \mid G_2)\right] W_{e_1}(G_1) + m_2^2 \binom{n_1}{2}$$

$$+ m_2 \left[m_2 + 2D_1^{(1)}(w \mid G_2)\right] W(G_1) + \binom{n_1}{2} D_1^{(1)}(w \mid G_2)^2$$

$$+ \left[m_2 \binom{n_1}{2} + \frac{1}{2} n_1 m_1\right] [D_1^{(2)}(w \mid G_2) + 3D_1^{(1)}(w \mid G_2)],$$
(ii) \( WW_{e_1}(G_1 \{ G_2 \}) = WW_{e_1}(G_1) + n_1 WW_{e_1}(G_2) + m_2 WW_{e_2}(G_1) \)
\[ + m_2^2 WW(G_1) + D_2^{(1)}(w | G_2) | W_{e_2}(G_1) + 2m_2 W(G_1) | \]
\[ + \left[ m_2 \left( \frac{n_1}{2} \right) + \frac{1}{2} m_1 n_1 \right] \times \left[ D_2^{(2)}(w | G_2) + D_2^{(1)}(w | G_2) \right] \]
\[ + \left( \frac{n_1}{2} \right) D_2^{(1)}(w | G_2)^2. \]

2.3. **Vertex-edge Wiener type invariants of clusters.** In this section, we determine the vertex-edge Wiener type invariants of the cluster \( G_1 \{ G_2 \} \).

**Theorem 2.3.** If \( \lambda \) is a positive integer, then the vertex-edge Wiener type invariants of \( G_1 \{ G_2 \} \) are given by:

(i) \( W_{ve_1}^{(\lambda)}(G_1 \{ G_2 \}) = W_{ve_1}^{(\lambda)}(G_1) + n_1 W_{ve_1}^{(\lambda)}(G_2) + 2\omega W_{e_1}^{(\lambda)}(G_1) + 2m_1 d^{(\lambda)}(w | G_2) \)
\[ + \sum_{i=0}^{\lambda} \binom{\lambda}{i} W_{ve_1}^{(i)}(G_1) d^{(\lambda-i)}(w | G_2) \]
\[ + 2 \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \left[ D_1^{(\lambda-i)}(w | G_2) + \omega d^{(\lambda-i)}(w | G_2) \right] \]
\[ + \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w | G_2) D_1^{(\lambda-i-j)}(w | G_2) \],

(ii) \( W_{ve_2}^{(\lambda)}(G_1 \{ G_2 \}) = W_{ve_2}^{(\lambda)}(G_1) + n_1 W_{ve_2}^{(\lambda)}(G_2) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} W_{ve_2}^{(i)}(G_1) d^{(\lambda-i)}(w | G_2) \)
\[ + 2 \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \left[ D_2^{(\lambda-i)}(w | G_2) \right] \]
\[ + \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w | G_2) D_2^{(\lambda-i-j)}(w | G_2) \].

**Proof.** (i) We partition the sum in the definition of \( W_{ve_1}^{(\lambda)}(G_1 \{ G_2 \}) \) into three sums as follows. The first sum \( S_1 \) is taken over all vertices \( u \in V(G_1^x) \) and edges \( e \in E(G_2^x) \), where \( x \in V(G_1) \). In this case, \( D_1(u, e | G_1 \{ G_2 \}) = D_1(u, e | G_2) \). So,
\[ S_1 = \sum_{x \in V(G_1)} \sum_{u \in V(G_2^x)} \sum_{e \in E(G_2^x)} D_1(u, e | G_2)^\lambda = n_1 W_{ve_1}^{(\lambda)}(G_2). \]

The second sum \( S_2 \) is taken over all vertices \( u \in V(G_2^x) \) and edges \( e \in E(G_1) \), where \( x \in V(G_1) \). In this case, \( D_1(u, e | G_1 \{ G_2 \}) = d(u, w | G_2) + D_1(x, e | G_1) \). So,
\[ S_2 = \sum_{x \in V(G_1)} \sum_{u \in V(G_2^x)} \sum_{e \in E(G_1)} [d(u, w | G_2) + D_1(x, e | G_1)]^\lambda. \]
We partition the sum $S_2$ into three sums $S_{21}$, $S_{22}$ and $S_{23}$ as follows. The sum $S_{21}$ is equal to $S_{21} = \sum_{x \in V(G_1)} \sum_{u = w} \sum_{e \in E(G_1)} D_1(x, e | G_1) = W^{(\lambda)}_{v \in \mathcal{V}}(G_1)$. The sum $S_{22}$ is equal to $S_{22} = \sum_{x \in V(G_1)} \sum_{u \in V(G^2)} \sum_{e \in E(G_1) \setminus x \notin V(e)} d(u, w | G_2) = 2m_1d^{(\lambda)}(w | G_2)$. The sum $S_{23}$ is equal to

$$S_{23} = \sum_{x \in V(G_1)} \sum_{u \in V(G^2) - \{w\}} \sum_{e \notin V(e)} \left[d(u, w | G_2) + D_1(x, e | G_1)\right]^{\lambda}$$

By adding the quantities $S_{21}$, $S_{22}$ and $S_{23}$, we obtain

$$S_2 = W^{(\lambda)}_{v \in \mathcal{V}}(G_1) + 2m_1d^{(\lambda)}(w | G_2) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}_{v \in \mathcal{V}}(G_1) d^{(\lambda-i)}(w | G_2).$$

The third sum $S_3$ is taken over all vertices $u \in V(G_2^\mathcal{S})$ and edges $e \in E(G_2^\mathcal{S})$, where $x, y \in V(G_1)$ and $x \neq y$. In this case, $D_1(u, e | G_1 \{G_2\}) = d(u, w | G_2) + d(x, y | G_1) = D_1(w, e | G_2)$. So,

$$S_3 = \sum_{x \in V(G_1)} \sum_{y \in V(G_1) - \{x\}} \sum_{u \in V(G_2^\mathcal{S})} \sum_{e \in E(G_2^\mathcal{S})} d(x, y | G_1) = 2\omega W^{(\lambda)}(G_1).$$

We partition the sum $S_3$ into four sums $S_{31}$, $S_{32}$, $S_{33}$ and $S_{34}$ as follows. The sum $S_{31}$ is equal to $S_{31} = \sum_{x \in V(G_1)} \sum_{y \in V(G_1) - \{x\}} \sum_{u = w} \sum_{e \in E(G_2^\mathcal{S})} d(x, y | G_1) = 2\omega W^{(\lambda)}(G_1)$. The sum $S_{32}$ is equal to

$$S_{32} = \sum_{x \in V(G_1)} \sum_{y \in V(G_1) - \{x\}} \sum_{u = w} \sum_{e \notin V(e)} d(x, y | G_1) = 2\omega W^{(\lambda)}(G_1) D^{(\lambda-i)}(w | G_2).$$

The sum $S_{33}$ is equal to

$$S_{33} = \sum_{x \in V(G_1)} \sum_{y \in V(G_1) - \{x\}} \sum_{u \in V(G_2^\mathcal{S}) - \{w\}} \sum_{e \notin V(e)} d(u, w | G_2) + d(x, y | G_1)$$

$$= 2\omega \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) d^{(\lambda-i)}(w | G_2).$$
The sum $S_{34}$ is equal to

$$S_{34} = \sum_{x \in V(G_1)} \sum_{y \in V(G_1) - \{x\}} \sum_{u \in V(G_2) - \{y\}} \sum_{e \in E(G_2); u \not\in V(e)} [d(u, w|G_2) + d(x, y|G_1)]$$

$$+ D_1(w, e|G_2)]^\lambda$$

$$= 2 \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w|G_2) D_1^{(\lambda-i-j)}(w|G_2).$$

By adding the quantities $S_{31}$, $S_{32}$, $S_{33}$ and $S_{34}$, we obtain

$$S_3 = 2\omega W^{(\lambda)}(G_1) + 2 \sum_{i=0}^{\lambda} \binom{\lambda}{i} W^{(i)}(G_1) \left[ D_1^{(\lambda-i)}(w|G_2) + \omega d^{(\lambda-i)}(w|G_2) \right]$$

$$+ \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} d^{(j)}(w|G_2) D_1^{(\lambda-i-j)}(w|G_2).$$

The formula for $W^{(\lambda)}_{ve}(G_1\{G_2\})$ is obtained by adding the quantities $S_1$, $S_2$ and $S_3$.

(iii) Using the similar argument as in the proof of part (i), we can get the desired result. \(\square\)

Using Theorem 2.3, the formulae for the vertex-edge Wiener and vertex-edge hyper-Wiener indices of $G_1\{G_2\}$ are obtained at once.

**Corollary 2.6.** For $i \in \{1, 2\}$, the first and second vertex-edge Wiener indices of $G_1\{G_2\}$ are given by

$$W_{ve}(G_1\{G_2\}) = n_2 W_{ve}(G_1) + n_1 W_{ve}(G_2) + 2n_2m_2 W(G_1) + 2n_2 \binom{n_1}{2} D_i^{(1)}(w|G_2)$$

$$+ \left[ 2m_2 \binom{n_1}{2} + n_1 m_1 \right] d^{(1)}(w|G_2).$$

**Corollary 2.7.** For $i \in \{1, 2\}$, the first and second vertex-edge hyper-Wiener indices of $G_1\{G_2\}$ are given by

$$WW_{ve}(G_1\{G_2\}) = n_2 WW_{ve}(G_1) + n_1 WW_{ve}(G_2) + n_2m_2 WW(G_1)$$

$$+ d^{(1)}(w|G_2) W_{ve}(G_1) + 2W(G_1) [n_2 D_i^{(1)}(w|G_2)]$$

$$+ m_2 d^{(1)}(w|G_2) + \left[ m_2 \binom{n_1}{2} + \frac{1}{2} n_1 m_1 \right] [d^{(1)}(w|G_2)]$$

$$+ d^{(2)}(w|G_2) + n_2 \binom{n_1}{2} \left[ D_i^{(1)}(w|G_2) + D_i^{(2)}(w|G_2) \right]$$

$$+ 2 \binom{n_1}{2} d^{(1)}(w|G_2) D_i^{(1)}(w|G_2).$$
Table 1. Some distance-based topological invariants of path and star.

<table>
<thead>
<tr>
<th>Graph G</th>
<th>$P_n$</th>
<th>$S_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W(G)$</td>
<td>$(\frac{n+1}{3})$</td>
<td>$(n-1)^2$</td>
</tr>
<tr>
<td>$WW(G)$</td>
<td>$(\frac{n+2}{4})$</td>
<td>$\frac{1}{2}(n-1)(3n-4)$</td>
</tr>
<tr>
<td>$TSZ(G)$</td>
<td>$(\frac{n+3}{5})$</td>
<td>$(n-1)(2n-3)$</td>
</tr>
<tr>
<td>$W_{e_0}(G)$</td>
<td>$(\frac{n}{3})$</td>
<td>$(\frac{n}{2})$</td>
</tr>
<tr>
<td>$WW_{e_0}(G)$</td>
<td>$2(\frac{n+1}{4})$</td>
<td>$(\frac{n}{2})$</td>
</tr>
<tr>
<td>$W_{e_1}(G)$</td>
<td>$\frac{n+3}{3}(\frac{n-1}{2})$</td>
<td>$2(\frac{n}{2})$</td>
</tr>
<tr>
<td>$WW_{e_1}(G)$</td>
<td>$\frac{n^2+5n+12}{6}(\frac{n-1}{2})$</td>
<td>$3(\frac{n}{2})$</td>
</tr>
<tr>
<td>$W_{e_{e_1}}(G)$</td>
<td>$2(\frac{n}{3})$</td>
<td>$2(\frac{n}{2})$</td>
</tr>
<tr>
<td>$WW_{e_{e_1}}(G)$</td>
<td>$2(\frac{n+1}{4})$</td>
<td>$2(\frac{n}{2})$</td>
</tr>
<tr>
<td>$W_{e_{e_2}}(G)$</td>
<td>$2(\frac{n+1}{3})$</td>
<td>$2(n-1)^2$</td>
</tr>
<tr>
<td>$WW_{e_{e_2}}(G)$</td>
<td>$2(\frac{n+2}{4})$</td>
<td>$(n-1)(3n-4)$</td>
</tr>
</tbody>
</table>

3. Examples and Corollaries

In this section, our results are illustrated by some examples. Interesting classes of graphs can be obtained by specializing components in clusters. Let $P_n$ and $S_n$ denote the $n$ vertex path and star, respectively. Some distance-based topological invariants of these graphs are listed in Table 1.

Our first example is about the $t$-fold bristled graph of a given graph. For a given graph $G$, its $t$-fold bristled graph $Brs_t(G)$ is obtained by attaching $t$ pendant vertices to each vertex of $G$. This graph can be represented as the cluster of $G$ and $S_{t+1}$, where the root vertex $w$ of $S_{t+1}$ is on its vertex of degree $t$. The $t$-fold bristled graph of a given graph is also known as its $t$-thorny graph. It is easy to see that,

$$d^{(1)}(w | S_{t+1}) = d^{(2)}(w | S_{t+1}) = d^{(3)}(w | S_{t+1}) = D_2^{(1)}(w | S_{t+1}) = D_2^{(2)}(w | S_{t+1}) = t,$$

$$D_1^{(1)}(w | S_{t+1}) = D_1^{(2)}(w | S_{t+1}) = 0.$$

Using these results, the results obtained in the previous section and Table 1, we can compute several Wiener-like topological invariants of $t$-fold bristled graphs.

**Corollary 3.1.** Let $G$ be a graph of order $n$ and size $m$ and let $t$ be a positive integer. Then

(i) $W(Brs_t(G)) = (t+1)^2 W(G) + nt(nt + n - 1)$;
(ii) $WW(Brs_t(G)) = (t+1)^2 WW(G) + 2t(t+1)W(G) + \frac{1}{2}nt(3nt + 2n - 3)$;
(iii) $TSZ(Brs_t(G)) = (t+1)^2 TSZ(G) + 2t(t+1)WW(G) + t(3t + 2)W(G) + nt(2nt + n - 2)$;
(iv) $W_{e_0}(Brs_t(G)) = W_{e_0}(G) + tW_{e_{e_1}}(G) + t^2 W(G) + \frac{1}{2}nt(nt + 2m - 1)$;
(v) $W_{e_1}(Brs_t(G)) = W_{e_4}(G) + tW_{e_{e_2}}(G) + t^2 W(G) + nt(nt + m - 1)$;
(vi) \( WW_{e_0}(Br_{s_1}(G)) = WW_{e_0}(G) + tWW_{v_{e_1}}(G) + t^2WW(G) + tW_{v_{e_1}}(G) + t^2W(G) + \frac{1}{2}nt(nt + 2m - 1); \)

(vii) \( WW_{e_2}(Br_{s_1}(G)) = WW_{e_1}(G) + tWW_{v_{e_2}}(G) + t^2WW(G) + tW_{v_{e_2}}(G) + 2t^2W(G) + \frac{1}{2}nt(nt + m - 1); \)

(viii) \( W_{v_{e_1}}(Br_{s_1}(G)) = (t + 1)W_{v_{e_1}}(G) + 2t(t + 1)W(G) + nt(nt + m - 1); \)

(ix) \( W_{v_{e_2}}(Br_{s_1}(G)) = (t + 1)W_{v_{e_2}}(G) + 2t(t + 1)W(G) + nt(2nt + n + m - 1); \)

(x) \( W_{v_{e_1}}(Br_{s_1}(G)) = (t + 1)W_{v_{e_1}}(G) + t(t + 1)W(G) + tW_{v_{e_1}}(G) + 2t^2W(G) + nt(nt + m - 1); \)

(xi) \( W_{v_{e_2}}(Br_{s_1}(G)) = (t + 1)W_{v_{e_2}}(G) + t(t + 1)W(G) + tW_{v_{e_2}}(G) + 2t(2t + 1)W(G) + nt(3nt + n + m - 2). \)

Our next example is about the bridge graph constructed on a given graph. Let \( G \) be a graph rooted at vertex \( w \) and let \( t \) be a positive integer. The bridge graph \( B_t(G, w) \) is the graph obtained by taking \( t \) copies of \( G \) and by connecting the vertex \( w \) of the \( i \)-th copy of \( G \) to the vertex \( w \) of the \( i + 1 \)-th copy of \( G \) by an edge for \( i = 1, 2, ..., t - 1 \). The bridge graph \( B_t(G, w) \) can be considered as the cluster of the \( t \)-vertex path \( P_t \) and the graph \( G \). So, using the results of the previous section and Table 1, we get the following results for bridge graphs.

**Corollary 3.2.** Let \( G \) be a rooted graph of order \( n \) and size \( m \) and let \( w \) be its root vertex. For positive integer \( t \), the following hold

(i) \( W(B_t(G, w)) = tW(G) + 2n(\binom{t}{1})d^{(1)}(w | G) + n^2(\binom{t+1}{3}); \)

(ii) \( WW(B_t(G, w)) = tWW(G) + 2n\left(\binom{t+1}{3}\right)d^{(1)}(w | G) + n^2\left(\binom{t+2}{4}\right) \]
\[ + \left\lceil \frac{t}{2} \right\rceil \left[ nd^{(1)}(w | G) + d^{(1)}(w | G)^2 + nd^{(2)}(w | G) \right]; \]

(iii) \( TSZ(B_t(G, w)) = tTSZ(G) + 2n\left(\binom{t+2}{4}\right)d^{(1)}(w | G) + \left(\binom{t+1}{3}\right)\left[ nd^{(2)}(w | G) + d^{(1)}(w | G)^2 + nd^{(1)}(w | G) \right] \]
\[ + 3d^{(2)}(w | G) + 2d^{(1)}(w | G) + \left(\binom{t}{2}\right) d^{(1)}(w | G) d^{(1)}(w | G) \]
\[ + d^{(2)}(w | G) + n^2\left(\binom{t+3}{5}\right); \]

(iv) \( W_{e_0}(B_t(G, w)) = tW_{e_0}(G) + 2(m + 1)\left(\binom{t}{2}\right) D^{(1)}_{1}(w | G) + m^2\left(\binom{t+1}{3}\right) + \left(\binom{t}{2}\right) \]
\[ + (2m + 1)\left(\binom{t}{3}\right) + 2m\left(\binom{t}{2}\right); \]
(v) \( W_{e_4}(B_t(G, w)) = tW_{e_4}(G) + 2(m + 1) \left(\frac{t}{2}\right) D_2^{(1)}(w \mid G) + m(m + 2) \left(\frac{t + 1}{3}\right) \]
\[+ \frac{1}{6}(t - 1)(t - 2)(t + 3); \]

(vi) \( WW_{e_4}(B_t(G, w)) = tWW_{e_4}(G) + \left(\frac{t}{2}\right) [(m + 1)D_1^{(2)}(w \mid G) + D_1^{(1)}(w \mid G)^2 \]
\[+ (m + 1)D_1^{(1)}(w \mid G)] + 2\left[m\left(\frac{t + 1}{3}\right) + \left(\frac{t}{3}\right)\right] \]
\[+ (m + 1)\left(\frac{t}{2}\right) D_1^{(1)}(w \mid G) + m^2 \left[\left(\frac{t + 2}{4}\right) + \left(\frac{t + 1}{3}\right)\right] \]
\[+ 2(m + 1)\left(\frac{t + 1}{4}\right) + 2m\left(\frac{t}{3}\right) + m(m + 2)\left(\frac{t}{2}\right); \]

(vii) \( WW_{e_4}(B_t(G, w)) = tWW_{e_4}(G) + \left(\frac{t}{2}\right) [(m + 1)D_2^{(2)}(w \mid G) + D_2^{(1)}(w \mid G)^2 \]
\[+ (m + 1)D_2^{(1)}(w \mid G)] + 2(m + 1)\left(\frac{t + 1}{3}\right) D_2^{(1)}(w \mid G) \]
\[+ m(m + 2)\left(\frac{t + 2}{4}\right) + \frac{1}{12}(t - 1)(t - 2)(t^2 + 5t + 12); \]

(viii) \( W_{e_1}(B_t(G, w)) = tW_{e_1}(G) + 2\left(\frac{t}{2}\right) [(m + 1)d^{(1)}(w \mid G) + nD_1^{(1)}(w \mid G) \]
\[+ 2n\left[m\left(\frac{t + 1}{3}\right) + \left(\frac{t}{3}\right)\right]; \]

(ix) \( W_{e_2}(B_t(G, w)) = tW_{e_2}(G) + 2\left(\frac{t}{2}\right) [(m + 1)d^{(1)}(w \mid G) + nD_2^{(1)}(w \mid G) \]
\[+ 2n(m + 1)\left(\frac{t + 1}{3}\right); \]

(x) \( WW_{e_1}(B_t(G, w)) = tWW_{e_1}(G) + 2\left(\frac{t}{3}\right)d^{(1)}(w \mid G) + 2\left(\frac{t + 1}{3}\right) [nD_1^{(1)}(w \mid G) \]
\[+ md^{(1)}(w \mid G)] + (m + 1)\left(\frac{t}{2}\right) [d^{(1)}(w \mid G) + d^{(2)}(w \mid G)] \]
\[+ n\left(\frac{t}{2}\right) [D_1^{(1)}(w \mid G) + D_1^{(2)}(w \mid G)] + nm\left(\frac{t + 2}{4}\right) \]
\[+ 2\left(\frac{t}{2}\right) d^{(1)}(w \mid G) D_1^{(1)}(w \mid G) + 2n\left(\frac{t + 1}{4}\right); \]
(xi) $WW_{ve2}(B_t(G, w)) = 2\left(\frac{t+1}{3}\right) \left[ nD_2^{(1)}(w | G) + (m+1)d^{(1)}(w | G) \right]$

$+ tWW_{ve2}(G) + (m+1)\left(\frac{t}{2}\right) \left[ d^{(1)}(w | G) + d^{(2)}(w | G) \right]$

$+ n\left(\frac{t}{2}\right) \left[ D_2^{(1)}(w | G) + D_2^{(2)}(w | G) \right]$

$+ 2\left(\frac{t}{2}\right) d^{(1)}(w | G)D_2^{(1)}(w | G) + n(m+2)\left(\frac{t+2}{4}\right).$

The results of parts (i) and (ii) of Corollary 3.2 have also been given in [20].

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