LAPLACIAN ENERGY OF UNION AND CARTESIAN PRODUCT
AND LAPLACIAN EQUIENERGETIC GRAPHS

HARISHCHANDRA S. RAMANE\textsuperscript{1}, GOURAMMA A. GUDODAGI\textsuperscript{1}, AND IVAN GUTMAN\textsuperscript{2}

ABSTRACT. The Laplacian energy of a graph $G$ with $n$ vertices and $m$ edges is defined as $\text{LE}(G) = \sum_{i=1}^{n} |\mu_i - 2m/n|$, where $\mu_1, \mu_2, \ldots, \mu_n$ are the Laplacian eigenvalues of $G$. If two graphs $G_1$ and $G_2$ have equal average vertex degrees, then $\text{LE}(G_1 \cup G_2) = \text{LE}(G_1) + \text{LE}(G_2)$. Otherwise, this identity is violated. We determine a term $\Xi$, such that $\text{LE}(G_1) + \text{LE}(G_2) - \Xi \leq \text{LE}(G_1 \cup G_2) \leq \text{LE}(G_1) + \text{LE}(G_2) + \Xi$ holds for all graphs. Further, by calculating $\text{LE}$ of the Cartesian product of some graphs, we construct new classes of Laplacian non-cospectral, Laplacian equienergetic graphs.

1. Introduction

Let $G$ be a finite, simple, undirected graph with $n$ vertices $v_1, v_2, \ldots, v_n$ and $m$ edges. In what follows, we say that $G$ is an $(n,m)$-graph. Let $A(G)$ be the adjacency matrix of $G$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues.

Let $D(G)$ be the diagonal matrix whose $(i,i)$-th entry is the degree of a vertex $v_i$. The matrix $C(G) = D(G) - A(G)$ is called the Laplacian matrix of $G$. The Laplacian polynomial of $G$ is defined as $\psi(G, \mu) = \det[\mu I - C(G)]$, where $I$ is an identity matrix. The eigenvalues of $C(G)$, denoted by $\mu_i = \mu_i(G)$, $i = 1, 2, \ldots, n$, are called the Laplacian eigenvalues of $G$ [16]. Two graphs are said to be Laplacian cospectral if they have same Laplacian eigenvalues. The adjacency eigenvalues and Laplacian eigenvalues satisfy the following conditions:

$$\sum_{i=1}^{n} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \mu_i = 2m.$$
The energy of a graph $G$ is defined as

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|.$$ 

It was introduced by one of the present authors in the 1970s, and since then has been much studied in both chemical and mathematical literature. For details see the book [15] and the references cited therein.

The Laplacian energy of a graph was introduced a few years ago [13] and is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i(G) - \frac{2m}{n} \right|.$$ 

This definition is chosen so as to preserve the main features of the ordinary graph energy $E$, see [18]. Basic properties and other results on Laplacian energy can be found in the survey [1], the recent papers [6–8,11,12,17,19], and the references cited therein.

2. LAPLACIAN ENERGY OF UNION OF GRAPHS

Let $G_1$ and $G_2$ be two graphs with disjoint vertex sets. Let for $i = 1, 2$, the vertex and edges sets of $G_i$ be, respectively, $V_i$ and $E_i$. The union of $G_1$ and $G_2$ is a graph $G_1 \cup G_2$ with vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. If $G_1$ is an $(n_1, m_1)$-graph and $G_2$ is an $(n_2, m_2)$-graph then $G_1 \cup G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2$ edges. It is easy to see that the Laplacian spectrum of $G_1 \cup G_2$ is the union of the Laplacian spectra of $G_1$ and $G_2$.

In [13] it was proven that if $G_1$ and $G_2$ have equal average vertex degrees, then $LE(G_1 \cup G_2) = LE(G_1) + LE(G_2)$. If the average vertex degrees are not equal, that is $\frac{2m_1}{n_1} \neq \frac{2m_2}{n_2}$, then it may be either $LE(G_1 \cup G_2) > LE(G_1) + LE(G_2)$ or $LE(G_1 \cup G_2) < LE(G_1) + LE(G_2)$ or, exceptionally, $LE(G_1 \cup G_2) = LE(G_1) + LE(G_2)$ [13].

In this section we study the Laplacian establish some additional relations between $LE(G_1 \cup G_2)$ and $LE(G_1) + LE(G_2)$.

**Theorem 2.1.** Let $G_1$ be an $(n_1, m_1)$-graph and $G_2$ be an $(n_2, m_2)$-graph, such that $\frac{2m_1}{n_1} > \frac{2m_2}{n_2}$. Then

$$LE(G_1) + LE(G_2) - \frac{4(n_2m_1 - n_1m_2)}{n_1 + n_2} \leq LE(G_1 \cup G_2) \leq LE(G_1) + LE(G_2) + \frac{4(n_2m_1 - n_1m_2)}{n_1 + n_2}.$$  

(2.1)
Proof. For the sake of simplicity, denote $G_1 \cup G_2$ by $G$. Then $G$ is an $(n_1 + n_2, m_1 + m_2)$-graphs. By the definition of Laplacian energy,

$$LE(G_1 \cup G_2) = \sum_{i=1}^{n_1 + n_2} \left| \mu_i(G) - \frac{2(m_1 + m_2)}{n_1 + n_2} \right|$$

$$= \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2(m_1 + m_2)}{n_1 + n_2} \right| + \sum_{i=n_1+1}^{n_1 + n_2} \left| \mu_i(G) - \frac{2(m_1 + m_2)}{n_1 + n_2} \right|$$

$$= \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2(m_1 + m_2)}{n_1 + n_2} \right| + \sum_{i=1}^{n_2} \left| \mu_i(G_2) - \frac{2(m_1 + m_2)}{n_1 + n_2} \right|$$

$$\leq \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2m_1}{n_1} \right| + n_1 \left| \frac{2m_1}{n_1} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right|$$

$$+ \sum_{i=1}^{n_2} \left| \mu_i(G_2) - \frac{2m_2}{n_2} \right| + n_2 \left| \frac{2m_2}{n_2} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right|$$

(2.2)

Since $n_2m_1 > n_1m_2$, Eq. (2.2) becomes

$$LE(G_1 \cup G_2) \leq LE(G_1) + n_1 \left( \frac{2m_1}{n_1} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right)$$

$$+ LE(G_2) + n_2 \left( - \frac{2m_2}{n_2} + \frac{2(m_1 + m_2)}{n_1 + n_2} \right)$$

$$= LE(G_1) + LE(G_2) + \frac{4(n_2m_1 - n_1m_2)}{n_1 + n_2}$$

which is an upper bound.

To obtain the lower bound, we just have to note that in full analogy to the above arguments,

$$LE(G_1 \cup G_2) \geq \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2m_1}{n_1} \right| - n_1 \left| \frac{2m_1}{n_1} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right|$$
\[ \sum_{i=1}^{n_2} \vert \mu_i(G_2) - \frac{2m_2}{n_2} \vert - n_2 \left( \frac{2m_2}{n_2} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right). \]  

(2.3)

Since \( n_2m_1 > n_1m_2 \), the Eq. (2.3) becomes

\[ LE(G_1 \cup G_2) \geq LE(G_1) - n_1 \left( \frac{2m_1}{n_1} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right) \]
\[ + LE(G_2) - n_2 \left( \frac{2m_2}{n_2} + \frac{2(m_1 + m_2)}{n_1 + n_2} \right) \]
\[ = LE(G_1) + LE(G_2) - \frac{4(n_2m_1 - n_1m_2)}{n_1 + n_2} \]

which is a lower bound.

\textbf{Corollary 2.1.} [13] Let \( G_1 \) be an \((n_1, m_1)\)-graph and \( G_2 \) be \((n_2, m_2)\)-graph such that \( \frac{2m_1}{n_1} = \frac{2m_2}{n_2} \). Then

\[ LE(G_1 \cup G_2) = LE(G_1) + LE(G_2). \]

\textbf{Corollary 2.2.} Let \( G_1 \) be an \( r_1 \)-regular graph on \( n_1 \) vertices and \( G_2 \) be an \( r_2 \)-regular graph on \( n_2 \) vertices, such that \( r_1 > r_2 \). Then

\[ LE(G_1) + LE(G_2) - \frac{2n_1n_2(r_1 - r_2)}{n_1 + n_2} \leq LE(G_1 \cup G_2) \]
\[ \leq LE(G_1) + LE(G_2) + \frac{2n_1n_2(r_1 - r_2)}{n_1 + n_2}. \]

\textit{Proof.} Result follows by setting \( m_1 = n_1r_1/2 \) and \( m_2 = n_2r_2/2 \) into Theorem 2.1. \( \square \)

\textbf{Theorem 2.2.} Let \( G \) be an \((n, m)\)-graph and \( \overline{G} \) be its complement, and let \( m > n(n-1)/4 \). Then

\[ LE(G) + LE(\overline{G}) - \left[ 4m - n(n-1) \right] \leq LE(G \cup \overline{G}) \leq LE(G) + LE(\overline{G}) + \left[ 4m - n(n-1) \right]. \]

\textit{Proof.} \( \overline{G} \) is a graph with \( n \) vertices and \( n(n-1)/2 - m \) edges. Substituting this into Eq. (2.1), the result follows. \( \square \)

\textbf{Theorem 2.3.} Let \( G \) be an \((n, m)\)-graph and \( G' \) be the graph obtained from \( G \) by removing \( k \) edges, \( 0 \leq k \leq m \). Then

\[ LE(G) + LE(G') - 2k \leq LE(G \cup G') \leq LE(G) + LE(G') + 2k. \]

\textit{Proof.} The number of vertices of \( G' \) is \( n \) and the number of edges is \( m-k \). Substituting this in Eq. (2.1), the result follows. \( \square \)
3. LAPLACIAN ENERGY OF CARTESIAN PRODUCT

Let $G$ be a graph with vertex set $V_1$ and $H$ be a graph with vertex set $V_2$. The Cartesian product of $G$ and $H$, denoted by $G \times H$ is a graph with vertex set $V_1 \times V_2$, such that two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G \times H$ if and only if either $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $H$ or $v_1 = v_2$ and $u_1$ is adjacent to $u_2$ in $G$ [14].

Lemma 3.1. [9] Let $A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$ be a symmetric $2 \times 2$ block matrix. Then the spectrum of $A$ is the union of the spectra of $A_0 + A_1$ and $A_0 - A_1$.

Theorem 3.1. If $\mu_1, \mu_2, \ldots, \mu_n$ are the Laplacian eigenvalues of a graph $G$, then the Laplacian eigenvalues of $G \times K_2$ are $\mu_1, \mu_2, \ldots, \mu_n$ and $\mu_1 + 2, \mu_2 + 2, \ldots, \mu_n + 2$.

Proof. The Laplacian matrix of $G \times K_2$ is
\[
C(G \times K_2) = \begin{bmatrix} C(G) + I & -I \\ -I & C(G) + I \end{bmatrix} = \begin{bmatrix} C_0 & C_1 \\ C_1 & C_0 \end{bmatrix}
\]

where $C(G)$ is the Laplacian matrix of $G$ and $I$ is an identity matrix of order $n$. By Lemma 3.1, the Laplacian spectrum of $G \times K_2$ is the union of the spectra of $C_0 + C_1$ and $C_0 - C_1$.

Here $C_0 + C_1 = C(G)$. Therefore, the eigenvalues of $C_0 + C_1$ are the Laplacian eigenvalues of $G$.

Because $C_0 - C_1 = C(G) + 2I$, the characteristic polynomial of $C_0 - C_1$ is
\[
\psi(C_0 - C_1, \mu) = \det [\mu I - (C_0 - C_1)] = \det [\mu I - (C(G) + 2I)]
\]
\[
= \det [(\mu - 2)I - C(G)] = \psi(G, \mu - 2).
\]

Therefore the eigenvalues of $C_0 - C_1$ are $\mu_i + 2, i = 1, 2, \ldots, n$. □

The Laplacian eigenvalues of the complete graph $K_n$ are $n$ ($n - 1$ times) and 0. The Laplacian eigenvalues of the complete bipartite regular graph $K_{k,k}$ are $2k, k$ ($2k - 2$ times) and 0. The Laplacian eigenvalues of the cocktail party graph $CP(k)$ (the regular graph on $n = 2k$ vertices and of degree $2k - 2$) are $2k$ ($k - 1$ times), $2k - 2$ ($k$ times) and 0 [16]. Applying Theorem 3.1, we directly arrive at the following example.

Example 3.1.

\[
LE(K_n \times K_2) = \begin{cases} 
4n - 4, & \text{if } n > 2, \\
2n, & \text{if } n \leq 2,
\end{cases}
\]

\[
LE(K_{k,k} \times K_2) = \begin{cases} 
8k - 4, & \text{if } k > 1, \\
6k - 2, & \text{if } k = 1,
\end{cases}
\]

\[
LE(CP(k) \times K_2) = 10k - 8, \quad \text{if } k \geq 2.
\]

Theorem 3.2. Let $G$ be an $(n,m)$-graph. Then

\[
2[LE(G) - n] \leq LE(G \times K_2) \leq 2[LE(G) + n].
\]
Proof. Let \( \mu_1, \mu_2, \ldots, \mu_n \) be the Laplacian eigenvalues of \( G \). Then by Theorem 3.1, the Laplacian eigenvalues of \( G \times K_2 \) are \( \mu_i, i = 1, 2, \ldots, n \) and \( \mu_i + 2, i = 1, 2, \ldots, n \). The graph \( G \times K_2 \) has \( 2n \) vertices and \( 2m + n \) edges. Therefore,

\[
LE(G \times K_2) = \sum_{i=1}^{n} \left| \mu_i - \frac{2(2m + n)}{2n} \right| + \sum_{i=1}^{n} \left| \mu_i + 2 - \frac{2(2m + n)}{2n} \right|
\]

(3.1)

Equation (3.1) can be rewritten as

\[
LE(G \times K_2) \leq \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} - 1 \right| + \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} + 1 \right|.
\]

which is an upper bound.

For lower bound, Eq. (3.1) can be rewritten as

\[
LE(G \times K_2) \geq \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| - n + \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| - n = 2LE(G) - 2n.
\]

Theorem 3.3. For a graph \( G \) with \( n \) vertices, \( LE(G \times K_2) \geq 2n \).

Proof. From Eq. (3.1)

\[
LE(G \times K_2) \geq \sum_{i=1}^{n} \left( \mu_i - \frac{2m}{n} - 1 \right) + \sum_{i=1}^{n} \left( \mu_i - \frac{2m}{n} + 1 \right)
\]

\[
= |2m - 2m - n| + |2m - 2m + n| = 2.
\]

□

4. LAPLACIAN EQUIENERGETIC GRAPHS

Two graphs \( G_1 \) and \( G_2 \) are said to be equienergetic if \( E(G_1) = E(G_2) \) [2]. For details see the book [15]. In analogy to this, two graphs \( G_1 \) and \( G_2 \) are said to be Laplacian equienergetic if \( LE(G_1) = LE(G_2) \).

Obviously Laplacian cospectral graphs are Laplacian equienergetic. Therefore we are interested in Laplacian non-cospectral graphs with equal number of vertices, having equal Laplacian energies. Stevanović [24] has constructed Laplacian equienergetic threshold graphs. Fritscher et al. [10] discovered a family of Laplacian equienergetic unicyclic graphs. We now report some additional classes of such graphs.

The line graph of the graph \( G \), denoted by \( L(G) \), is a graph whose vertices corresponds to the edges of \( G \) and two vertices in \( L(G) \) are adjacent if and only if the corresponding edges are adjacent in \( G \) [14]. The \( k \)-th iterated line graph of \( G \) is defined
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as $L^k(G) = L(L^{k-1}(G))$ where $L^0(G) \equiv G$ and $L^1(G) \equiv L(G)$. If $G$ is a regular graph of order $n_0$ and of degree $r_0$, then $L(G)$ is a regular graph of order $n_1 = n_0 r_0 / 2$ and of degree $r_1 = 2r_0 - 2$. Consequently, the order and degree of $L^k(G)$ are [3,4]:

$$n_k = \frac{1}{2} n_{k-1} r_{k-1} \quad \text{and} \quad r_k = 2r_{k-1} - 2$$

where $n_i$ and $r_i$ stand for the order and degree of $L^i(G)$, $i = 0, 1, 2, \ldots$. Therefore [3,4],

\begin{equation}
(4.1) \quad n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} \left(2^i r_0 - 2^{i+1} + 2\right)
\end{equation}

and

\begin{equation}
(4.2) \quad r_k = 2^k r_0 - 2^{k+1} + 2.
\end{equation}

**Theorem 4.1.** [23] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph $G$ of order $n$ and of degree $r$, then the adjacency eigenvalues of $L(G)$ are

$$\lambda_i + r - 2 \quad i = 1, 2, \ldots, n, \quad \text{and} \quad -2 \quad n(r - 2)/2 \text{ times.}$$

**Theorem 4.2.** [22] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph $G$ of order $n$ and of degree $r$, then the adjacency eigenvalues of $\overline{G}$, the complement of $G$, are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \ldots, n$.

**Theorem 4.3.** [16] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph $G$ of order $n$ and of degree $r$, then its Laplacian eigenvalues are $r - \lambda_i$, $i = 1, 2, \ldots, n$.

For $G$ being a regular graph of degree $r \geq 3$, and for $k \geq 2$, expressions for $E(L^k(G))$ and $E(L^k(G))$ were reported in [20,21].

**Theorem 4.4.** If $G$ is a regular graph of order $n$ and of degree $r \geq 4$, then

$$LE(L^2(G) \times K_2) = 4nr(r - 2).$$

**Proof.** Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of $G$. Then by Theorem 4.1, the adjacency eigenvalues of $L(G)$ are

\begin{equation}
(4.3) \quad \lambda_i + r - 2 \quad i = 1, 2, \ldots, n, \quad \text{and} \quad -2 \quad n(r - 2)/2 \text{ times.}
\end{equation}

Since $L(G)$ is a regular graph of order $nr/2$ and of degree $2r - 2$, by Eq. (4.3) the adjacency eigenvalues of $L^2(G)$ are

\begin{equation}
(4.4) \quad \lambda_i + 3r - 6 \quad i = 1, 2, \ldots, n, \quad \text{and} \quad 2r - 6 \quad n(r - 2)/2 \text{ times, and} \quad -2 \quad nr(r - 2)/2 \text{ times.}
\end{equation}
Since $L^2(G)$ is a regular graph of order $nr(r - 1)/2$ and of degree $4r - 2$, by Theorem 4.3 and Eq. (4.4), the Laplacian eigenvalues of $L^2(G)$ are

\[
\begin{align*}
    r - \lambda_i & \quad i = 1, 2, \ldots, n, \quad \text{and} \\
    2r & \quad n(r - 2)/2 \text{ times, and} \\
    4r - 4 & \quad nr(r - 2)/2 \text{ times.}
\end{align*}
\]

(4.5)

Using Theorem 3.1 and Eq. (4.5), the Laplacian eigenvalues of $L^2(G) \times K_2$ are

\[
\begin{align*}
    r - \lambda_i & \quad i = 1, 2, \ldots, n, \quad \text{and} \\
    2r & \quad n(r - 2)/2 \text{ times, and} \\
    4r - 4 & \quad nr(r - 2)/2 \text{ times, and} \\
    r - \lambda_i + 2 & \quad i = 1, 2, \ldots, n, \quad \text{and} \\
    2r + 2 & \quad n(r - 2)/2 \text{ times, and} \\
    4r - 2 & \quad nr(r - 2)/2 \text{ times.}
\end{align*}
\]

(4.6)

The graph $L^2(G) \times K_2$ is a regular graph of order $nr(r - 1)$ and of degree $4r - 5$. By Eq. (4.6), the Laplacian energy of $L^2(G) \times K_2$ is computed as

\[
\begin{align*}
    LE(L^2(G) \times K_2) & = \sum_{i=1}^{n} \left| r - \lambda_i - (4r - 5) \right| + \left| 2r - (4r - 5) \right| \frac{n(r - 2)}{2} \\
    & + \left| 4r - 4 - (4r - 5) \right| \frac{nr(r - 2)}{2} + \sum_{i=1}^{n} \left| r - \lambda_i + 2 - (4r - 5) \right| \\
    & + \left| 2r + 2 - (4r - 5) \right| \frac{n(r - 2)}{2} + \left| 4r - 2 - (4r - 5) \right| \frac{nr(r - 2)}{2} \\
    & = \sum_{i=1}^{n} \left| - \lambda_i - 3r + 5 \right| + \left| - 2r + 5 \right| \frac{n(r - 2)}{2} \\
    & + \left| 1 \right| \frac{nr(r - 2)}{2} + \sum_{i=1}^{n} \left| - \lambda_i - 3r + 7 \right| \\
    & + \left| - 2r + 7 \right| \frac{n(r - 2)}{2} + \left| 3 \right| \frac{nr(r - 2)}{2}.
\end{align*}
\]

(4.7)

If $d_{\max}$ is the greatest vertex degree of a graph, then all its adjacency eigenvalues belongs to the interval $[-d_{\max}, d_{\max}]$ [5]. In particular, the adjacency eigenvalues of a regular graph of degree $r$ satisfy the condition $-r \leq \lambda_i \leq r, i = 1, 2, \ldots, n$. 

If \( r \geq 4 \) then \( \lambda_i + 3r - 5 > 0, \lambda_i + 3r - 7 > 0, 2r - 5 > 0 \), and \( 2r - 7 > 0 \). Therefore by Eq. (4.7), and bearing in mind that \( \sum_{i=1}^{n} \lambda_i = 0 \),

\[
LE(L^2(G) \times K_2) = \sum_{i=1}^{n} \lambda_i + n(3r - 5) + (2r - 5) \frac{n(r-2)}{2} + \frac{nr(r-2)}{2}
\]
\[
+ \sum_{i=1}^{n} \lambda_i + n(3r - 7) + (2r - 7) \frac{n(r-2)}{2} + \frac{3nr(r-2)}{2}
\]
\[
= 4nr(r - 2).
\]

\[ \square \]

**Corollary 4.1.** Let \( G \) be a regular graph of order \( n_0 \) and of degree \( r_0 \geq 4 \). Let \( n_k \) and \( r_k \) be the order and degree, respectively of the \( k \)-th iterated line graph \( L^k(G) \) of \( G \), \( k \geq 2 \). Then

\[
LE(L^k(G) \times K_2) = 4n_k - 2r_k - 2(r_k - 2 - 2) = 4n_k - 1(r_k - 1 - 2),
\]

\[
LE(L^k(G) \times K_2) = 4n_0(r_0 - 2) \prod_{i=0}^{k-2} (2^i r_0 - 2^{i+1} + 2),
\]

\[
LE(L^k(G) \times K_2) = 8(n_k - n_{k-1}) = 8n_k \left( \frac{r_k - 2}{r_k + 2} \right).
\]

From Eq. (4.8) we see that the energy of \( L^k(G) \times K_2 \), \( k \geq 2 \) is fully determined by the order \( n \) and degree \( r \geq 4 \) of \( G \).

**Theorem 4.5.** If \( G \) is a regular graph of order \( n \) and of degree \( r \geq 3 \), then

\[
LE \left( L^2(G) \times K_2 \right) = (nr - 4)(4r - 6) - 4.
\]

**Proof.** Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the adjacency eigenvalues of a regular graph \( G \) of order \( n \) and of degree \( r \geq 3 \). Then the adjacency eigenvalues of \( L^2(G) \) are as given by Eq. (4.4).

Since \( L^2(G) \) is a regular graph of order \( nr(r-1)/2 \) and of degree \( 4r - 2 \), by Theorem 4.2 and Eq. (4.4), the adjacency eigenvalues of \( L^2(G) \) are

\[
-\lambda_i - 3r + 5 \quad i = 2, 3, \ldots, n, \quad \text{and}
\]
\[
-2r + 5 \quad n(r - 2)/2 \quad \text{times, and}
\]
\[
1 \quad nr(r - 2)/2 \quad \text{times, and}
\]
\[
(nr(r - 1)/2 - 4r + 5).
\]
Since $\overline{L^2(G)}$ is a regular graph of order $nr(r-1)/2$ and of degree $(nr(r-1)/2)-4r+5$, by Theorem 4.3 and Eq. (4.9), the Laplacian eigenvalues of $\overline{L^2(G)}$ are
\[ (nr(r-1)/2) - r - \lambda_i \quad i = 2, 3, \ldots, n, \quad \text{and} \]
\[ (nr(r-1)/2) - 2r \quad n(r-2)/2 \text{ times, and} \]
\[ (nr(r-1)/2) - 4r + 4 \quad nr(r-2)/2 \text{ times, and} \]
\[ 0. \]

Using Theorem 3.1 and Eq. (4.10), the Laplacian eigenvalues of $\overline{L^2(G) \times K_2}$ are
\[ (nr(r-1)/2) - r - \lambda_i \quad i = 2, 3, \ldots, n, \quad \text{and} \]
\[ (nr(r-1)/2) - 2r \quad n(r-2)/2 \text{ times, and} \]
\[ (nr(r-1)/2) - 4r + 4 \quad nr(r-2)/2 \text{ times, and} \]
\[ 0 \quad 1 \text{ time}, \quad \text{and} \]
\[ (nr(r-1)/2) - r - \lambda_i + 2 \quad i = 2, 3, \ldots, n, \quad \text{and} \]
\[ (nr(r-1)/2) - 2r + 2 \quad n(r-2)/2 \text{ times, and} \]
\[ (nr(r-1)/2) - 4r + 6 \quad nr(r-2)/2 \text{ times, and} \]
\[ 2. \]

The graph $\overline{L^2(G) \times K_2}$ is a regular graph of order $nr(r-1)$ and of degree $(nr(r-1)/2)-4r+6$. By Eq. (4.11),
\[
LE\left(\overline{L^2(G) \times K_2}\right) = \sum_{i=2}^{n} \left| \frac{nr(r-1)}{2} - r + \lambda_i - \left( \frac{nr(r-1)}{2} - 4r + 6 \right) \right| \]
\[ + \left| \frac{nr(r-1)}{2} - 2r - \left( \frac{nr(r-1)}{2} - 4r + 6 \right) \right| \frac{n(r-2)}{2} \]
\[ + \left| \frac{nr(r-1)}{2} - 4r + 4 - \left( \frac{nr(r-1)}{2} - 4r + 6 \right) \right| \frac{nr(r-2)}{2} \]
\[ + \left| 0 - \left( \frac{nr(r-1)}{2} - 4r + 6 \right) \right| \]
\[ + \sum_{i=2}^{n} \left| \frac{nr(r-1)}{2} - r + \lambda_i + 2 - \left( \frac{nr(r-1)}{2} - 4r + 6 \right) \right| \]
\[ + \left| \frac{nr(r-1)}{2} - 2r + 2 - \left( \frac{nr(r-1)}{2} - 4r + 6 \right) \right| \frac{n(r-2)}{2} \]
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\[ + \left| \frac{nr(r-1)}{2} - 4r + 6 - \left( \frac{nr(r-1)}{2} - 4r + 6 \right) \right| \frac{nr(r-2)}{2} \]

\[ + \left| 2 - \left( \frac{nr(r-1)}{2} - 4r + 6 \right) \right| \]

\[ = \sum_{i=2}^{n} |\lambda_i + 3r - 6| + |2r - 6| \frac{n(r-2)}{2} \]

\[ + |2r - 4| \frac{n(r-2)}{2} + |0| \frac{nr(r-2)}{2} + \left| \frac{-nr(r-1)}{2} + 4r - 6 \right| + \sum_{i=2}^{n} |\lambda_i + 3r - 4| \]

(4.12)

\[ + |2r - 4| \frac{n(r-2)}{2} + |0| \frac{nr(r-2)}{2} + \left| \frac{-nr(r-1)}{2} + 4r - 4 \right| . \]

All adjacency eigenvalues of a regular graph of degree \( r \) satisfy the condition \(-r \leq \lambda_i \leq r, i = 1, 2, \ldots, n \) [5]. Therefore if \( r \geq 3 \), then \( \lambda_i + 3r - 6 \geq 0, \lambda_i + 3r - 4 \geq 0, 2r - 6 \geq 0, 2r - 4 \geq 0, (-nr(r-1)/2) + 4r - 6 < 0 \), and \((-nr(r-1)/2) + 4r - 4 < 0 \).

Then from Eq. (4.12), and bearing in mind that \( \sum_{i=2}^{n} \lambda_i = -r \), we get

\[ LE \left( L^2(G) \times K_2 \right) = \sum_{i=2}^{n} \lambda_i + (n-1)(3r - 6) + (r-3)n(r-2) + nr(r-2) \]

\[ + \frac{nr(r-1)}{2} - 4r + 6 + \sum_{i=2}^{n} \lambda_i + (n-1)(3r - 4) \]

\[ + (r-2)n(r-2) + \frac{nr(r-1)}{2} - 4r + 4 \]

\[ = 2(nr - 4)(2r - 3) - 4. \]

□

**Corollary 4.2.** Let \( G \) be a regular graph of order \( n_0 \) and of degree \( r_0 \geq 3 \). Let \( n_k \) and \( r_k \) be the order and degree, respectively of the \( k \)-th iterated line graph \( L^k(G) \) of \( G \), \( k \geq 2 \). Then

\[ LE \left( L^k(G) \times K_2 \right) = (n_{k-2}r_{k-2} - 4)(4r_{k-2} - 6) - 4 \]

\[ = (2n_{k-1} - 4)(2r_{k-1} - 2) - 4, \]

(4.13) \[ LE \left( L^k(G) \times K_2 \right) = \left[ \frac{n_0}{2k-2} \prod_{i=0}^{k-2} \left( 2^i r_0 - 2^{i+1} + 2 \right) - 4 \right] (2^k r_0 - 2^{k+1} + 2) - 4, \]
\[ LE \left( L^k(G) \times K_2 \right) = \frac{8nk}{2 + rk} - 4(r_k + 1). \]

From Eq. (4.13) we see that the energy of \( L^k(G) \times K_2 \), \( k \geq 2 \) is fully determined by the order \( n \) and degree \( r \geq 3 \) of \( G \).

**Theorem 4.6.** Let \( G_1 \) and \( G_2 \) be two Laplacian non-cospectral, regular graphs of the same order and of the same degree \( r \geq 4 \). Then for any \( k \geq 2 \), \( L^k(G_1) \times K_2 \) and \( L^k(G_2) \times K_2 \) is a pair of Laplacian non-cospectral, Laplacian equienergetic graphs possessing same number of vertices and same number of edges.

**Proof.** If \( G \) is any graph with \( n \) vertices and \( m \) edges, then \( G \times K_2 \) has \( 2n \) vertices and \( 2m + n \) edges. Hence by repeated applications of Eqs. (4.1) and (4.2), \( L^k(G_1) \times K_2 \) and \( L^k(G_2) \times K_2 \) have same number of vertices and same number of edges. By Eqs. (4.5) and (4.6), if \( G_1 \) and \( G_2 \) are not Laplacian cospectral, then \( L^k(G_1) \times K_2 \) and \( L^k(G_2) \times K_2 \) are not Laplacian cospectral for all \( k \geq 1 \). Finally, Eq. (4.8) implies that \( L^k(G_1) \times K_2 \) and \( L^k(G_2) \times K_2 \) are Laplacian equienergetic. \( \square \)

**Theorem 4.7.** Let \( G_1 \) and \( G_2 \) be two Laplacian non-cospectral, regular graphs of the same order and of the same degree \( r \geq 3 \). Then for any \( k \geq 2 \), \( L^k(G_1) \times K_2 \) and \( L^k(G_2) \times K_2 \) is a pair of Laplacian non-cospectral, Laplacian equienergetic graphs possessing same number of vertices and same number of edges.

**Proof.** The proof is similar to that of Theorem 4.6 by using Eqs. (4.10), (4.11), and (4.13). \( \square \)

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**References**


1Department of Mathematics,
Karnatak University,
Dharwad – 580003, India
E-mail address: hsramane@yahoo.com
E-mail address: gouri.gudodagi@gmail.com

2Faculty of Science,
University of Kragujevac,
Kragujevac, Serbia, and
State University of Novi Pazar,
Novi Pazar, Serbia
E-mail address: gutman@kg.ac.rs