CHEN-LIKE INEQUALITIES ON LIGHTLIKE HYPERSURFACE OF A LORENTZIAN PRODUCT MANIFOLD WITH QUARTER-SYMMETRIC NONMETRIC CONNECTION

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Abstract. In this paper, we introduce $k$-Ricci curvature and $k$-scalar curvature on lightlike hypersurface of a Lorentzian product manifold with quarter-symmetric nonmetric connection. Using these curvatures, we establish some Chen-type inequalities for lightlike hypersurface of a Lorentzian product manifold with quarter-symmetric nonmetric connection. Considering the equality case, we obtain some results.

1. Introduction

In [16], Golab introduced the idea of a quarter-symmetric linear connections in a differential manifold. Later, the properties of Riemannian manifolds with quarter-symmetric metric (nonmetric) connection have been studied by some authors [19,24].

Warped products were first defined by Bishop and O’Neill in [6]. In [2], Atçeken and Kılıç introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. In [20], Kılıç and Oğuzhan considered lightlike hypersurfaces with respect to a quarter-symmetric nonmetric connection which is determined by the product structure. They also gave some equivalent conditions for integrability of distributions with respect to the Levi-Civita connection of semi-Riemannian manifold and the quarter-symmetric nonmetric connection, and obtained some results.

In 1993, B. Y. Chen [9] introduced a new Riemannian invariant for a Riemannian manifold $M$ as follows:

$$\delta_{M}(p) = \tau(p) - \inf(K)(p),$$

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where $\tau(p)$ is scalar curvature of $M$ and 
\[
\inf(K)(p) = \inf\{K(\Pi) : K(\Pi) \text{ is a plane section of } T_pM\}.
\]
In [9], B. Chen established a sharp inequality for submanifolds in a real space form involving $\delta_M$ and the main extrinsic invariant, namely the squared mean curvature.

Afterwards, B. Y. Chen and some geometers studied similar problems for non-degenerate submanifolds of different spaces such as in [8,9,17,28]. Later, Mihai and Özgür in [22] proved Chen inequalities for submanifolds of real space forms endowed with a semi-symmetric metric connection.

In degenerate submanifolds, M. Gülbahar, E. Kılıç and S. Keleş introduced $k$-Ricci curvature, $k$-scalar curvature, $k$-degenerate Ricci curvature, $k$-degenerate scalar curvature and they established some inequalities that characterize lightlike hypersurface of a Lorentzian manifold in [17]. After, they established some inequalities involving $k$-Ricci curvature, $k$-scalar curvature, the screen scalar curvature on a screen homothetic lightlike hypersurface of a Lorentzian manifold and they computed Chen-Ricci inequality and Chen inequality on a screen homothetic lightlike hypersurface of a Lorentzian manifold in [18].

In this paper, we study Chen-type inequalities for screen homothetic lightlike hypersurface of a real product space form $\tilde{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature $c$, endowed with quarter-symmetric nonmetric connection. Considering these inequalities, we obtain the relation between Ricci curvature and scalar curvature endowed with the quarter-symmetric nonmetric connection.

2. Preliminaries

Let $M$ be a hypersurface of an $(n+1)$-dimensional, $n > 1$, semi-Riemannian manifold $\tilde{M}$ with semi-Riemannian metric $\tilde{g}$ of index $1 \leq \nu \leq n$. We consider 
\[
T_xM^\perp = \left\{Y_x \in T_x\tilde{M} \mid \tilde{g}_x(Y_x, X_x) = 0, \text{ for all } X_x \in T_xM \right\},
\]
for any $x \in M$. Then we say that $M$ is a lightlike (null, degenerate) hypersurface of $\tilde{M}$ or equivalently, the immersion 
\[
i : M \to \tilde{M}
\]
of $M$ in $\tilde{M}$ is lightlike (null, degenerate) if $T_xM \cap T_xM^\perp \neq \{0\}$ at any $x \in M$. Henceforth we identify $i(M)$ with $M$ and we denote the differential $di$, immersing a vector field $X$ in $M$ to a vector field $\phi X$ in $\tilde{M}$, by $\phi$. Thus the induced metric tensor $g = \tilde{g}_{\beta\alpha}$ is defined by 
\[
g(X, Y) = \tilde{g}(\phi X, \phi Y), \quad \text{for all } X, Y \in \Gamma(TM).
\]

An orthogonal complementary vector bundle of $TM^\perp$ in $TM$ is non-degenerate subbundle of $TM$ called the screen distribution on $M$ and denoted by $S(TM)$. We have the following splitting into orthogonal direct sum: 
\[
(2.1) \quad TM = S(TM) \perp TM^\perp.
\]
The subbundle $S(TM)$ is non-degenerate, so is $S(TM)\perp$, and the following holds:

\begin{equation}
\widetilde{T_M} = S(TM) \perp S(TM)\perp,
\end{equation}

where $S(TM)\perp$ is the orthogonal complementary vector bundle to $S(TM)$ in $\widetilde{T_M}|_M$.

Let $\text{tr}(TM)$ denote the complementary vector bundle of $TM\perp$ in $S(TM)\perp$. Then we have

\begin{equation}
S(TM)\perp = TM\perp \oplus \text{tr}(TM).
\end{equation}

Let $U$ be a coordinate neighbourhood in $M$ and $\xi$ be a basis of $\Gamma(TM\perp|_U)$. Then there exists a basis $N$ of $\text{tr}(TM)|_U$ satisfying the following conditions:

$\tilde{g}(N,\xi) = 1$,

and

$\tilde{g}(N,N) = \tilde{g}(W,N) = 0$, for all $W \in \Gamma(S(TM)|_U)$.

The subbundle $\text{tr}(TM)$ is called a \textit{lightlike transversal vector bundle} of $M$. We note that $\text{tr}(TM)$ is never orthogonal to $TM$. From (2.1), (2.2) and (2.3) we have

$\widetilde{T_M}|_M = S(TM) \perp (TM\perp \oplus \text{tr}(TM)) = TM \oplus \text{tr}(TM)$.

Let $\nabla$ be the Levi-Civita connection of $\widetilde{M}$ and $P$ be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$. The Gauss and Weingarten formulas are given

\begin{equation}
\begin{aligned}
\overset{\circ}{\nabla}_XY &= \overset{\circ}{\nabla}_XY + B(X,Y)N, \\
\overset{\circ}{\nabla}_XY &= -\overset{\circ}{A}_NX + \omega(X)N, \\
\overset{\circ}{\nabla}_XPY &= \overset{\circ}{\nabla}_XPY + C(X,PY)\xi, \\
\overset{\circ}{\nabla}_X\xi &= -\overset{\circ}{A}_\xi X - \omega(X)\xi,
\end{aligned}
\end{equation}

for any $X,Y \in \Gamma(TM)$, where $\overset{\circ}{\nabla}$ and $\nabla$ are the induced linear connection on $TM$ and $S(TM)$, respectively; $B$ and $C$ are the local second fundamental forms on $TM$ and $S(TM)$, respectively; $\overset{\circ}{A}_N$ and $\overset{\circ}{A}_\xi$ are the shape operators on $TM$ and $S(TM)$, respectively; and $\omega$ is a 1-form on $TM$ [14,15]. Also, the local second fundamental forms $B$ and $C$ of $TM$ and $S(TM)$, respectively; are related to their shape operators $\overset{\circ}{A}_N$ and $\overset{\circ}{A}_\xi$ by

$B(X,Y) = g(\overset{\circ}{A}_\xi X,Y)$,

$C(X,PY) = g(\overset{\circ}{A}_N X, PY)$. 

If $B = 0$, then the lightlike hypersurface $M$ is called totally geodesic in $\tilde{M}$. A point $p \in M$ is said to be umbilical if

$$B(X, Y)_p = H g_p(X, Y), \quad X, Y \in \Gamma(T_p M),$$

where $H \in R$. The lightlike hypersurface $M$ is called totally umbilical in $\tilde{M}$ if every points of $M$ is umbilical [14].

The mean curvature $\mu$ of $M$ with respect to an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\Gamma(S(TM))$ is defined in [5] as follows:

$$\mu = \frac{1}{n} \text{tr}(B) = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i B(e_i, e_i), \quad g(e_i, e_i) = \varepsilon_i.$$

A Lightlike hypersurface $(M, g)$ of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is called screen locally conformal if the shape operators $\tilde{A}_N$ and $\tilde{A}_\xi$ of $M$ and $S(TM)$, respectively, are related by

$$\tilde{A}_N = \varphi \tilde{A}_\xi,$$

where $\varphi$ is a non-vanishing smooth function on a neighbourhood $\mathcal{U}$ on $M$. In particular, $M$ is called screen homothetic if $\varphi$ is non-zero constant [3].

We denote by $\tilde{R}$ the curvature tensor of $\tilde{M}$ with respect to Levi-Civita connection $\tilde{\nabla}$ and by $\tilde{R}$ that of $M$ with respect to induced connection $\nabla$. Then the Gauss equations of $M$ is given by

$$\tilde{R}(X, Y)Z = \tilde{R}(X, Y)Z + A_{h(X, Z)}Y - A_{h(Y, Z)}X$$

$$+ (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z),$$

for $X, Y, Z, W \in \Gamma(TM)$.

Let $M$ be a two-dimensional non-degenerate plane. The number

$$K_{ij} = \frac{g(R(e_j, e_i)e_i, e_j)}{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)^2}$$

is called the sectional curvature of the plane section spanned by $e_i$ and $e_j$ at $p \in M$ [15].

Let $p \in M$ and $\xi$ be null vector of $T_p M$. A plane $\Pi$ of $T_p M$ is said to be null plane if it contains $\xi$ and $e_i$ such that $g(\xi, e_i) = 0$ and $g(e_i, e_i) = \varepsilon_i = \pm 1$. The null sectional curvature of $\Pi$ is given in [4] as follows

$$K_{i}^{null} = \frac{g(R_p(e_i, \xi)e_i, e_i)}{g_p(e_i, e_i)}.$$
The Ricci tensor $\tilde{\text{Ric}}$ of $\tilde{M}$ and the induced Ricci type tensor $R^{(0,2)}$ of $M$ are defined by

$$\tilde{\text{Ric}}(X, Y) = \text{trace}\{Z \to \tilde{R}(Z, X)Y\}, \quad \text{for all } X, Y \in \Gamma(T\tilde{M}),$$

$$R^{(0,2)}(X, Y) = \text{trace}\{Z \to R(Z, X)Y\}, \quad \text{for all } X, Y \in \Gamma(TM),$$

where

$$R^{(0,2)}(X, Y) = \sum_{i=1}^{n} \varepsilon_i g(R(e_i, X)Y, e_i) + g(R(\xi, X)Y, N),$$

for the quasi-orthonormal frame $\{e_1, \ldots, e_n, \xi\}$ of $T_pM$.

If $M$ admits that an induced symmetric Ricci tensor $\text{Ric}$ and Ricci tensor satisfy

$$\text{Ric}(X, Y) = kg(X, Y),$$

where $k$ is a constant, then $M$ is called an Einstein hypersurface [15].

### 3. Lorentzian Product Manifolds

In this section, we use the same notations and terminologies as in [20].

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two $(m_1 + 1)$ and $(m_2 + 1)$ dimensional Lorentzian manifolds with constant indexes $q_1 > 0$, $q_2 > 0$, respectively, and $\tilde{M} = (M_1 \times M_2, \tilde{g})$ be $(m_1 + m_2 + 2)$-dimensional differentiable manifold with a tensor field $F$ of type $(1, 1)$ on $\tilde{M}$ such that

$$F^2 = I. \tag{3.1}$$

Let $\pi : M_1 \times M_2 \to M_1$ and $\sigma : M_1 \times M_2 \to M_2$ be the projections which are given by $\pi(x, y) = x$ and $\sigma(x, y) = y$ for any $(x, y) \in M_1 \times M_2$. Then $\tilde{M} = M_1 \times M_2$ is called an almost product manifold with almost product structure $F$. If we put

$$\pi = \frac{1}{2}(I + F), \quad \sigma = \frac{1}{2}(I - F),$$

then we have

$$\pi^2 = \pi, \quad \sigma^2 = \sigma, \quad \pi\sigma = \sigma\pi = 0, \quad \pi + \sigma = I, \quad F = \pi - \sigma,$$

where $\pi$ and $\sigma$ define two complementary distributions [20].

If an almost product manifold $\tilde{M}$ admits a Lorentzian metric $\tilde{g}$ such that

$$\tilde{g}(FX, FY) = \tilde{g}(X, Y), \tag{3.2}$$

for any vector fields $X, Y \in \Gamma(T\tilde{M})$, then $\tilde{M} = M_1 \times M_2$ is called Lorentzian almost product manifold. From (3.1) and (3.2), we can easily see that

$$\tilde{g}(FX, Y) = \tilde{g}(X, FY).$$

If, for any vector fields $X, Y$ on $\tilde{M}$,

$$(\tilde{\nabla}_X F)Y = 0, \text{ that is } \tilde{\nabla}_X FY = F(\tilde{\nabla}_X Y),$$
then \( \tilde{M} \) is called a Lorentzian product manifold, where \( \tilde{\nabla} \) is the Levi-Civita connection on \( \tilde{M} \) (see, [20]).

Now, let \( M_1 \) and \( M_2 \) be real space forms with constant sectional curvatures \( c_1 \) and \( c_2 \) respectively. Then the Riemannian curvature tensor \( \tilde{\nabla} \) of \( \tilde{M} = M_1(c_1) \times M_2(c_2) \) is given by

\[
\tilde{\nabla}(X,Y)Z = \frac{1}{16}(c_1 + c_2)\left\{ \tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y \right. \\
\left. + \tilde{g}(F_Y,Z)FX - \tilde{g}(F_X,Z)FY \right\} \\
+ \frac{1}{16}(c_1 - c_2)\left\{ \tilde{g}(F_Y,Z)X - \tilde{g}(F_X,Z)Y \right. \\
\left. + \tilde{g}(Y,Z)FX - \tilde{g}(X,Z)FY \right\},
\]

(3.3)

for any \( X, Y, Z \in \Gamma(TM) \) [29].

Let \( (\tilde{M}, \tilde{g}, F) \) be Lorentzian product manifold and \( \tilde{\nabla} \) a Levi-Civita connection on \( \tilde{M} \). A linear connection \( \tilde{\nabla} \) is said to be \textit{quarter-symmetric nonmetric connection} if the torsion tensor \( \tilde{T} \) is of the form

\[
\tilde{T}(X,Y) = \tilde{\pi}(Y)FX - \tilde{\pi}(X)FY,
\]

where \( \tilde{\pi} \) is a 1-form on \( \tilde{M} \) with \( \tilde{Q} \) as associated vector field, that is

\[
\tilde{g}(\tilde{Q},X) = \tilde{\pi}(X).
\]

A linear connection \( \tilde{\nabla} \) is called a nonmetric connection if

\[
(\tilde{\nabla}_X \tilde{g})(Y,Z) \neq 0.
\]

Let \( M \) be a lightlike hypersurface of a Lorentzian product manifold \( (\tilde{M}, \tilde{g}) \). For any \( X \in \Gamma(TM) \) we can write

\[
FX = fX + w(X)N,
\]

(3.4)

where \( f \) is a \((1,1)\) tensor field and \( w \) is a 1-form on \( M \) given by \( w(X) = \tilde{g}(FX, \xi) = \tilde{g}(X,F\xi) \).

Following [16], a quarter-symmetric non-metric connection \( \tilde{\nabla} \) on \( \tilde{M} \) is given by

\[
\tilde{\nabla}_X Y = \tilde{\nabla}_X Y + \tilde{\pi}(Y)FX,
\]

(3.5)

for any vector fields \( X \) and \( Y \) of \( M \).

From (3.5) the curvature tensor \( \tilde{R} \) of the quarter-symmetric nonmetric connection \( \tilde{\nabla} \) is given by

\[
\tilde{R}(X,Y)Z = \tilde{\nabla}(X,Y)Z + \tilde{\lambda}(X,Z)FY - \tilde{\lambda}(Y,Z)FX,
\]

(3.6)
for any vector fields \(X, Y \in \Gamma(TM)\), where \(\tilde{\lambda}\) is a \((0,2)\) tensor given by \(\tilde{\lambda}(X,Z) = (\tilde{\nabla}_X \pi)(Z) - \pi(Z)\pi(FX)\).

Let \(M\) be a lightlike hypersurface of a Lorentzian product manifold \((\tilde{M},\tilde{g})\) with quarter-symmetric nonmetric connection \(\tilde{\nabla}\). Then the Gauss and Weingarten formulas with respect to \(\tilde{\nabla}\) are given by, respectively,

\[
\begin{align*}
\tilde{\nabla}_X Y &= \nabla_X Y + \tilde{B}(X,Y)N, \\
\tilde{\nabla}_X N &= -\tilde{A}_N X + \tilde{\tau}(X)N,
\end{align*}
\]

for any \(X, Y \in \Gamma(TM)\).

From (2.4), (3.4), (3.5), (3.7) and (3.8) we obtain

\[
\begin{align*}
\nabla_X Y &= \tilde{\nabla}_X Y + \tilde{\pi}(Y)fX, \\
\tilde{B}(X,Y) &= B(X,Y) + \tilde{\pi}(Y)w(X), \\
\tilde{A}_N X &= A_N X - \tilde{\pi}(N)fX, \\
\tilde{\tau}(X) &= \tau(X) + \tilde{\pi}(N)w(X),
\end{align*}
\]

for any \(X, Y \in \Gamma(TM)\).

Using (3.7) we have

\[
(3.9) \quad \tilde{g}(R(X,Y,Z,PW) = \tilde{R}(X,Y,Z,PW) + \tilde{B}(Y,Z)\tilde{C}(X,PW) - \tilde{B}(X,Z)\tilde{C}(Y,PW),
\]

for any any \(X, Y, Z, W \in \Gamma(TM)\).

From (3.6) and (3.9)

\[
(3.10) \quad \tilde{g}(\tilde{\nabla}_X Y Z, PW) = \tilde{g}(\tilde{\nabla}_X Y Z, PW) + \tilde{B}(Y,Z)\tilde{C}(X,PW) - \tilde{B}(X,Z)\tilde{C}(Y,PW) + \tilde{\lambda}(X,Z)g(FY, PW) - \tilde{\lambda}(Y,Z)g(FX, PW),
\]

for any any \(X, Y, Z, W \in \Gamma(TM)\).

From now on, we will consider a Lorentzian product manifold \(\tilde{M}\) endowed with a quarter-symmetric nonmetric connection \(\tilde{\nabla}\) and the Levi-Civita connection denoted by \(\circ\).

4. CHEN–RICCI INEQUALITY

In this section, we use the same notations and terminologies as in [17].

Let \(M\) be an \((n+1)\)-dimensional lightlike hypersurface of a Lorentzian product manifold \(\tilde{M} = M_1 \times M_2\) with a quarter-symmetric nonmetric connection and \(\{e_1, \ldots, e_n, \xi\}\) be a basis of \(\Gamma(TM)\) where \(\{e_1, \ldots, e_n\}\) is an orthonormal basis of \(\Gamma(S(TM))\) and \(n = m_1 + m_2\). For \(k \leq n\), we set \(\pi_{k,\xi} = \text{Span}\{e_1, \ldots, e_k, \xi\}\) is a \((k+1)\) dimensional degenerate plane section and \(\pi_k = \text{Span}\{e_1, \ldots, e_k\}\) is \(k\)-dimensional non degenerate plane section. Define \(k\)-degenerate Ricci curvature and \(k\)-Ricci curvature at a unit
vector $X \in \Gamma(TM)$ as follows:

$$\text{Ric}_{\pi_k, \xi}(X) = R^{(0,2)}(X, X) = \sum_{j=1}^{k} g(R(e_j, X)X, e_j) + \bar{g}(R(\xi, X)X, N),$$

$$\text{Ric}_{\pi_k}(X) = R^{(0,2)}(X, X) = \sum_{j=1}^{k} g(R(e_j, X)X, e_j),$$

respectively [17]. Furthermore, $k$-degenerate scalar curvature and $k$-scalar curvature at $p \in M$ are given by

$$\tau_{\pi_k, \xi}(p) = \sum_{i,j=1}^{k} K_{ij} + \sum_{i=1}^{k} K^\text{null}_i + K_{iN},$$

$$\tau_{\pi_k}(p) = \sum_{i,j=1}^{k} K_{ij},$$

respectively [17]. For $k = n$, $\pi_n = \text{Span}\{e_1, \ldots, e_n\} = \Gamma(S(TM))$, we have the screen Ricci curvature and the screen scalar curvature given by

$$\text{Ric}_{S(TM)}(e_1) = \text{Ric}_{\pi_n}(e_1) = \sum_{j=1}^{n} K_{1j} = K_{12} + \cdots + K_{1n},$$

and

$$\tau_{S(TM)} = \sum_{i,j=1}^{n} K_{ij},$$

respectively [17].

From (3.3) and (3.10) we can write

$$\tau_{S(TM)}(p) = \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1)) + \frac{1}{8} (c_1 - c_2) (izF) + \sum_{i,j=1}^{n} m_{ij}$$

$$+ \sum_{i,j=1}^{n} \tilde{B}_{ij} \tilde{C}_{jj} - \tilde{B}_{ij} \tilde{C}_{ji},$$

(4.1)

where $\tilde{B}_{ij} = \tilde{B}(e_i, e_j)$, $\tilde{C}_{ij} = \tilde{C}(e_i, e_j)$ and $m(e_i, e_j) = m_{ij} = \tilde{\lambda}(e_i, e_j) g(Fe_i, e_j) - \tilde{\lambda}(e_j, e_j) g(Fe_i, e_i)$, for $i, j \in \{1, \ldots, n\}$.

Let $M$ be a screen homothetic lightlike hypersurface of an $(n + 2)$-dimensional Lorentzian space form $\tilde{M}(c)$. Then, from (4.1) we get

$$\tau_{S(TM)}(p) = \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1)) + \frac{1}{8} (c_1 - c_2) (izF)$$

$$+ \sum_{i,j=1}^{n} m_{ij} + \varphi n^2 \mu^2 - \varphi \sum_{i,j=1}^{n} (\tilde{B}_{ij})^2.$$

(4.2)
Since the sectional curvature of screen homothetic lightlike hypersurface is symmetric, we can denote the screen scalar curvature by \( r_{S(TM)} \) as follows:

\[
(4.3) \quad r_{S(TM)}(p) = \sum_{1 \leq i < j \leq n} K_{ij} = \frac{1}{2} \sum_{i,j=1}^{n} K_{ij} = \frac{1}{2} r_{S(TM)}(p).
\]

By (4.3), (4.2) equality become

\[
2r_{S(TM)}(p) = \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n-1)) + \frac{1}{8} (c_1 - c_2)(izF)
\]

\[
+ \sum_{i,j=1}^{n} m_{ij} + \varphi n^2 \mu^2 - \varphi \sum_{i,j=1}^{n} (\bar{B}_{ij})^2.
\]

\[
(4.4)
\]

**Theorem 4.1.** Let \( M \) be a screen homothetic lightlike hypersurface of a real product space form \( \tilde{M}(c) = M_1(c_1) \times M_2(c_2) \) of constant sectional curvature \( c \), endowed with quarter-symmetric nonmetric connection \( \tilde{\nabla} \). Then, the following statements are true.

(i) For \( X \in S^1(TM) = \{ X \in S(TM) : \langle X, X \rangle = 1 \} \)

\[
\text{Ric}_{S(TM)}(X) \leq \frac{1}{4} \varphi n^2 \mu^2 + \frac{1}{32} (c_1 + c_2) (2(izF)^2 + n(n-1)) - \frac{1}{2} \sum_{2 \leq i < j \leq n} m_{ij}
\]

\[
+ \frac{1}{2} \left( \sum_{i=1}^{n} m_{ii} + \sum_{1 \leq j < i \leq n} m_{ij} + \sum_{j=2}^{n} m(X, e_j) \right).
\]

\[
(4.5)
\]

(ii) The equality case of (4.5) is satisfied by \( X \in T_p^1(M) \) if and only if

\[
\bar{B}(X, Y) = 0, \quad \text{for all } Y \in T_p(M) \text{ orthogonal to } X,
\]

\[
(4.6)
\]

\[
\bar{B}(X, X) = \frac{n}{2} \mu.
\]

(iii) The equality case of (4.5) holds for all \( X \in T_p^1(M) \) if and only if either \( p \) is a totally geodesic point or \( n = 2 \) and \( p \) is a totally umbilical point.

**Proof.** From (4.4) we get

\[
\frac{1}{4} \varphi n^2 \mu^2 = r_{S(TM)}(p) - \frac{1}{32} (c_1 + c_2) ((izF)^2 + n(n-1)) - \frac{1}{16} (c_1 - c_2)(izF)
\]

\[
- \frac{1}{2} \sum_{i,j=1}^{n} m_{ij} + \frac{1}{4} \varphi (\bar{B}_{11} - \bar{B}_{22} - \cdots - \bar{B}_{nn})^2 + \varphi \sum_{j=2}^{n} (\bar{B}_{1j})^2
\]

\[
- \varphi \sum_{2 \leq i < j \leq n} \left( \bar{B}_{ij} \bar{B}_{jj} - (\bar{B}_{ij})^2 \right).
\]

\[
(4.7)
\]
Using (3.3) and (3.10) we also have

\[
\phi \sum_{2 \leq i < j \leq n} \left( \bar{B}_{ij}^2 - (\bar{B}_{ij})^2 \right) = \sum_{2 \leq i < j \leq n} K_{ij} - \sum_{2 \leq i < j \leq n} \bar{K}_{ij}
\]

\[
= \sum_{2 \leq i < j \leq n} K_{ij} - \frac{1}{2}(c_1 + c_2)((iz)^2 - 2(iF) \bar{g}(Fe_1, e_1))
\]

\[
- \frac{1}{16}(c_1 - c_2)((izF) - (n - 1)\bar{g}(Fe_1, e_1))
\]

\[
- \frac{1}{32}(c_1 + c_2)(n - 2)^2 - \sum_{2 \leq i < j \leq n} m_{ij}.
\]

(4.8)

From (4.7) and (4.8) we obtain

\[
\text{Ric}_{\text{S}(TM)}(e_1) = \frac{1}{4}\phi n^2 \mu^2 \phi - \frac{1}{4}\phi \left( \bar{B}_{11} - \bar{B}_{22} - \cdots - \bar{B}_{nn} \right)^2 - \phi \sum_{j=2}^{n} (\bar{B}_{1j})^2
\]

\[
+ \frac{1}{32}(c_1 + c_2)(2(iF)\bar{g}(Fe_1, e_1) + 3n - 4) - \sum_{2 \leq i < j \leq n} m_{ij}
\]

\[
+ \frac{1}{16}(c_1 - c_2)(n - 1)\bar{g}(Fe_1, e_1)
\]

\[
+ \frac{1}{2} \left( \sum_{i=1}^{n} m_{ii} + \sum_{1 \leq i < j \leq n} m_{ij} + \sum_{j=2}^{n} m_{1j} \right).
\]

(4.9)

If we put \(e_1 = X\) as any vector of \(T_p^l(M)\) in (4.9) we obtain (4.5).

The equality case of (4.5) holds for \(X \in T_p^l(M)\) if and only if

\[
\bar{B}_{i2} = \bar{B}_{i3} = \cdots = \bar{B}_{in} = 0 \quad \text{and} \quad \bar{B}_{11} = \bar{B}_{22} + \cdots + \bar{B}_{nn},
\]

(4.10)
equivalent to (4.6).

Now we prove the statement (iii). Assuming the equality case of (4.5) for all \(X \in T_p^l(M)\), in view of (4.10), we have

\[
\bar{B}_{ij} = 0, \quad i \neq j,
\]

(4.11)

and

\[
2\bar{B}_{ii} = \bar{B}_{11} + \bar{B}_{22} + \cdots + \bar{B}_{nn}, \quad i \in \{1, \ldots, n\}.
\]

(4.12)

From (4.12) we have \(2\bar{B}_{11} = 2\bar{B}_{22} = \cdots = 2\bar{B}_{nn} = \bar{B}_{11} + \bar{B}_{22} + \cdots + \bar{B}_{nn}\) which implies that

\[
(n - 2)(\bar{B}_{11} + \bar{B}_{22} + \cdots + \bar{B}_{nn}) = 0.
\]

Thus, either \(\bar{B}_{11} + \bar{B}_{22} + \cdots + \bar{B}_{nn} = 0\) or \(n = 2\). If \(\bar{B}_{11} + \bar{B}_{22} + \cdots + \bar{B}_{nn} = 0\), then in view of (4.12), we get \(\bar{B}_{ii} = 0\) for all \(i \in \{1, \ldots, n\}\). This together with (4.11) gives \(\bar{B}_{ij} = 0\) for all \(i, j \in \{1, \ldots, n\}\), that is, \(p\) is a totally geodesic point. If \(n = 2\), then
from (4.12), \(2\bar{B}_{11} = 2\bar{B}_{22} = \bar{B}_{11} + \bar{B}_{22}\), which shows that \(p\) is a totally umbilical point. The proof of the converse part is straightforward. \(\square\)

We recall the following algebraic Lemma from [27].

**Lemma 4.1.** Let \(a_1, a_2, \ldots, a_n\), be \(n\)-real number \((n > 1)\), then

\[
\frac{1}{n} \left( \sum_{i=1}^{n} a_i \right)^2 \leq \sum_{i=1}^{n} a_i^2
\]

with equality if and only if \(a_1 = a_2 = \cdots = a_n\).

**Theorem 4.2.** Let \(M\) be a screen homothetic lightlike hypersurface of a real product space form \(\tilde{M}(c) = M_1(c_1) \times M_2(c_2)\) of constant sectional curvature \(c\), endowed with quarter-symmetric nonmetric connection \(\tilde{\nabla}\)

\[
\tau_{S(TM)}(p) \leq \varphi n(n-1)\mu^2 + \frac{1}{16}(c_1 + c_2)\left((izF)^2 + n(n-1)\right)
\]

\[
+ \frac{1}{8}(c_1 - c_2)(izF) + \sum_{i,j=1}^{n} m_{ij},
\]

with equality if and only if \(p\) is a totally umbilical point.

**Proof.** From (4.2) we have

\[
\varphi n^2 \mu^2 = \tau_{S(TM)}(p) + \varphi \sum_{i=1}^{n} (B_{ii})^2 + \varphi \sum_{i \neq j} (B_{ij})^2 - \sum_{i,j=1}^{n} m_{ij}
\]

\[
- \frac{1}{16}(c_1 + c_2)\left((izF)^2 + n(n-1)\right) - \frac{1}{8}(c_1 - c_2)(izF).
\]

Using Lemma 4.1 we get

\[
n\mu^2 \leq \sum_{i=1}^{n} (B_{ii})^2.
\]

Considering (4.14) and (4.15) we obtain (4.13). Equality case of (4.13) holds if and only if

\[
\bar{B}_{11} = \bar{B}_{22} = \cdots = \bar{B}_{nn},
\]

the shape operator \(A^*_\xi\) take the form:

\[
A^*_\xi = \begin{bmatrix}
\bar{B}_{11} & 0 & \cdots & 0 & 0 \\
0 & \bar{B}_{11} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \bar{B}_{11} & 0 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix},
\]

which shows that \(M\) is totally umbilical. This completes the proof of the theorem. \(\square\)
Also, the components of the second fundamental form \( \bar{B} \) and the screen second fundamental form \( \bar{C} \) satisfy
\[
(4.17) \quad \sum_{i,j=1}^{n} \bar{B}_{ij} \bar{C}_{ji} = \frac{1}{2} \left\{ \sum_{i,j=1}^{n} (\bar{B}_{ij} + \bar{C}_{ji})^2 - \sum_{i,j=1}^{n} (\bar{B}_{ij})^2 + (\bar{C}_{ji})^2 \right\},
\]
and
\[
(4.18) \quad \sum_{i,j} \bar{B}_{ii} \bar{C}_{jj} = 1 - \frac{1}{2} \left\{ \left( \sum_{i,j} \bar{B}_{ii} + \bar{C}_{jj} \right)^2 - \left( \sum_{i} \bar{B}_{ii} \right)^2 - \left( \sum_{j} \bar{C}_{jj} \right)^2 \right\}.
\]

**Theorem 4.3.** Let \( M \) be lightlike hypersurface of a real product space form \( \bar{M}(c) = M_1(c_1) \times M_2(c_2) \) of constant sectional curvature \( c \), endowed with quarter-symmetric nonmetric connection \( \bar{\nabla} \). Then

(i) \[
\tau_{S(TM)}(p) \leq n \mu \text{ trace } A_N + \frac{1}{2} \sum_{i,j=1}^{n} \left( (\bar{B}_{ij})^2 + (\bar{C}_{ji})^2 \right) + \sum_{i,j=1}^{n} m_{ij}
\]
\[
+ \frac{1}{16} (c_1 + c_2) (izF)^2 + \frac{1}{8} (c_1 - c_2)(izF).
\]

The equality case of (4.19) holds for all \( p \in M \) if and only if either \( M \) is a screen homothetic lightlike hypersurface with \( \varphi = -1 \) or \( M \) is a totally geodesic lightlike hypersurface.

(ii) \[
\tau_{S(TM)}(p) \geq n \mu \text{ trace } A_N - \frac{1}{2} \sum_{i,j=1}^{n} \left( (\bar{B}_{ij})^2 + (\bar{C}_{ji})^2 \right) + \sum_{i,j=1}^{n} m_{ij}
\]
\[
+ \frac{1}{16} (c_1 + c_2) (izF)^2 + \frac{1}{8} (c_1 - c_2)(izF).
\]

The equality case of (4.20) holds for all \( p \in M \) if and only if either \( M \) is a screen homothetic lightlike hypersurface with \( \varphi = 1 \) or \( M \) is a totally geodesic lightlike hypersurface.

(iii) The equalities case of (4.19) and (4.20) hold at \( p \in M \) if and only if \( p \) is a totally geodesic point.

**Proof.** Using (4.1) and (4.17), we get
\[
\tau_{S(TM)}(p) = \sum_{i,j=1}^{n} \bar{B}_{ii} \bar{C}_{jj} - \frac{1}{2} \sum_{i,j=1}^{n} (\bar{B}_{ij} + \bar{C}_{ji}) + \frac{1}{2} \sum_{i,j=1}^{n} \left( (\bar{B}_{ij})^2 + (\bar{C}_{ji})^2 \right)
\]
\[
+ \frac{1}{16} (c_1 + c_2) (izF)^2 + \frac{1}{8} (c_1 - c_2)(izF) + \sum_{i,j=1}^{n} m_{ij},
\]
which yields (4.19).
Since
\[ \frac{1}{2} ((B_{ij})^2 + (C_{ji})^2) = \frac{1}{4} (B_{ij} + C_{ji})^2 + \frac{1}{4} (B_{ij} - C_{ji})^2, \]
we obtain
\[
\tau_{S(TM)}(p) = \sum_{i,j=1}^{n} \bar{B}_{ij} C_{jj} - \frac{1}{2} \sum_{i,j=1}^{n} \left( (\bar{B}_{ij})^2 + (\bar{C}_{ji})^2 \right) + \frac{1}{2} \sum_{i,j=1}^{n} (\bar{B}_{ij} - \bar{C}_{ji})^2
\]
\[ + \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1)) + \frac{1}{8} (c_1 - c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}, \]
which yields (4.20). From (4.19), (4.20), (4.21) and (4.23) it is easy to get (i), (ii) and (iii) statements.

By Theorem 4.3 we have the following corollary. □

**Corollary 4.1.** Let \( M \) be a screen homothetic lightlike hypersurface of a real product space form \( \tilde{M}(c) = M_1(c_1) \times M_2(c_2) \) of constant sectional curvature \( c \), endowed with quarter-symmetric nonmetric connection \( \tilde{\nabla} \). Then, we have
\[
\tau_{S(TM)}(p) \leq \varphi n^2 \mu^2 + \left( \frac{1 + \varphi^2}{2} \right) \sum_{i,j=1}^{n} (\bar{B}_{ij})^2 + \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1))
\]
\[ + \frac{1}{8} (c_1 - c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}, \]
and
\[
\tau_{S(TM)}(p) \geq \varphi n^2 \mu^2 - \left( \frac{1 + \varphi^2}{2} \right) \sum_{i,j=1}^{n} (\bar{B}_{ij})^2 + \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1))
\]
\[ + \frac{1}{8} (c_1 - c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}. \]

**Theorem 4.4.** Let \( M \) be lightlike hypersurface of a real product space form \( \tilde{M}(c) = M_1(c_1) \times M_2(c_2) \) of constant sectional curvature \( c \), endowed with quarter-symmetric nonmetric connection \( \tilde{\nabla} \). Then, we have
\[
\tau_{S(TM)}(p) \leq \frac{1}{2} \left( \text{trace } \bar{A} \right)^2 - \frac{1}{2} \left( \text{trace } A_N \right)^2 - \frac{1}{4} \sum_{i,j=1}^{n} (\bar{B}_{ij} + \bar{C}_{ji})^2
\]
\[ + \frac{1}{4} \sum_{i,j=1}^{n} (\bar{B}_{ij} - \bar{C}_{ji})^2 + \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1))
\]
\[ + \frac{1}{8} (c_1 - c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}, \]
(4.24)
where

\[
\bar{A} = \begin{bmatrix}
\bar{B}_{11} + \bar{C}_{11} & \bar{B}_{12} + \bar{C}_{21} & \cdots & \bar{B}_{1n} + \bar{C}_{n1} \\
\bar{B}_{21} + \bar{C}_{12} & \bar{B}_{22} + \bar{C}_{22} & \cdots & \bar{B}_{2n} + \bar{C}_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{B}_{n1} + \bar{C}_{1n} & \bar{B}_{n2} + \bar{C}_{2n} & \cdots & \bar{B}_{nn} + \bar{C}_{nn}
\end{bmatrix}.
\]

The equality case of (4.24) holds for all \( p \in M \) if and only if \( M \) is minimal.

**Proof.** From (4.1), (4.17) and (4.18) we get

\[
\tau_{S(TM)}(p) = \frac{1}{2} \left( \sum_{i,j} \bar{B}_{ii} + C_{jj} \right)^2 - \frac{1}{2} \left( \sum_{i} \bar{B}_{ii} \right)^2 - \frac{1}{2} \left( \sum_{j} C_{jj} \right)^2 - \frac{1}{2} \sum_{i,j=1}^{n} (\bar{B}_{ij} + \bar{C}_{ji})^2 + \frac{1}{2} \sum_{i,j=1}^{n} ((\bar{B}_{ij})^2 + (\bar{C}_{ji})^2) + \sum_{i,j=1}^{n} m_{ij}
\]

\[
+ \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1)) + \frac{1}{8} (c_1 - c_2)(izF).
\]

(4.26)

From (4.22) we have

\[
= -\frac{1}{4} \sum_{i,j=1}^{n} (\bar{B}_{ij} + \bar{C}_{ji})^2 + \frac{1}{4} \sum_{i,j=1}^{n} (\bar{B}_{ij} - \bar{C}_{ji})^2.
\]

(4.27)

If we put (4.27) in (4.26), we obtain

\[
\tau_{S(TM)}(p) = \frac{1}{2} \left( \sum_{i,j} \bar{B}_{ii} + C_{jj} \right)^2 - \frac{1}{2} \left( \sum_{i} \bar{B}_{ii} \right)^2 - \frac{1}{2} \left( \sum_{j} C_{jj} \right)^2 - \frac{1}{4} \sum_{i,j=1}^{n} (\bar{B}_{ij} + \bar{C}_{ji})^2 + \frac{1}{4} \sum_{i,j=1}^{n} (\bar{B}_{ij} - \bar{C}_{ji})^2 + \sum_{i,j=1}^{n} m_{ij}
\]

\[
+ \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1)) + \frac{1}{8} (c_1 - c_2)(izF).
\]

The equality case of (4.24) satisfies then

\[
\sum_{i} \bar{B}_{ii} = 0.
\]

This shows that \( M \) is minimal. \(\square\)

By Theorem 4.4 we have the following corollary.
Corollary 4.2. Let $M$ be a screen homothetic lightlike hypersurface of a real product space form $\widetilde{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature $c$, endowed with quarter-symmetric nonmetric connection $\nabla$

$$\tau_{S(TM)}(p) \leq \frac{(2\varphi + 1)}{2} n^2 \mu^2 - \varphi \sum_{i,j=1}^{n} (\bar{B}_{ij})^2 + \frac{1}{16} (c_1 + c_2) \left( (izF)^2 + n(n - 1) \right)$$

(4.28)

$$+ \frac{1}{8} (c_1 - c_2)(izF) + \sum_{i,j=1}^{n} m_{ij}.$$ 

The equality case of (4.28) holds for all $p \in M$ if and only if $M$ is minimal.

Theorem 4.5. Let $M$ be lightlike hypersurface of a real product space form $\widetilde{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature $c$, endowed with quarter-symmetric nonmetric connection $\nabla$. Then, we have

$$\tau_{S(TM)}(p) \leq \frac{n - 1}{2n} \left( \text{trace } \bar{A} \right)^2 - \frac{1}{2} (\text{trace } A_N)^2 - \frac{1}{2} n^2 \mu^2 - \frac{1}{2} \sum_{i \neq j} (\bar{B}_{ij} + \bar{C}_{ji})^2$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \left( (\bar{B}_{ij})^2 + (\bar{C}_{ji})^2 \right) + \frac{1}{16} (c_1 + c_2) \left( (izF)^2 + n(n - 1) \right)$$

(4.29)

$$+ \frac{1}{8} (c_1 - c_2)(izF) + \sum_{i,j=1}^{n} m_{ij},$$

where $\bar{A}$ is equal to (4.25).

The equality case of (4.29) holds for all $p \in M$ if and only if $n\mu = - \text{trace } A_N$.

Proof. From (4.26)

$$\tau_{S(TM)}(p) = \frac{1}{2} \left( \text{trace } \bar{A} \right)^2 - \frac{1}{2} (\text{trace } A_N)^2 - \frac{1}{2} n^2 \mu^2 - \frac{1}{2} \sum_{i} (\bar{B}_{ii} + \bar{C}_{ii})^2$$

$$- \frac{1}{2} \sum_{i \neq j} (\bar{B}_{ij} + \bar{C}_{ji})^2 + \frac{1}{2} \sum_{i,j=1}^{n} \left( (\bar{B}_{ij})^2 + (\bar{C}_{ji})^2 \right) + \sum_{i,j=1}^{n} m_{ij}$$

(4.30)

$$+ \frac{1}{16} (c_1 + c_2) \left( (izF)^2 + n(n - 1) \right) + \frac{1}{8} (c_1 - c_2)(izF).$$

Using Lemma 4.1 and equality case of (4.30), we have

$$\tau_{S(TM)}(p) \leq \frac{1}{2} \left( \text{trace } \bar{A} \right)^2 - \frac{1}{2} (\text{trace } A_N)^2 - \frac{1}{2} n^2 \mu^2 - \frac{1}{2n} \sum_{i} (\bar{B}_{ii} + \bar{C}_{ii})^2$$

$$- \frac{1}{2} \sum_{i \neq j} (\bar{B}_{ij} + \bar{C}_{ji})^2 + \frac{1}{2} \sum_{i,j=1}^{n} (\bar{B}_{ij})^2 + (\bar{C}_{ji})^2 + \sum_{i,j=1}^{n} m_{ij}$$

$$+ \frac{1}{16} (c_1 + c_2) \left( (izF)^2 + n(n - 1) \right) + \frac{1}{8} (c_1 - c_2)(izF),$$
which implies (4.29). The equality case of (4.29) holds, then
\[(4.31) \quad \bar{B}_{11} + \bar{C}_{11} = \cdots = \bar{B}_{nn} + \bar{C}_{nn}.\]
From (4.31) we get
\[ (1 - n) \bar{B}_{11} + \bar{B}_{22} + \cdots + \bar{B}_{nn} + (1 - n) \bar{C}_{11} + \bar{C}_{22} + \cdots + \bar{C}_{nn} = 0, \]
\[ \bar{B}_{11} + (1 - n) \bar{B}_{22} + \cdots + \bar{B}_{nn} + \bar{C}_{11} + (1 - n) \bar{C}_{22} + \cdots + \bar{C}_{nn} = 0, \]
\[ \vdots \]
\[ \bar{B}_{11} + \bar{B}_{22} + \cdots + (1 - n) \bar{B}_{nn} + \bar{C}_{11} + \bar{C}_{22} + \cdots + (1 - n) \bar{C}_{nn} = 0. \]
By the above equations, we have
\[ (n - 1)^2 (\text{trace } A_N + n \mu) = 0. \]
Since \( n \neq 1 \), we obtain \( n \mu = - \text{trace } A_N \).

By Theorem 4.5 we have the following corollary.

**Corollary 4.3.** Let \( M \) be screen homothetic lightlike hypersurface of a real product space form \( \tilde{M}(c) = M_1(c_1) \times M_2(c_2) \) of constant sectional curvature \( c \), endowed with quarter-symmetric nonmetric connection \( \tilde{\nabla} \). Then
\[ \tau_{S(TM)}(p) \leq \varphi n (n - 1) \mu^2 - \frac{1 + \varphi^2}{2} n \mu^2 - \frac{1 + \varphi^2}{2} \sum_{i \neq j} (\bar{B}_{ij})^2 + \frac{1 + \varphi^2}{2} \sum_{i,j=1}^n (\bar{B}_{ij})^2 \]
\[ + \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1)) + \frac{1}{8} (c_1 - c_2) (izF) + \sum_{i,j=1}^n m_{ij}. \]
(4.32)

The equality case of (4.32) holds for all \( p \in M \) if and only if either \( \varphi = -1 \) or \( M \) is minimal.

**Theorem 4.6.** Let \( M \) be lightlike hypersurface of a real product space form \( \tilde{M}(c) = M_1(c_1) \times M_2(c_2) \) of constant sectional curvature \( c \), endowed with quarter-symmetric nonmetric connection \( \tilde{\nabla} \). Then
\[ \tau_{S(TM)}(p) \geq \frac{1}{2} (\text{trace } \tilde{A})^2 - \frac{1}{2} (\text{trace } A_N)^2 - \frac{1}{2} n(n - 1) \mu^2 - \frac{1}{2} \sum_{i,j=1}^n (\bar{B}_{ij} + \bar{C}_{ji})^2 \]
\[ + \frac{1}{2} \sum_{i \neq j} (\bar{B}_{ij})^2 + \frac{1}{2} \sum_{i,j=1}^n (\bar{C}_{ji})^2 + \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1)) \]
\[ + \frac{1}{8} (c_1 - c_2) (izF) + \sum_{i,j=1}^n m_{ij}. \]
(4.33)

The equality case of (4.33) holds for all \( p \in M \) if and only if \( p \) is a totally umbilical point.
Proof. From (4.26)
\[
\tau_{S(TM)}(p) = \frac{1}{2} (\text{trace } \bar{A})^2 - \frac{1}{2} (\text{trace } A_N)^2 - \frac{1}{2} n^2 \mu^2 + \frac{1}{2} \sum_i (\bar{B}_i)^2 + \frac{1}{2} \sum_{i \neq j} (\bar{B}_{ij})^2 \\
+ \frac{1}{2} \sum_{i,j=1}^n (\bar{C}_{ij})^2 - \frac{1}{2} \sum_{i,j=1}^n (\bar{B}_{ij} + \bar{C}_{ji})^2 + \sum_{i,j=1}^n m_{ij}
\]
(4.34)
\[
+ \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1)) + \frac{1}{8} (c_1 - c_2)(izF).
\]
Using Lemma 4.1 and equality case of (4.34) we have
\[
\tau_{S(TM)}(p) \geq \frac{1}{2} (\text{trace } \bar{A})^2 - \frac{1}{2} (\text{trace } A_N)^2 - \frac{1}{2} n^2 \mu^2 + \frac{1}{2n} \left( \sum_i \bar{B}_i \right)^2 \\
+ \frac{1}{2} \sum_{i \neq j} (\bar{B}_{ij})^2 + \frac{1}{2} \sum_{i,j=1}^n (\bar{C}_{ij})^2 - \frac{1}{2} \sum_{i,j=1}^n (\bar{B}_{ij} + \bar{C}_{ji})^2 + \sum_{i,j=1}^n m_{ij}
\]
\[
+ \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1)) + \frac{1}{8} (c_1 - c_2)(izF),
\]
which implies (4.33). Equality case of (4.33) holds if and only if $\bar{B}_{11} = \cdots = \bar{B}_{nn}$ the shape operator $A_1^*$ take the form as (4.16), which shows that $M$ is totally umbilical. This completes the proof of the theorem. □

By Theorem 4.6 we have the following corollary.

**Corollary 4.4.** Let $M$ be screen homothetic lightlike hypersurface of a real product space form $\bar{M}(c) = M_1(c_1) \times M_2(c_2)$ of constant sectional curvature $c$, endowed with quarter-symmetric nonmetric connection $\bar{\nabla}$. Then
\[
\tau_{S(TM)}(p) \geq \frac{(2\varphi + 1)}{2} n^2 \mu^2 - \frac{1}{2} n(n - 1) \mu^2 - \frac{(2\varphi + 1)}{2} \sum_{i,j=1}^n (\bar{B}_{ij})^2 \\
+ \frac{1}{16} (c_1 + c_2) ((izF)^2 + n(n - 1)) + \frac{1}{8} (c_1 - c_2)(izF) + \sum_{i,j=1}^n m_{ij}.
\]
(4.35)
The equality case of (4.35) holds for all $p \in M$ if and only if $p$ is a totally umbilical point.

**References**


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