NORMALIZED LAPLACIAN SPECTRUM OF DIFFERENT TYPE OF CORonas OF TWO REGULAR GRAPHS

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Abstract. In this paper, we determine the full normalized Laplacian spectrum of the corona, edge corona and neighborhood corona of a connected regular graph with an arbitrary regular graph in terms of the normalized Laplacian eigenvalues of the original graphs. Moreover, applying these results we find some non-regular normalized Laplacian co-spectral graphs.

1. Introduction

There are several kinds of spectrums associated with a graph, for example, adjacency spectrum, Laplacian spectrum, signless Laplacian spectrum, normalized Laplacian spectrum etc. Normalized Laplacian spectrum determines the bipartiteness from the largest eigenvalue and the number of connected components from the second smallest eigenvalue [5]. F Chung [5] introduced the normalized Laplacian matrix of a graph $G$, denoted by $\mathcal{L}(G)$, which is a square matrix with rows and columns are indexed by vertices of $G$, and for any two vertices $u$ and $v$ of $G$ the $(u,v)^{th}$ entry of it is given by

$$\mathcal{L}(u,v) = \begin{cases} 
1, & \text{if } u = v \text{ and } d_v \neq 0, \\
-\frac{1}{\sqrt{d_u d_v}}, & \text{if } u \text{ and } v \text{ are adjacent}, \\
0, & \text{otherwise}.
\end{cases}$$

where $d_u$ and $d_v$ are degree of $u$ and $v$ respectively. If $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix (that is $A(u,v) = 1$ if and only if vertex $u$ is adjacent to vertex $v$ and 0 otherwise) of $G$, then we can write $\mathcal{L}(G) = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$ with the convention that $D(G)^{-1}(u,u) = 0$ if

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$d_a = 0$. We denote the characteristic polynomial $\det(\lambda I - \mathcal{L})$ of $\mathcal{L}(G)$ by $f_G(\lambda)$. The roots of $f_G(\lambda)$ are known as the normalized Laplacian eigenvalues of $G$. The multiset of the normalized Laplacian eigenvalues of $G$ is called the normalized Laplacian spectrum of $G$. Since $\mathcal{L}(G)$ is a symmetric and positive semi-definite matrix, its eigenvalues, denoted by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, are all real, non-negative and can be arranged in non-decreasing order $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$. Two graphs $G$ and $H$ are called cospectral if $A(G)$ and $A(H)$ have the same spectrum. Analogously graphs $G$ and $H$ are called normalized Laplacian cospectral or simply $\mathcal{L}$-cospectral if the spectrum of $\mathcal{L}(G)$ and $\mathcal{L}(H)$ are the same. In [5] Chung proved that all normalized Laplacian eigenvalues lie within the interval $[0, 2]$ and 0 is always a normalized Laplacian eigenvalue of any graph $G$. She also determined normalized Laplacian spectrum of different kind of graphs like complete graphs, bipartite graphs, hypercubes etc. In [1], Banerjee and Jost investigated how the normalized Laplacian spectrum is affected by operations like motif doubling, graph splitting or joining. In [3], Butler and Grout produced (exponentially) large families of non-bipartite, non-regular graphs which are mutually cospectral, and also gave an example of a graph which is cospectral with its complement but is not self-complementary. In [11], Li studied the effect on the second smallest normalized Laplacian eigenvalue by grafting some pendant paths. In this paper we are interested on finding normalized Laplacian spectrum of some coronas of graphs, which are defined below.

**Definition 1.1.** For $i = 1, 2$, let $G_i$ be the graph with $n_i$ vertices and $m_i$ edges. Then

(i) The corona [8] of $G_1$ and $G_2$, denoted by $G_1 \circ G_2$, is the graph obtained by taking one copy of $G_1$ and $n_1$ copies of $G_2$, and then joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$ for $i = 1, 2, \ldots, n_1$. The corona $G_1 \circ G_2$ has $n_1(n_2 + 1)$ vertices and $m_1 + n_1(m_2 + n_2)$ edges.

(ii) The edge corona [10] of $G_1$ and $G_2$, denoted by $G_1 \diamond G_2$, is the graph obtained by taking one copy of $G_1$ and $m_1$ copies of $G_2$, and then joining two end vertices of the $i^{th}$ edge of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$ for $i = 1, 2, \ldots, m_1$. The edge corona $G_1 \diamond G_2$ has $n_1 + m_1n_2$ vertices and $m_1 + 2m_1n_2 + m_1m_2$ edges.

(iii) The neighborhood corona [7] of $G_1$ and $G_2$, denoted by $G_1 \ast G_2$, is the graph obtained by taking one copy of $G_1$ and $n_1$ copies of $G_2$, and then joining every neighbor of the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$ for $i = 1, 2, \ldots, n_1$. The neighborhood corona $G_1 \ast G_2$ has $n_1(n_2 + 1)$ vertices and $m_1(2n_2 + 1) + n_1m_2$ edges.

Many researchers have worked on the corona, edge corona and neighborhood corona of two graphs. Barik et al. [2] provided complete information about the adjacency spectrum of $G_1 \circ G_2$ for a connected graph $G_1$ and a regular graph $G_2$ and the Laplacian spectrum of $G_1 \circ G_2$ for arbitrary graphs $G_1$ and $G_2$. Hou and Shiu [10] found the adjacency spectrum of $G_1 \circ G_2$ for a connected regular graph $G_1$ and a regular graph $G_2$ and the Laplacian spectrum of the same for a connected regular graph $G_1$ and a graph $G_2$. In [14], Wang and Zhou gave complete information about the signless Laplacian spectrum of $G_1 \circ G_2$ for a graph $G_1$ and a regular graph $G_2$ and the signless
Laplacian spectrum of $G_1 \circ G_2$ for a connected regular graph $G_1$ and a regular graph $G_2$. In [7], Gopalapillai determined the adjacency spectrum and Laplacian spectrum of $G_1 \circ G_2$ for a regular graph $G_1$ and an arbitrary graph $G_2$. Here we compute the full normalized Laplacian spectrum of $G_1 \circ G_2$, $G_1 \cdot G_2$ and $G_1 \circ G_2$ for two regular graphs $G_1$ and $G_2$, with $G_1$ connected.

To prove our results we need the following matrix products and few results on them. Recall that, the Kronecker product of matrices $A = (a_{ij})$ of size $m \times n$ and $B$ of size $p \times q$, denoted by $A \otimes B$, is defined to be the $mp \times nq$ partition matrix $(a_{ij}B)$. It is known [9] that for matrices $M$, $N$, $P$ and $Q$ of suitable sizes, $MN \otimes PQ = (M \otimes P)(N \otimes Q)$. This implies that for nonsingular matrices $M$ and $N$, $(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}$. It is also known that [9], for square matrices $M$ and $N$ of order $k$ and $s$ respectively, $(det(M \otimes N) = (det M)^s(det N)^k$. For two matrices $A$ and $B$, of same size $m \times n$, the Hadamard product $A \cdot B$ of $A$ and $B$ is a matrix of the same size $m \times n$ with entries given by $(A \cdot B)_{ij} = (A)_{ij} \cdot (B)_{ij}$ (entrywise multiplication). Hadamard product is commutative, that is $A \cdot B = B \cdot A$.

We also need the lemma below in the proof of our results.

**Lemma 1.1** (Schur Complement [6]). If $Q$ is a non-singular square matrix and the order of all four matrices $M$, $N$, $P$ and $Q$ satisfy the rules of operations on matrices, then we have,

$$\begin{bmatrix} M & N \\ P & Q \end{bmatrix} = |Q||M - NQ^{-1}P|.$$

**2. Our Results**

Throughout the paper for any integer $k$, $I_k$ denotes the identity matrix of size $k$ and $J_k$ denotes the column vector of size $k$, whose all entries are 1. In the lemma below we represent the normalized Laplacian matrix of corona, edge corona and neighborhood corona of two regular graphs in terms of Kronecker product and Hadamard product of matrices.

**Lemma 2.1.** For $i = 1, 2$, let $G_i$ be $r_i$-regular graph with order $n_i$ and size $m_i$. Then we have the following:

(i) $L(G_1) = I_{n_1} - \frac{1}{r_1} A(G_1)$, $L(G_2) = I_{n_2} - \frac{1}{r_2} A(G_2)$.

(ii) $L(G_1 \circ G_2) = \begin{bmatrix} L(G_1) \cdot B(G_1) \\ -(C_{n_2}^T \otimes I_{n_1}) (L(G_2) \cdot B(G_2)) \otimes I_{n_1} \end{bmatrix}$,

where $C_{n_2}$ is the column vector of size $n_2$ with all entries equal to $\frac{1}{\sqrt{(r_1+n_2)(r_2+1)}}$, $B(G_1)$ is the $n_1 \times n_1$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_1}{r_1+n_2}$ and $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{r_2+1}$.

(iii) $L(G_1 \cdot G_2) = \begin{bmatrix} L(G_1) \cdot B(G_1) \\ -(R(G_1) \otimes C_{n_2}^T) I_{m_1} \otimes (L(G_2) \cdot B(G_2)) \end{bmatrix}$,

where $C_{n_2}$ is the column vector of size $n_2$ with all entries equal to $\frac{1}{\sqrt{(r_1+r_2)(r_2+2)}}$.
For any two graphs $G_1$ and $G_2$, the adjacency matrix of $G_1 \lor G_2$ is given by \[^{[2]}\]

$$A(G_1 \lor G_2) = \begin{pmatrix}
A(G_1) & J^T_{n_2} \otimes I_{n_1} \\
J^T_{n_2} \otimes I_{n_1} & A(G_2) \otimes I_{n_1}
\end{pmatrix}.$$ 

Since $G_1$ is a $r_1$ regular graph with $n_1$ vertices and $G_2$ is a $r_2$ regular graph with $n_2$ vertices, the normalized Laplacian matrix of $G_1 \lor G_2$ is

$${\mathcal{L}}(G_1 \lor G_2) = I_{n_1(n_2+1)} - \left( \begin{array}{c|c}
n_1 & O \\
\hline
O & \frac{1}{\sqrt{r_1+n_2}} I_{n_2} \otimes I_{n_1}
\end{array} \right) \times \left( \begin{array}{c|c}
A(G_1) & J^T_{n_2} \otimes I_{n_1} \\
J^T_{n_2} \otimes I_{n_1} & A(G_2) \otimes I_{n_1}
\end{array} \right) \times \left( \begin{array}{c|c}
n_1 & O \\
\hline
O & \frac{1}{\sqrt{r_1+n_2}} I_{n_2} \otimes I_{n_1}
\end{array} \right) = \left( I_{n_1} - \frac{1}{r_1+n_2} A(G_1) \right) \left( \begin{array}{c|c}
-n^T_{n_2} \otimes I_{n_1} & I_{n_2} - \frac{1}{r_2+1} A(G_2) \\
I_{n_2} - \frac{1}{r_2+1} A(G_2) & I_{n_1}
\end{array} \right).$$

(iii) For any two graphs $G_1$ and $G_2$, the adjacency matrix of $G_1 \lor G_2$ is given by \[^{[1]}\]

$$A(G_1 \lor G_2) = \begin{pmatrix}
A(G_1) & R(G_1) \otimes J^T_{n_2} \\
(R(G_1) \otimes J^T_{n_2})^T & I_{m_1} \otimes A(G_2)
\end{pmatrix}.$$ 

Since $G_1$ is a $r_1$ regular graph with $n_1$ vertices, $m_1$ edges and $G_2$ is a $r_2$ regular graph with $n_2$ vertices, $m_2$ edges, the normalized Laplacian matrix of $G_1 \lor G_2$ is

$${\mathcal{L}}(G_1 \lor G_2) = I_{n_1+m_1 n_2} - \left( \begin{array}{c|c}
n_1 & O \\
\hline
O & \frac{1}{\sqrt{r_1+1} n_2} I_{m_1} \otimes \frac{1}{\sqrt{r_2+2}} I_{n_2}
\end{array} \right) \times \left( \begin{array}{c|c}
A(G_1) & R(G_1) \otimes J^T_{n_2} \\
(R(G_1) \otimes J^T_{n_2})^T & I_{m_1} \otimes A(G_2)
\end{array} \right) \times \left( \begin{array}{c|c}
n_1 & O \\
\hline
O & \frac{1}{\sqrt{r_1+1} n_2} I_{m_1} \otimes \frac{1}{\sqrt{r_2+2}} I_{n_2}
\end{array} \right) = \left( I_{n_1} - \frac{1}{r_1+1} n_2 A(G_1) \right) \left( \begin{array}{c|c}
-R(G_1) \otimes C^T_{n_2} & I_{m_1} \otimes I_{n_2} \\
I_{m_1} \otimes I_{n_2} & -\frac{1}{r_2+2} A(G_2)
\end{array} \right).$$
(iv) For any two graphs $G_1$ and $G_2$, the adjacency matrix of $G_1 \ast G_2$ is given by [7]:

$$A(G_1 \ast G_2) = \begin{pmatrix}
A(G_1) & J_{n_2}^T \otimes A(G_1) \\
(J_{n_2}^T \otimes A(G_1))^T & A(G_2) \otimes I_{n_1}
\end{pmatrix}.$$

Since $G_1$ is a $r_1$ regular graph with $n_1$ vertices and $G_2$ is a $r_2$ regular graph with $n_2$ vertices, the normalized Laplacian matrix of $G_1 \ast G_2$ will be

$$\mathcal{L}(G_1 \ast G_2) = I_{n_1(n_2+1)} - \left(\frac{1}{\sqrt{r_1+r_1n_2}}I_{n_1} \otimes O \right) \times \left(\frac{1}{\sqrt{r_2+r_2n_2}}I_{n_2} \otimes I_{n_1} \right) \times \left(\frac{1}{\sqrt{r_1+r_1n_2}}I_{n_2} \otimes I_{n_1} \right).$$

$$= \begin{pmatrix}
I_{n_1} - \frac{1}{r_1+r_1n_2}A(G_1) & -C_{n_2}^T \otimes A(G_1) \\
-C_{n_2}^T \otimes A(G_1) & I_{n_2} - \frac{1}{r_2+r_2n_2}A(G_2) \otimes I_{n_1}
\end{pmatrix} \times \begin{pmatrix}
\mathcal{L}(G_1) \otimes I_{n_1} \\
\mathcal{L}(G_2) \otimes B(G_2) \otimes I_{n_1}
\end{pmatrix}.$$

\[\square\]

**Notation 2.1.** Let $G$ be a graph on $n$ vertices, $B$ and $C$ be matrices of size $n \times n$ and $n \times 1$ respectively. For any parameter $\lambda$, we have the notation

$$\chi_G(B, C, \lambda) = C^T(\lambda I_n - (\mathcal{L}(G) \otimes B))^{-1}C.$$

We note that the notation is similar to the notion ‘coronal’ which was introduced by McLeman[13].

**Theorem 2.1.** For $i = 1, 2$, let $G_i$ be $r_i$-regular graph on $n_i$ vertices with $G_1$ connected. Then the normalized Laplacian spectrum of $G_1 \circ G_2$ consists of

(i) the eigenvalue $\frac{1+r_2\delta_j}{r_2+1}$ with multiplicity $n_1$ for every eigenvalue $\delta_j$, $j = 2, \ldots, n_2$, of $\mathcal{L}(G_2)$;

(ii) two simple eigenvalues

$$\frac{(2n_2+n_2r_2+r_1+r_1\mu_1+r_1r_2\mu_1) \pm \sqrt{(2n_2+n_2r_2+r_1+r_1\mu_1+r_1r_2\mu_1)^2-4r_1\mu_1(r_2+1)(r_1+n_2)^2}}{2(r_2+1)(r_1+n_2)},$$

for each eigenvalue $\mu_i$, $i = 1, 2, \ldots, n_1$, of $\mathcal{L}(G_1)$.

**Proof.** The normalized Laplacian characteristic polynomial of $G_1 \circ G_2$ is

$$f_{G_1 \circ G_2}(\lambda) = \det(\lambda I_{n_1(n_2+1)} - \mathcal{L}(G_1 \circ G_2))$$

$$= \det(\lambda I_{n_1} - (\mathcal{L}(G_1) \otimes B(G_1))$$

$$= \det(\lambda I_{n_1} - (\mathcal{L}(G_2) \otimes B(G_2)) \otimes I_{n_1})$$

$$= \det(\lambda I_{n_2} - (\mathcal{L}(G_2) \otimes B(G_2)) \otimes I_{n_1}).$$
\[
= \det((\lambda I_{n_2} - (L(G_2) \bullet B(G_2))) \otimes I_{n_1})
\]
\[
\times \det(\lambda I_{n_1} - (L(G_1) \bullet B(G_1)) - (C_{n_2}^T \otimes I_{n_1})
\]
\[
\times ((\lambda I_{n_2} - (L(G_2) \bullet B(G_2))) \otimes I_{n_1})^{-1}(C_{n_2}^T \otimes I_{n_1})^T
\]  
by Lemma 1.1
\[
= \det(\lambda I_{n_2} - (L(G_2) \bullet B(G_2)))^{n_1}
\]
\[
\times \det(\lambda I_{n_1} - (L(G_1) \bullet B(G_1)) - (C_{n_2}^T (\lambda I_{n_2} - (L(G_2) \bullet B(G_2)))^{-1} C_{n_2} \otimes I_{n_1})
\]
\[
= \det(\lambda I_{n_2} - (L(G_2) \bullet B(G_2)))^{n_1}
\]
\[
\times \det((\lambda - \chi G_2(B(G_2), C_{n_2}, \lambda)) I_{n_1} - (L(G_1) \bullet B(G_1)))
\]  
from Notation 2.1.

Since \(L(G_2) \bullet B(G_2) = I_{n_2} - \frac{1}{r_2+1} A(G_2)\), one gets \(L(G_2) \bullet B(G_2) = \frac{1}{r_2+1}(I_{n_2} + r_2 L(G_2))\).

As \(G_2\) is regular, the sum of all entries on every row of \(L(G_2)\) is zero. That means
\[
L(G_2) C_{n_2} = \left(1 - \frac{r_2}{r_2+1}\right) C_{n_2} = 0 C_{n_2}.
\]

Then
\[
(L(G_2) \bullet B(G_2)) C_{n_2} = \left(1 - \frac{r_2}{r_2+1}\right) C_{n_2} = \frac{1}{r_2+1} C_{n_2}
\]

and
\[
(\lambda I_{n_2} - (L(G_2) \bullet B(G_2))) C_{n_2} = \left(\lambda - \frac{1}{r_2+1}\right) C_{n_2}.
\]

Also, \(C_{n_2}^T C_{n_2} = \frac{n_2}{(r_1+n_2)(r_2+1)}\). Therefore,
\[
\chi G_2(B(G_2), C_{n_2}, \lambda) = C_{n_2}^T (\lambda I_{n_2} - (L(G_2) \bullet B(G_2)))^{-1} C_{n_2}
\]
\[
= \frac{C_{n_2}^T C_{n_2}}{(\lambda - \frac{1}{r_2+1}) (r_1+n_2)(r_2+1) \left(\lambda - \frac{1}{r_2+1}\right)}.
\]

Again, since \(L(G_1) \bullet B(G_1) = I_{n_1} - \frac{1}{r_1+n_2} A(G_1)\), then \(L(G_1) \bullet B(G_1) = \frac{1}{r_1+n_2}(n_2 I_{n_2} + r_1 L(G_1))\).

Now, if \(\delta_j\) is an eigenvalue of \(L(G_2)\) and \(\mu_i\) is an eigenvalue of \(L(G_1)\) then
\[
f_{G_1 \circ G_2}(\lambda) = \prod_{j=1}^{n_2} \left(\lambda - \frac{1 + r_2 \delta_j}{r_2+1}\right)^{n_1}
\]
\[
\times \prod_{i=1}^{n_1} \left(\lambda - \frac{n_2}{(r_1+n_2)(r_2+1)} \left(\lambda - \frac{1}{r_2+1}\right) - \frac{n_2 + r_1 \mu_i}{r_1+n_2}\right).
\]

(i) Since the only pole of \(\chi G_2(B(G_2), C_{n_2}, \lambda) = \frac{n_2}{(r_1+n_2)(r_2+1)} (\lambda - \frac{1}{r_2+1})\) is \(\lambda = \frac{1}{r_2+1}\) and 0 is an eigenvalue of \(L(G_2)\), then \(\frac{1 + r_2 \delta_j}{r_2+1}\) is an eigenvalue of \(L(G_1 \circ G_2)\) with multiplicity \(n_1\) for \(j = 2, \ldots, n_2\).

(ii) The remaining \(2n_1\) eigenvalues are obtained by solving the equation
\[
\lambda - \frac{n_2}{(r_1+n_2)(r_2+1)} \left(\lambda - \frac{1}{r_2+1}\right) - \frac{n_2 + r_1 \mu_i}{r_1+n_2} = 0.
\]
or
\[(r_2 + 1)(r_1 + n_2)\lambda^2 - (2n_2 + n_2r_2 + r_1 + r_1\mu_i + r_1r_2\mu_i)\lambda + r_1\mu_i = 0.\]
So the eigenvalues are,
\[\lambda_i = \frac{(2n_2 + n_2r_2 + r_1 + r_1\mu_i + r_1r_2\mu_i)\pm \sqrt{(2n_2 + n_2r_2 + r_1 + r_1\mu_i + r_1r_2\mu_i)^2 - 4r_1\mu_i(r_2 + 1)(r_1 + n_2)}}{2(r_2 + 1)(r_1 + n_2)},\]
for \(i = 1, 2, \ldots, n_1.\)

\[\square\]

**Theorem 2.2.** For \(i = 1, 2,\) let \(G_i\) be \(r_i\)-regular graph on \(n_i\) vertices and \(m_i\) edges with \(G_1\) connected. Then the normalized Laplacian spectrum of \(G_1 \circ G_2\) consists of

(i) the eigenvalue \(\frac{2+4r_2\delta_i}{r_2+2}\) with multiplicity \(m_1\) for every eigenvalue \(\delta_j, j = 2, \ldots, n_2,\) of \(\mathcal{L}(G_2)\);

(ii) two simple eigenvalues
\[\frac{(2+4n_2+r_2n_2+2\mu_i+2m_i)\pm \sqrt{(2+4n_2+r_2n_2+2\mu_i)^2-4\mu_i(n_2+2)(r_2+2)(n_2+1)}}{2(r_2+2)(n_2+1)},\]
for each eigenvalue \(\mu_i, i = 1, 2, \ldots, n_1,\) of \(\mathcal{L}(G_1)\);

(iii) the eigenvalue \(\frac{2}{r_2+2}\) with multiplicity \((m_1-n_1)\) if \(n_1 < m_1.\)

**Proof.** The normalized Laplacian characteristic polynomial of \(G_1 \circ G_2\) is
\[f_{G_1 \circ G_2}(\lambda) = \det(\lambda I_{n_1+m_1n_2} - \mathcal{L}(G_1 \circ G_2))\]
\[= \det\left(\lambda I_{n_1} - (\mathcal{L}(G_1) \cdot B(G_1)) \begin{pmatrix} R(G_1) \otimes C_{n_2}^T & R(G_1) \otimes C_{n_2}^T \\ R(G_1) \otimes C_{n_2}^T & I_{m_1} \otimes (\mathcal{L}(G_2) \cdot B(G_2)) \end{pmatrix}\right)\]
\[= \det\left(\lambda I_{n_1} - (\mathcal{L}(G_1) \cdot B(G_1)) \begin{pmatrix} R(G_1) \otimes C_{n_2}^T & R(G_1) \otimes C_{n_2}^T \\ R(G_1) \otimes C_{n_2}^T & I_{m_1} \otimes (\mathcal{L}(G_2) \cdot B(G_2)) \end{pmatrix}\right)\]
\[\times (I_{m_1} \otimes (\mathcal{L}(G_2) \cdot B(G_2)))^{-1}(R(G_1) \otimes C_{n_2}^T)^T \text{[by Lemma 1.1]}\]
\[= \det(\lambda I_{n_2} - (\mathcal{L}(G_2) \cdot B(G_2)))^{m_1} \times \det(\lambda I_{n_1} - (\mathcal{L}(G_1) \cdot B(G_1)))\]
\[- (R(G_1) I_{m_1} R(G_1)^T) \otimes (C_{n_2}^T (\mathcal{L}(G_2) \cdot B(G_2))^{-1} C_{n_2})\]
\[[\text{It is well known [6] that } R(G_1) R(G_1)^T = A(G_1) + r_1 I_{n_1} \text{ and } A(G_1) = r_1 (I_{n_1} - \mathcal{L}(G_1)), \text{ so one gets } R(G_1) R(G_1)^T = r_1 (2I_{n_1} - \mathcal{L}(G_1))]\]
\[= \det(\lambda I_{n_2} - (\mathcal{L}(G_2) \cdot B(G_2)))^{m_1} \det(\lambda I_{n_1} - (\mathcal{L}(G_1) \cdot B(G_1)))\]
\[- r_1 (2I_{n_1} - \mathcal{L}(G_1)) \otimes \chi_{G_2}(B(G_2), C_{n_2}, \lambda) \text{[from Notation 2.1]}\]
\[= \det(\lambda I_{n_2} - (\mathcal{L}(G_2) \cdot B(G_2)))^{m_1} \det((\lambda - 2r_1 \chi_{G_2}(B(G_2), C_{n_2}, \lambda)) I_{n_1}\]
\[- (\mathcal{L}(G_1) \cdot B(G_1)) + r_1 \chi_{G_2}(B(G_2), C_{n_2}, \lambda) \mathcal{L}(G_1))\]

Since \(\mathcal{L}(G_2) \cdot B(G_2) = I_{n_2} - \frac{1}{r_2+2} A(G_2),\) we get \(\mathcal{L}(G_2) \cdot B(G_2) = \frac{1}{r_2+2}(2I_{n_2} + r_2 \mathcal{L}(G_2)).\)

The sum of all entries on every row of \(\mathcal{L}(G_2)\) is zero because \(G_2\) is regular. That
means \( \mathcal{L}(G_2)C_{n_2} = (1 - \frac{r_2}{r_2})C_{n_2} = 0C_{n_2} \). Then

\[ (\mathcal{L}(G_2) \cdot B(G_2))C_{n_2} = \left(1 - \frac{r_2}{r_2 + 2}\right)C_{n_2} = \frac{2}{r_2 + 2}C_{n_2} \]

and

\[ (\lambda I_{n_2} - (\mathcal{L}(G_2) \cdot B(G_2)))C_{n_2} = \left(\lambda - \frac{2}{r_2 + 2}\right)C_{n_2}. \]

Also, \( C_n^{T}C_{n_2} = \frac{n_2}{(r_1 + r_1n_2)(r_2 + 2)} \). Hence,

\[ \chi_{G_2}(B(G_2), C_{n_2}, \lambda) = C_n^{T}(\lambda I_{n_2} - (\mathcal{L}(G_2) \cdot B(G_2)))^{-1}C_{n_2} \]

\[ = \frac{n_2}{(r_1 + r_1n_2)(r_2 + 2)\left(\lambda - \frac{2}{r_2 + 2}\right)}. \]

As \( \mathcal{L}(G_1) \cdot B(G_1) = I_{n_1} - \frac{1}{r_1 + r_1n_2}A(G_1) \), we have

\[ \mathcal{L}(G_1) \cdot B(G_1) = \frac{1}{r_1 + r_1n_2}(r_1n_2I_{n_2} + r_1\mathcal{L}(G_1)) = \frac{1}{n_2 + 1}(n_2I_{n_2} + \mathcal{L}(G_1)). \]

Now, if \( \delta_j \) is an eigenvalue of \( \mathcal{L}(G_2) \) and \( \mu_i \) is an eigenvalue of \( \mathcal{L}(G_1) \) then,

\[ f_{G_1 \circ G_2}(\lambda) = \prod_{j=1}^{n_2} \left(\lambda - \frac{2 + r_2\delta_j}{r_2 + 2}\right)^{m_1} \prod_{i=1}^{n_1} \left(\lambda - \frac{2n_2}{(n_2 + 1)(r_2 + 2)\left(\lambda - \frac{2}{r_2 + 2}\right)}\right)^{n_2\mu_i} \frac{-n_2 + \mu_i}{n_2 + 1} \frac{n_2\mu_i}{(n_2 + 1)(r_2 + 2)\left(\lambda - \frac{2}{r_2 + 2}\right)}. \]

(i) Since the only pole of \( \chi_{G_2}(B(G_2), C_{n_2}, \lambda) = \frac{n_2}{(r_1 + r_1n_2)(r_2 + 2)\left(\lambda - \frac{2}{r_2 + 2}\right)} \) is \( \lambda = \frac{2}{r_2 + 2} \) and 0 is an eigenvalue of \( \mathcal{L}(G_2) \), one gets that \( \frac{2 + r_2\delta_j}{r_2 + 2} \) is an eigenvalue of \( \mathcal{L}(G_1 \circ G_2) \) with multiplicity \( m_1 \) for \( j = 2, \ldots, n_2 \).

(ii) The 2\( n_1 \) eigenvalues are obtained by solving the equation

\[ \lambda - \frac{2n_2}{(n_2 + 1)(r_2 + 2)\left(\lambda - \frac{2}{r_2 + 2}\right)} - \frac{n_2 + \mu_i}{n_2 + 1} + \frac{n_2\mu_i}{(n_2 + 1)(r_2 + 2)\left(\lambda - \frac{2}{r_2 + 2}\right)} = 0. \]

or

\[ (n_2 + 1)(r_2 + 2)\lambda^2 - (2 + 4n_2 + r_2n_2 + r_2\mu_i + 2\mu_i)\lambda + \mu_i(n_2 + 2) = 0. \]

So the eigenvalues are,

\[ \lambda_i = \frac{(2 + 4n_2 + r_2n_2 + r_2\mu_i + 2\mu_i) + \sqrt{(2 + 4n_2 + r_2n_2 + r_2\mu_i + 2\mu_i)^2 - 4\mu_i(n_2 + 2)(r_2 + 2)(n_2 + 1)}}{2(r_2 + 2)(n_2 + 1)} \]

for \( i = 1, 2, \ldots, n_1 \).
(iii) Since $G_1$ is connected regular graph, then $n_1 \leq m_1$. If $n_1 = m_1$ then all eigenvalues are obtained by (i) and (ii). If $n_1 < m_1$ then the remaining $n_1 + m_1 - n_1 = n_2$ (where $n_2 = m_1 - n_1$) normalized Laplacian eigenvalues of $G$ must come from the only pole $\lambda = \frac{2}{r_2+2}$ of $\chi_{G_2}(B(G_2), C_{n_2}, \lambda) = \frac{n_2}{(r_1+r_1m_2)(r_2+2)(\lambda - \frac{2}{r_2+2})}$. \hfill $\square$

**Theorem 2.3.** For $i = 1, 2$, let $G_i$ be $r_i$-regular graph on $n_i$ vertices with $G_1$ connected. Then the normalized Laplacian spectrum of $G_1 \star G_2$ consists of

(i) the eigenvalue $\frac{r_1+r_2\delta}{r_2+r_1}$ with multiplicity $n_1$ for every eigenvalue $\delta_j$, $j = 2, \ldots, n_2$, of $\mathcal{L}(G_2)$;

(ii) two simple eigenvalues

$$\frac{r_1+2r_2n_2+2r_1n_2+r_1\mu_i+r_2\mu_i}{2(r_2+r_1)(n_2+1)}$$

for each eigenvalue $\mu_i$, $i = 1, 2, \ldots, n_1$, of $\mathcal{L}(G_1)$.

**Proof.** The normalized Laplacian characteristic polynomial of $G_1 \star G_2$ is

$$f_{G_1 \star G_2}(\lambda) = \det(\lambda I_{n_1(n_2+1)} - \mathcal{L}(G_1 \star G_2))$$

$$= \det \left( \begin{array}{c} \lambda I_{n_1} - (\mathcal{L}(G_1) \bullet B(G_1)) \\ (C^T_{n_2} \otimes A(G_1))^T \\ \end{array} \right) I_{n_1n_2} - (\mathcal{L}(G_2) \bullet B(G_2))^2$$

$$= \det(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))) \otimes I_{n_1}$$

$$= \det(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))^2).$$

Since $\mathcal{L}(G_2) \bullet B(G_2) = I_{n_2} - \frac{1}{r_2+r_1} A(G_2)$, we have $\mathcal{L}(G_2) \bullet B(G_2) = \frac{1}{r_2+r_1} (r_1 I_{n_2} + r_2 \mathcal{L}(G_2))$. As $G_2$ is regular, the sum of all entries on every row of $\mathcal{L}(G_2)$ is zero. That means, $\mathcal{L}(G_2)C_{n_2} = (1 - \frac{r_2}{r_2+r_1})C_{n_2} = 0C_{n_2}$. Then

$$\mathcal{L}(G_2) \bullet B(G_2)C_{n_2} = \left( 1 - \frac{r_2}{r_2+r_1} \right) C_{n_2} = \frac{r_1}{r_2+r_1} C_{n_2}$$

and

$$\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)) = \left( \lambda - \frac{r_1}{r_2+r_1} \right) C_{n_2}.$$
Also, $C_{n_2}^TC_{n_2} = \frac{n_2}{(r_1 + r_1n_2)(r_2 + r_1)}$. Hence,
\[
\chi_{G_2}(B(G_2), C_{n_2}, \lambda) = C_{n_2}^T(M_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))^{-1}C_{n_2}
= \frac{n_2}{(r_1 + r_1n_2)(r_2 + r_1)}(\lambda - \frac{r_1}{r_2 + r_1}).
\]
Again, since $\mathcal{L}(G_1) \bullet B(G_1) = I_{n_1} - \frac{1}{r_1 + r_1n_2}A(G_1)$, we get
\[
\mathcal{L}(G_1) \bullet B(G_1) = \frac{1}{r_1 + r_1n_2}(r_1n_2I_{n_2} + r_1\mathcal{L}(G_1))
= \frac{1}{n_2 + 1}(n_2I_{n_2} + \mathcal{L}(G_1)).
\]
Now, if $\delta_j$ is an eigenvalue of $\mathcal{L}(G_2)$ and $\mu_i$ is an eigenvalue of $\mathcal{L}(G_1)$ then,
\[
f_{G_1 \ast G_2}(\lambda) = \prod_{j=1}^{n_2} \left(\lambda - \frac{1 + r_2\delta_j}{r_2 + 1}\right)^{n_1}
\times \prod_{j=1}^{n_2} \left(\lambda - \frac{n_2 + \mu_i}{n_2 + 1} - \frac{n_2r_1(\mu_i^2 - 2\mu_i + 1)}{(n_2 + 1)(r_2 + r_1)}\right).
\]
(i) Since the only pole of $\chi_{G_2}(B(G_2), C_{n_2}, \lambda) = \frac{n_2}{(r_1 + r_1n_2)(r_2 + r_1)(\lambda - \frac{r_1}{r_2 + r_1})}$ is $\lambda = \frac{r_1}{r_2 + r_1}$ and $0$ is an eigenvalue of $\mathcal{L}(G_2)$, then $\frac{r_1 + r_2\delta_j}{r_2 + r_1}$ is an eigenvalue of $\mathcal{L}(G_1 \ast G_2)$ with multiplicity $n_1$ for $j = 2, \ldots, n_2$.
(ii) The remaining $2n_1$ eigenvalues are obtained by solving the equation
\[
\lambda - \frac{n_2 + \mu_i}{n_2 + 1} - \frac{n_2r_1(\mu_i^2 - 2\mu_i + 1)}{(n_2 + 1)(r_2 + r_1)} = 0.
\]
or
\[
(n_2 + 1)(r_2 + r_1)\lambda^2 - (r_1 + 2r_1n_2 + r_2n_2 + r_1\mu_i + r_2\mu_i)\lambda + r_1\mu_i(1 - n_2\mu_i + 2n_2) = 0.
\]
So the eigenvalues are,
\[
\lambda_i = \frac{(r_1 + 2r_1n_2 + r_2n_2 + r_1\mu_i + r_2\mu_i) \pm \sqrt{(r_1 + 2r_1n_2 + r_2n_2 + r_1\mu_i + r_2\mu_i)^2 - 4r_1\mu_i(r_2 + r_1)(n_2 + 1)(1 - n_2\mu_i + 2n_2)}}{2(r_2 + r_1)(n_2 + 1)}
\]
for $i = 1, 2, \ldots, n_1$.

\[\square\]

\textbf{Example 2.1.} Let us consider $G_1 = C_4$ and $G_2 = K_3$. Then the normalized Laplacian eigenvalues of $G_1$ are $1 - \cos \frac{2\pi k}{4}$ for $k = 0, 1, 2, 3$ and the normalized Laplacian eigenvalues of $G_2$ are $0$ and $\frac{2}{3}$ with multiplicity $2$.

Now using the result of Theorem 2.1, we get the normalized Laplacian spectrum of $G_1 \circ G_2$, as
\[
\left\{ \frac{4}{3} \text{ (multiplicity 8)}, 0, \frac{14}{15}, \frac{10 \pm \sqrt{70}}{15} \text{ (multiplicity 2)}, \frac{13 \pm \sqrt{109}}{15} \right\}.
\]
From Theorem 2.2, we get the normalized Laplacian spectrum of $G_1 \diamond G_2$, as
\[
\left\{ \frac{5}{4} \text{ (multiplicity 8)}, 0, \frac{5}{4}, \frac{3 \pm \sqrt{2}}{4} \text{ (multiplicity 2)}, \frac{7 \pm \sqrt{3}}{8} \right\}.
\]
Finally by Theorem 2.3, we get the normalized Laplacian spectrum of $G_1 \star G_2$, as
\[
\left\{ \frac{5}{4} \text{ (multiplicity 8)}, 0, \frac{5}{4}, \frac{1}{2} \text{ (multiplicity 2)}, 1 \text{ (multiplicity 2)}, \frac{7 \pm \sqrt{33}}{8} \right\}.
\]

Remark 2.1. If $G_1$ and $G_2$ are two regular graphs then we find from Theorems 2.1, 2.2 and 2.3, that the normalized Laplacian spectrum of all the coronas depend only on the degrees of regularities, number of vertices, number of edges, and normalized Laplacian eigenvalues of $G_1$ and $G_2$. Thus for $i = 1, 2$, if $G_i$ and $H_i$ are $\mathcal{L}$-cospectral regular graphs then $G_1 \circ G_2$ (resp. $G_1 \circ G_2$ and $G_1 \star G_2$) is $\mathcal{L}$-cospectral with $H_1 \circ H_2$ (resp. $H_1 \circ H_2$ and $H_1 \star H_2$).

Now we apply the results of the paper and determine some normalized Laplacian cospectral graphs. Since for an $r$-regular graph $G$ we have $\mathcal{L}(G) = I_n - \frac{1}{r}A(G)$, the Lemma below is immediate.

Lemma 2.2. Two regular graphs are $\mathcal{L}$-cospectral if and only if they are cospectral.

In the literature there are several regular cospectral graphs, for example see [15]. In Theorem 2.4 below we construct non-regular $\mathcal{L}$-cospectral graphs using coronas. Proof of this theorem follows from Remark 2.1 and Lemma 2.2.

Theorem 2.4. If $G_1$ and $H_1$ (not necessarily distinct) are $\mathcal{L}$-cospectral regular graphs, and $G_2$ and $H_2$ (not necessarily distinct) and not necessarily different from $G_1$ and $H_1$, are $\mathcal{L}$-cospectral regular graphs, then $G_1 \circ G_2$ (resp. $G_1 \circ G_2$ and $G_1 \star G_2$) and $H_1 \circ H_2$ (resp. $H_1 \circ H_2$ and $H_1 \star H_2$) are $\mathcal{L}$-cospectral graphs.

Example 2.2. Applying Theorem 2.4 here we construct $\mathcal{L}$-cospectral graphs. We consider regular cospectral graphs $G_1$ and $H_1$ [15] as given in Figure 1.

![Figure 1. Two cospectral regular graphs.](image)

We also consider graphs $G_2$ and $H_2$ both of which are copies of $K_2$. Now by Theorem 2.4 graphs $G_1 \circ K_2$ and $H_1 \circ K_2$ given in Figure 2a and Figure 2b respectively are...
\(\mathcal{L}\)-cospectral. Similarly graphs \(G_1 \circ K_2\) (resp. \(G_1 \star K_2\)) and \(H_1 \circ K_2\) (resp. \(H_1 \star K_2\)) are also \(\mathcal{L}\)-cospectral graphs.

![non-regular nonisomorphic \(\mathcal{L}\)-cospectral graphs](image)

**Figure 2.** Non-regular nonisomorphic \(\mathcal{L}\)-cospectral graphs.

**REFERENCES**


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